

CHAPTER 2

MODELING OF DYNAMIC SYSTEMS

PROBLEMS

Problems for Section 2-1

2-1. Find the equation of the motion of the mass-spring system shown in Fig. 2P-1. Also calculate the natural frequency of the system.

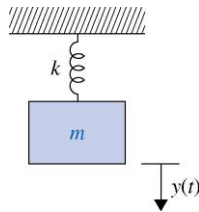
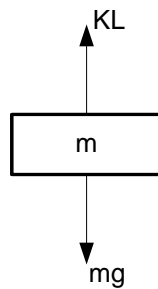


Figure 2P-1

When the mass is added to spring, then the spring will stretch from position O to position L.



The total potential energy is:

$$U_s = \frac{1}{2} K (L + y)^2$$

where y is a displacement from equilibrium position L.

The gravitational energy is:

$$U_g = -mgy$$

The kinetic energy of the mass-spring system is calculated by:

$$T = \frac{1}{2} m \dot{y}^2$$

As we know that $T + U_g + U_s = \text{constant}$, then

$$\frac{1}{2} m \dot{y}^2 - mgy + \frac{1}{2} K (L + y)^2 = \text{constant}$$

By differentiating from above equation, we have:

$$m \dot{y} \ddot{y} - mg \dot{y} + \dot{y} K (L + y) = 0$$

$$m \ddot{y} + Ky \dot{y} + \dot{y} (KL - mg) = 0$$

since $KL = mg$, therefore:

$$\dot{y} (m \ddot{y} + Ky) = 0$$

As \dot{y} cannot be zero because of vibration, then $m \ddot{y} + Ky = 0$

Alternative using Newton's Law:

At equilibrium from the Free-body-diagram we have

$$KL = mg$$

Measured from equilibrium position using $y(t)$, we have

$$F = Ma$$

$$-Ky(t) = M \ddot{y}(t)$$

or

$$M \ddot{y}(t) + Ky(t) = 0$$

2-2. Find its single spring-mass equivalent in the five-spring one-mass system shown in Fig. 2P-2. Also calculate the natural frequency of the system.

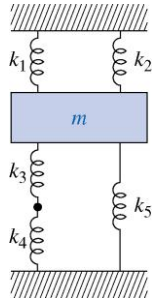
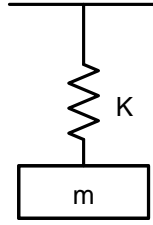
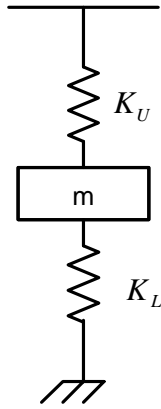


Figure 2P-2



$$K_u = K_1 + K_2$$

$$K_L = \frac{K_3 K_4}{K_3 + K_4} + K_5$$

$$K = K_u + K_L$$

$$= K_1 + K_2 + K_5 + \frac{K_3 K_4}{K_3 + K_4}$$

$$= \frac{K_1 K_3 + K_2 K_3 + K_5 K_3 + K_1 K_4 + K_2 K_4 + K_4 K_5 + K_3 K_4}{K_3 + K_4}$$

$$\omega_n = \sqrt{\frac{K}{m}}$$

2-3. Find the equation of the motion for a simple model of a vehicle suspension system hitting a bump. As shown in Fig. 2P-3 the mass of wheel and its mass moment of inertia are m and J , respectively. Also calculate the natural frequency of the system.

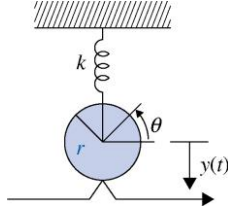


Figure 2P-3

a) **Rotational kinetic energy:** $T_{rot} = \frac{1}{2} J \dot{\theta}^2$

Translational kinetic energy: $T_T = \frac{1}{2} m \dot{y}^2$

Relation between translational displacement and rotational displacement:

$$y = r\theta$$

$$\dot{y} = r\dot{\theta}$$

$$T_{Rot} = \frac{1}{2} \frac{J}{r^2} \dot{y}^2$$

Potential energy: $U = \frac{1}{2} K y^2$

From conservation of energy $T_{Rot} + T_T + U = \text{constant}$, then:

$$\frac{1}{2} \frac{J}{r^2} \dot{y}^2 + \frac{1}{2} m \dot{y}^2 + \frac{1}{2} K y^2 = \text{constant}$$

By differentiating, we have:

$$\frac{J}{r^2} \dot{y} \ddot{y} + m \dot{y} \ddot{y} + K y \dot{y} = 0$$

$$\dot{y} \left(\frac{J}{r^2} \ddot{y} + m \ddot{y} + K y \right) = 0$$

Since \dot{y} cannot be zero, then $J \frac{\ddot{y}}{r^2} + m \ddot{y} + K y = 0$

Alternatively using Newton's law, take a moment about point P , assuming motion is counterclockwise, and as the wheel goes above the bump, y is upwards. Also we assume the system starts from equilibrium (in the vertical direction) where the spring force and the weight of the system cancel each other. So mg does not appear

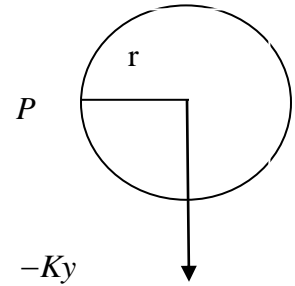
in the equations. As the wheel moves up the spring compresses by y measured from the equilibrium.

Assuming positive direction is counterclockwise, we have

$$\sum Mom_p = -Kyr = J\ddot{\theta} + mr\ddot{y}$$

$$J \frac{\ddot{y}}{r^2} + m\ddot{y} + Ky = 0$$

b)



Natural frequency is the coefficient of y divided by the coefficient of \ddot{y}

$$\omega_n = \sqrt{\frac{K}{m + \frac{J}{r^2}}} = r \sqrt{\frac{K}{mr^2 + J}}$$

2-4. Write the force equations of the linear translational systems shown in Fig. 2P-4.

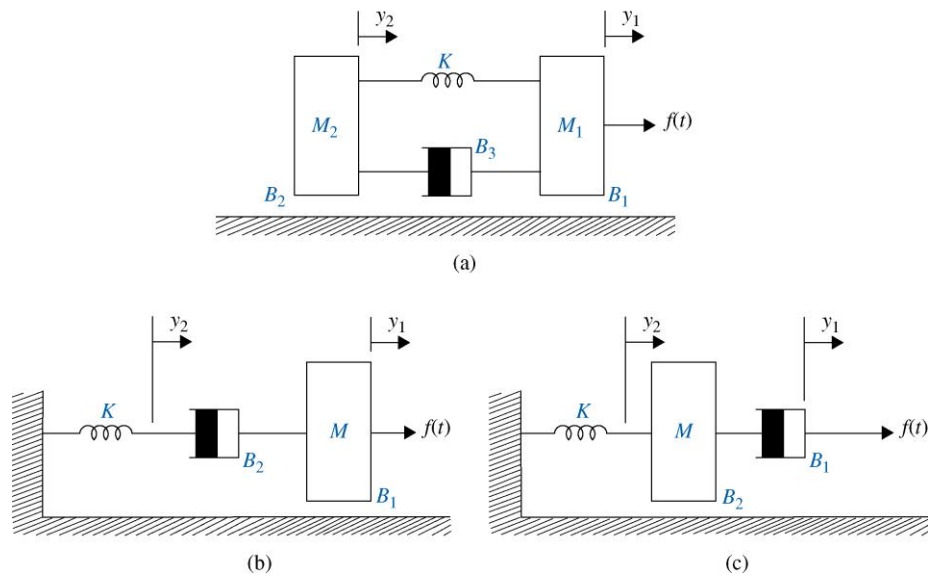
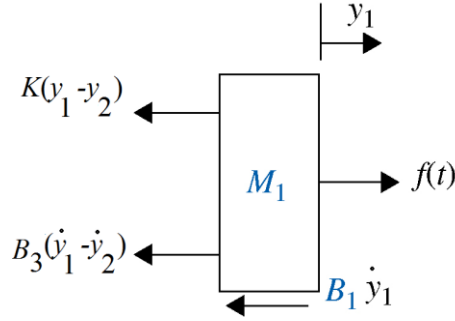
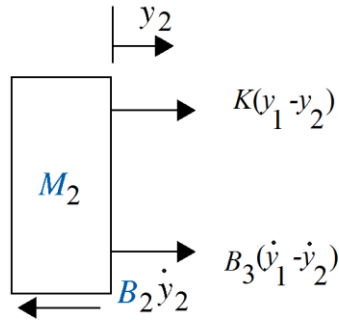


Figure 2P-4

(a) Force equations:



$$M_1 \ddot{y}_1 = -B_1 \dot{y}_1 - B_3 (\dot{y}_1 - \dot{y}_2) - K(y_1 - y_2) + f(t)$$



$$M_2 \ddot{y}_2 = -B_2 \dot{y}_2 + B_3 (\dot{y}_1 - \dot{y}_2) + K(y_1 - y_2)$$

$$f(t) = M_1 \frac{d^2 y_1}{dt^2} + B_1 \frac{dy_1}{dt} + B_3 \left(\frac{dy_1}{dt} - \frac{dy_2}{dt} \right) + K(y_1 - y_2)$$

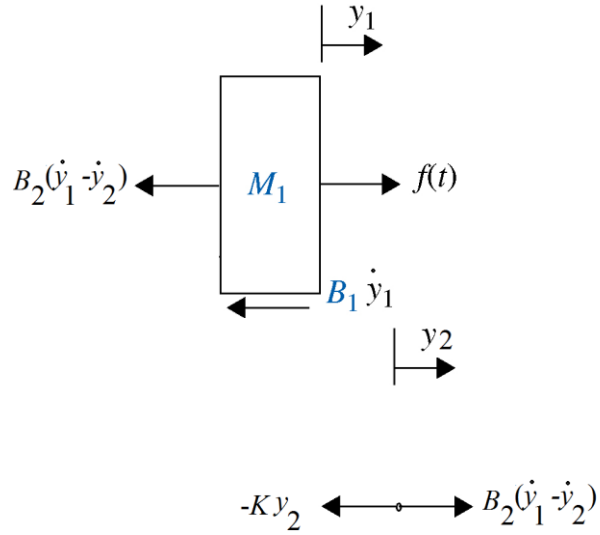
$$B_3 \left(\frac{dy_1}{dt} - \frac{dy_2}{dt} \right) + K(y_1 - y_2) + M_2 \frac{d^2 y_2}{dt^2} + B_2 \frac{dy_2}{dt}$$

Rearrange the equations as follows:

$$\frac{d^2 y_1}{dt^2} = -\frac{(B_1 + B_3)}{M_1} \frac{dy_1}{dt} + \frac{B_3}{M_1} \frac{dy_2}{dt} - \frac{K}{M_1} (y_1 - y_2) + \frac{f}{M_1}$$

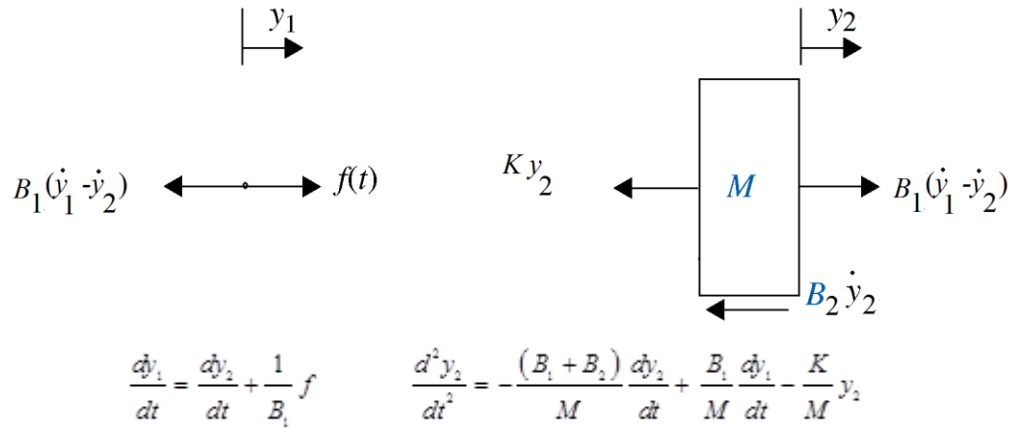
$$\frac{d^2 y_2}{dt^2} = \frac{B_3}{M_2} \frac{dy_1}{dt} - \frac{(B_2 + B_3)}{M_2} \frac{dy_2}{dt} + \frac{K}{M_2} (y_1 - y_2)$$

(b) Force equations:



$$\frac{d^2 y_1}{dt^2} = -\frac{(B_1 + B_2)}{M} \frac{dy_1}{dt} + \frac{B_2}{M} \frac{dy_2}{dt} + \frac{1}{M} f \quad \frac{dy_2}{dt} = \frac{dy_1}{dt} - \frac{K}{B_2} y_2$$

(c) **Force equations:**



$$\frac{d\dot{y}_1}{dt} = \frac{d\dot{y}_2}{dt} + \frac{1}{B_1} f \quad \frac{d^2 y_2}{dt^2} = -\frac{(B_1 + B_2)}{M} \frac{d\dot{y}_2}{dt} + \frac{B_1}{M} \frac{d\dot{y}_1}{dt} - \frac{K}{M} y_2$$

2-5. Write the force equations of the linear translational system shown in Fig. 2P-5.

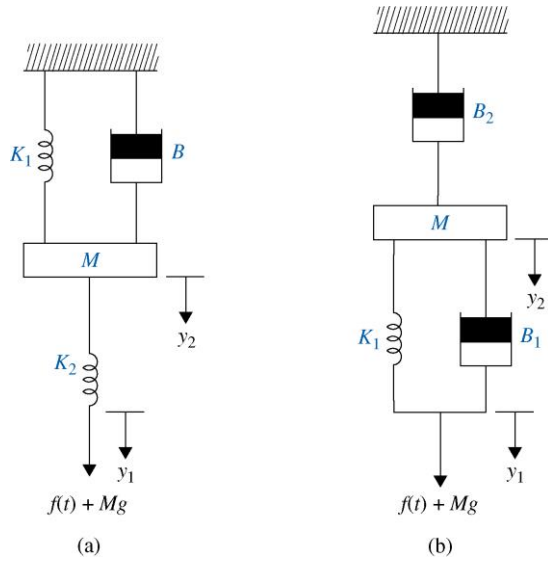


Figure 2P-5

(a) Force equations:

$$y_1 = \frac{1}{K_2} (f + Mg) + y_2 \quad \frac{d^2 y_2}{dt^2} = -\frac{B}{M} \frac{dy_2}{dt} - \frac{K_1 + K_2}{M} y_2 + \frac{K_2}{M} y_1$$

(b) Force equations:

$$M\ddot{y}_2 = K_1(y_1 - y_2) + B_1(\dot{y}_1 - \dot{y}_2) - B_2\dot{y}_2$$

$$f(t) + Mg = K_1(y_1 - y_2) + B_1(\dot{y}_1 - \dot{y}_2)$$

$$M\ddot{y}_2 + B_2\dot{y}_2 = f(t) + Mg$$

$$\frac{dy_1}{dt} = \frac{1}{B_1} [f(t) + Mg] + \frac{dy_2}{dt} - \frac{K_1}{B_1} (y_1 - y_2) \quad \frac{d^2 y_2}{dt^2} = \frac{B_1}{M} \left(\frac{dy_1}{dt} - \frac{dy_2}{dt} \right) + \frac{K_1}{M} (y_1 - y_2) - \frac{B_2}{M} (\dot{y}_1 - \dot{y}_2) - \frac{B_2}{M} \frac{dy_2}{dt}$$

2-6. Consider a train consisting of an engine and a car, as shown in Fig. 2P-6.

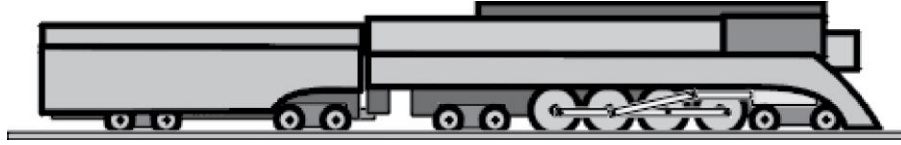
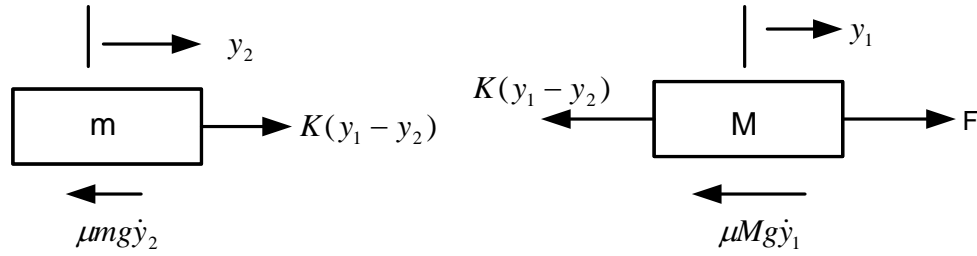


Figure 2P-6

A controller is applied to the train so that it has a smooth start and stop, along with a constant-speed ride. The mass of the engine and the car are M and m , respectively. The two are held together by a spring with the stiffness coefficient of K . F represents the force applied by the engine, and μ represents the coefficient of rolling friction. If the train only travels in one direction:

- Draw the free-body diagram.
- Find the equations of motion.

a)



b) From Newton's Law:

$$M\ddot{y}_1 = F - K(y_1 - y_2) - \mu Mg\dot{y}_1$$

$$\ddot{y}_2 = K(y_1 - y_2) - \mu mg\dot{y}_2$$

2-7 A vehicle towing a trailer through a spring-damper coupling hitch is shown in Fig. 2P-7. The following parameters and variables are defined: M is the mass of the trailer; K_h , the spring constant of the hitch; B_h , the viscous-damping coefficient of the hitch; B_t , the viscous-friction coefficient of the trailer; $y_1(t)$, the displacement of the towing vehicle; $y_2(t)$, the displacement of the trailer; and $f(t)$, the force of the towing vehicle.

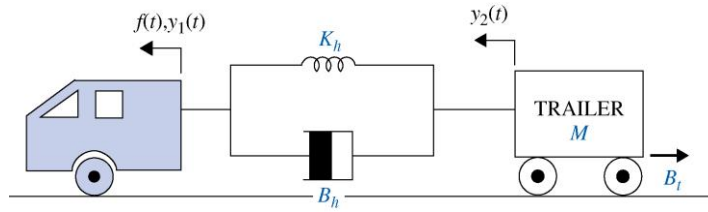


Figure 2P-7

Write the differential equation of the system.

Force equations:

$$f(t) = K_h(y_1 - y_2) + B_h\left(\frac{dy_1}{dt} - \frac{dy_2}{dt}\right) \quad K_h(y_1 - y_2) + B_h\left(\frac{dy_1}{dt} - \frac{dy_2}{dt}\right) = M \frac{d^2 y_2}{dt^2} + B_t \frac{dy_2}{dt}$$

2-8. Assume that the displacement angle of the pendulums shown in Fig. 2P-8 are small enough that the spring always remains horizontal. If the rods with the length of L are massless and the spring is attached to the rods $7/8$ from the top, find the state equation of the system.

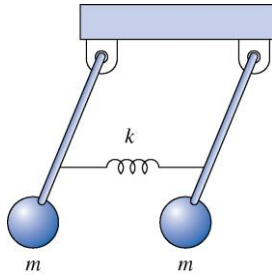
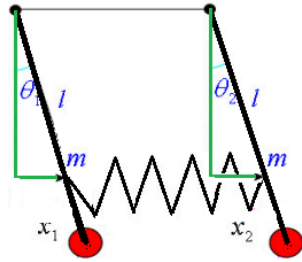
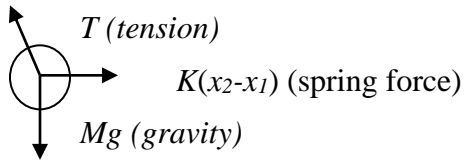


Figure 2P-8



For the left pendulum, assuming motion in counter-clockwise direction, the FBD is:



Note: $T = mg \cos \theta_1$

$$x_1 = \frac{7l}{8} \sin \theta_1$$

Also:

$$x_2 = \frac{7l}{8} \sin \theta_2$$

Taking a moment about the left fixed point we get:

$$m\ddot{\theta}_1 + mgl \sin \theta_1 - K \left(\frac{7l}{8} \right)^2 (\sin \theta_2 - \sin \theta_1) \cos \theta_2 = 0$$

Where $(7/8)l \cos \theta_2$ is the moment arm for the spring force.

$$\Rightarrow m\ddot{\theta}_1 + mgl \sin \theta_1 + K \left(\frac{7l}{8} \right)^2 (\sin \theta_1 - \sin \theta_2) \cos \theta_2 = 0$$

For the right pendulum, we can also write:

$$\Rightarrow m\ddot{\theta}_2 + mgl \sin \theta_2 + K \left(\frac{7l}{8} \right)^2 (\sin \theta_2 - \sin \theta_1) \cos \theta_1 = 0$$

since the angles are small:

$$\sin \theta_1 \approx \theta_1; \sin \theta_2 \approx \theta_2;$$

$$\cos \theta_1 \approx 1; \cos \theta_2 \approx 1$$

Hence,

$$\ddot{\theta}_1 + gl\theta_1 + K \left(\frac{7l}{8} \right)^2 (\theta_1 - \theta_2) = 0$$

$$\ddot{\theta}_2 + gl\theta_2 + K \left(\frac{7l}{8} \right)^2 (\theta_2 - \theta_1) = 0$$

2-9. Fig. 2P-9 shows an inverted pendulum on a cart.

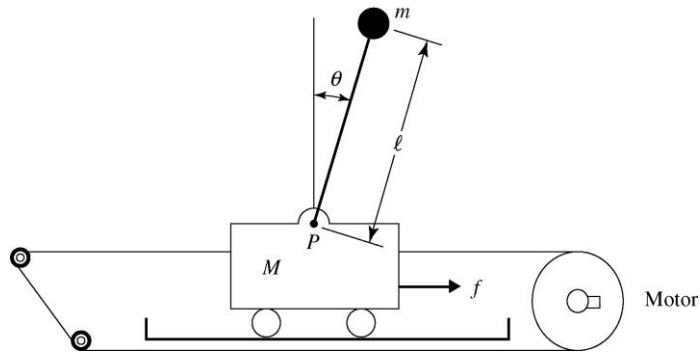


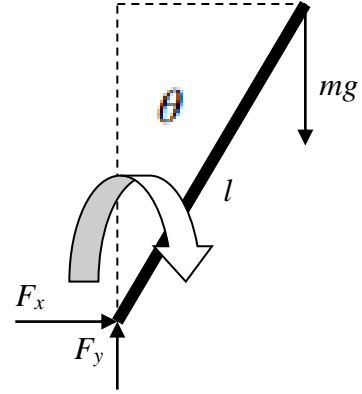
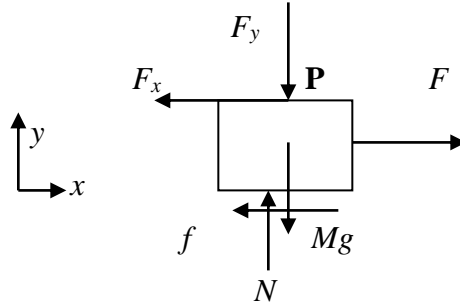
Figure 2P-9

If the mass of the cart is represented by M and the force f is applied to hold the bar at the desired position, then

(a) Draw the free-body diagram.

(b) Determine the dynamic equation of the motion.

a)



b) If we consider the coordinate of centre of gravity of mass m as (x_g, y_g) ,
Then $x_g = x + l \sin \theta$ and $y_g = l \cos \theta$

From force balance, we have:

$$\begin{cases} F - f - F_x = M\ddot{x} \\ N - Mg - F_y = 0 \end{cases}$$

$$\begin{cases} F_x = m\ddot{x}_g \\ F_y - mg = m\ddot{y}_g \end{cases}$$

Combining the equations we have:

$$M\ddot{x} + m\ddot{x}_g = F - f$$

$$N - (M + m)g = m\ddot{y}_g$$

Assuming viscous damping for friction (rough assumption) $f = B\dot{x}$:

$$M\ddot{x} + m\ddot{x}_g + B\dot{x} = F$$

Note: $\ddot{x}_g = \ddot{x} - l\dot{\theta}^2 \sin \theta + l\ddot{\theta} \cos \theta$ and $\ddot{y}_g = -l\dot{\theta}^2 \cos \theta - l\ddot{\theta} \sin \theta$

Hence,

$$M\ddot{x} + m\ddot{x} + B\dot{x} + ml(\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta) = F \quad (1)$$

For the pendulum, if we take a moment about the point mass mg , we have:

$$ml^2\ddot{\theta} = -F_x l \cos \theta + F_y l \sin \theta$$

Where using:

$$F_x = m\ddot{x}_g = m(\ddot{x} - l\dot{\theta}^2 \sin \theta + l\ddot{\theta} \cos \theta)$$

$$F_y - mg = m\ddot{y}_g = -m(l\dot{\theta}^2 \cos \theta + l\ddot{\theta} \sin \theta)$$

We get:

$$ml^2\ddot{\theta} = -m(\ddot{x} - l\dot{\theta}^2 \sin\theta + l\ddot{\theta} \cos\theta)l \cos\theta - m(l\dot{\theta}^2 \cos\theta + l\ddot{\theta} \sin\theta)l \sin\theta \quad (2)$$

Simplifying equations (1) and (2) we arrive at the two equations of the system:

$$\begin{aligned} \Rightarrow (M + m)\ddot{x} + B\dot{x} &= F + ml(\dot{\theta}^2 \sin\theta - \ddot{\theta} \cos\theta) \\ ml^2\ddot{\theta} &= mgl \sin\theta - ml\ddot{x} \cos\theta \end{aligned}$$

For small angles, linearized model of the system becomes

$$\begin{aligned} (M + m)\ddot{x} + ml\ddot{\theta} + B\dot{x} &= F \\ l\ddot{\theta} + \ddot{x} &= g\theta \end{aligned}$$

2-10. A two-stage inverted pendulum on a cart is shown in Fig. 2P-10.

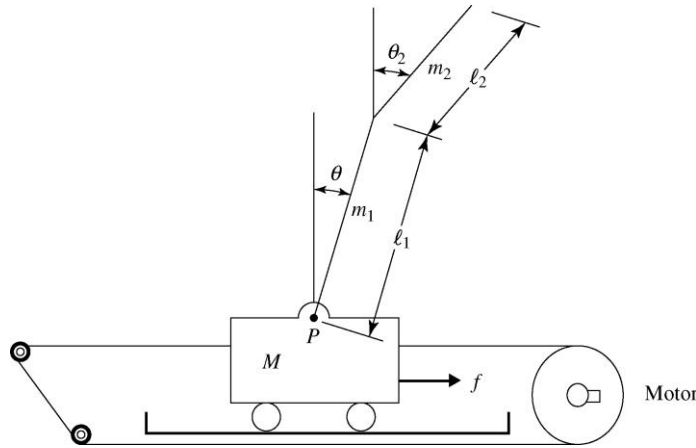


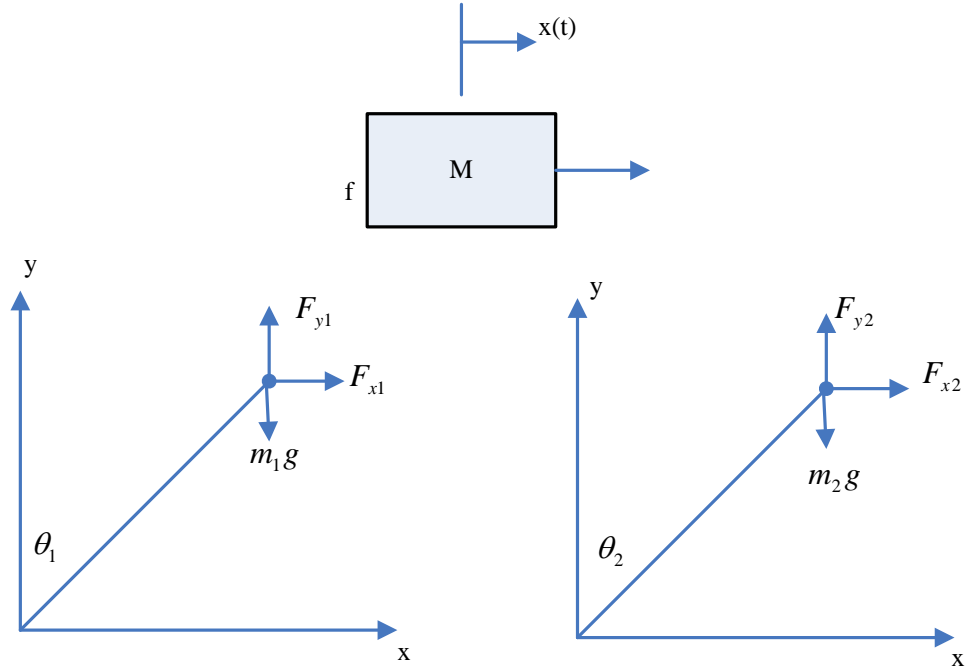
Figure 2P-10

If the mass of the cart is represented by M and the force f is applied to hold the bar at the desired position, then

- Draw the free-body diagram of mass M .
- Determine the dynamic equation of the motion.

The easiest way to find the equations for this challenge problem is using Lagranges's approach which is outside the scope of this text.

- for simplicity friction is ignored.



b) Kinetic energy

(i) For lower pendulum:

$$T_1 = \frac{1}{2} J_1 \dot{\theta}_1^2 + \frac{1}{2} m_1 \left\{ \left[\frac{d}{dt} (l_1 \sin \theta_1) \right]^2 + \left[\frac{d}{dt} (l_1 \cos \theta_1) \right]^2 \right\}$$

For upper pendulum:

$$T_2 = \frac{1}{2} J_2 \dot{\theta}_2^2 + \frac{1}{2} m_2 \left\{ \left[\frac{d}{dt} (l_2 \sin \theta_2) \right]^2 + \left[\frac{d}{dt} (l_2 \cos \theta_2) \right]^2 \right\}$$

For the cart: $T_3 = \frac{1}{2} M \dot{x}^2$

(ii) Potential energy:

For lower pendulum: $U_1 = m_1 g l_1 \cos \theta_1$

For upper pendulum: $U_2 = m_2 g l_2 \cos \theta_2$

For the cart: $U_3 = 0$

(iii) Total kinetic energy: $T_1 = T_1 + T_2 + T_3$

Total potential energy: $U = U_1 + U_2 + U_3$ The Lagrange's equation of motion is:

$$\begin{cases} \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}} \right) - \frac{\partial T}{\partial x} + \frac{\partial U}{\partial x} = f \\ \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}_1} \right) - \frac{\partial T}{\partial \theta_1} + \frac{\partial U}{\partial \theta_1} = 0 \\ \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}_2} \right) - \frac{\partial T}{\partial \theta_2} + \frac{\partial U}{\partial \theta_2} = 0 \end{cases}$$

Substituting T and U into the Lagrange's equation of motion gives:

$$\begin{cases} (m_1 + m_2 + M) \ddot{x} + m_1 l_1 \ddot{\theta}_1 \cos \theta_1 + m_2 l_2 \ddot{\theta}_2 \cos \theta_2 = m_1 l_1 \dot{\theta}_1^2 \sin \theta_1 + m_2 l_2 \dot{\theta}_2^2 \sin \theta_2 + f \\ m_1 l_1 \ddot{x} \cos \theta_1 + m_1 l_1 \dot{\theta}_1^2 \sin \theta_1 + (J_1 + m_1 l_1^2) \ddot{\theta}_1 = m_1 l_1 g \sin \theta_1 \\ m_2 l_2 \ddot{x} \cos \theta_2 + m_2 l_2 \dot{\theta}_1^2 \sin \theta_1 + (J_2 + m_2 l_2^2) \ddot{\theta}_2 = m_2 l_2 g \sin \theta_2 \end{cases}$$

Upon linearization about small angles we get

$$\begin{cases} (m_1 + m_2 + M) \ddot{x} + m_1 l_1 \ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 = m_1 l_1 \dot{\theta}_1^2 \theta_1 + m_2 l_2 \dot{\theta}_2^2 \theta_2 + f \\ m_1 l_1 \ddot{x} + m_1 l_1 \dot{\theta}_1^2 \theta_1 + (J_1 + m_1 l_1^2) \ddot{\theta}_1 = m_1 l_1 g \theta_1 \\ m_2 l_2 \ddot{x} + m_2 l_2 \dot{\theta}_1^2 \theta_1 + (J_2 + m_2 l_2^2) \ddot{\theta}_2 = m_2 l_2 g \theta_2 \end{cases}$$

2-11. Fig. 2P-11 shows a well-known “ball and beam” system in control systems. A ball is located on a beam to roll along the length of the beam. A lever arm is attached to the one end of the beam and a servo gear is attached to the other end of the lever arm. As the servo gear turns by an angle θ , the lever arm goes up and down, and then the angle of the beam is changed by α . The change in angle causes the ball to roll along the beam. A controller is desired to manipulate the ball's position.

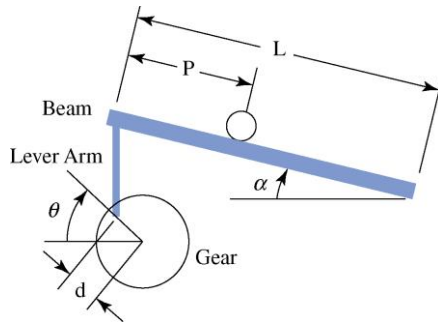


Figure 2P-11

Assuming:

m = mass of the ball

r = radius of the ball

d = lever arm offset

g = gravitational acceleration

L = length of the beam

J = ball's moment of inertia

p = ball position coordinate

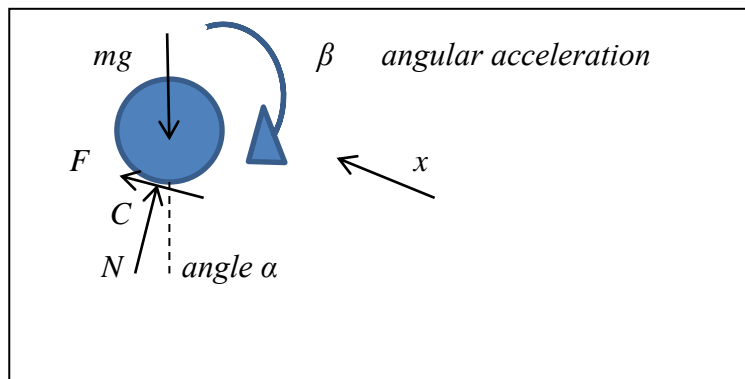
α = beam angle coordinate

θ = servo gear angle

Determine the dynamic equation of the motion.

Solution:

Considering the FBD of the ball:



- a) For a given α , the acceleration at point C will have two components due to rotation of the beam; that is $a_{cx} = -p\dot{\alpha}^2$ the centripetal and tangential $a_{cy} = p\ddot{\alpha}$ accelerations created by rotation of the bar. Also, we assume a case of rolling without slipping. Acceleration of the center of mass of the ball relative to the

rotating axis x, y, z is $a_x = \ddot{p} - p\dot{\alpha}^2$
 $a_y = p\ddot{\alpha} + 2\dot{\alpha}\dot{p}$ where $\ddot{p} = -r\beta$ (rolling without slipping and

β is the angular acceleration of the ball – the minus sign accounts for the assumed directions).

Note, in the case α is fixed, then $a_x = \ddot{p} = -r\beta$, which is in line with the rolling without slipping assumption in a fixed incline case.

From the equation of motion in x direction and by taking a moment about the center of mass of the ball (see a second year dynamics of rigid bodies text in case you need to verify the following formula), we get:

$$\sum F_x = ma_x = m(\ddot{p} - p\dot{\alpha}^2) = F - mg \sin \theta$$

$$\sum M_{c.m.} = J\beta = -\frac{J\ddot{p}}{r} = rF$$

Combining the above we have

$$\left(\frac{J}{r^2} + m\right)\ddot{p} + mg \sin \alpha - mp\dot{\alpha}^2 = 0$$

We can further linearize these equations to arrive at:

$$\alpha = \frac{d}{L}\theta$$

Then

$$\left(\frac{J}{r^2} + m\right)\ddot{p} + mg \sin\left(\frac{d\dot{\theta}}{L}\right) - mp\frac{d}{L}\dot{\theta}^2 = 0$$

If we linearize the equation about beam angle $\alpha = 0$, then $\sin \alpha \approx \alpha$ and $\sin \theta \approx \theta$

Then:

$$\left(\frac{J}{r^2} + m\right)\ddot{p} = -mg \frac{d}{L}\theta$$

2-12. The motion equations of an aircraft are a set of six nonlinear coupled differential equations. Under certain assumptions, they can be decoupled and linearized into the longitudinal and lateral equations. Fig. 2P-12 shows a simple model of airplane during its flight. Pitch control is a longitudinal problem, and an autopilot is designed to control the pitch of the airplane.

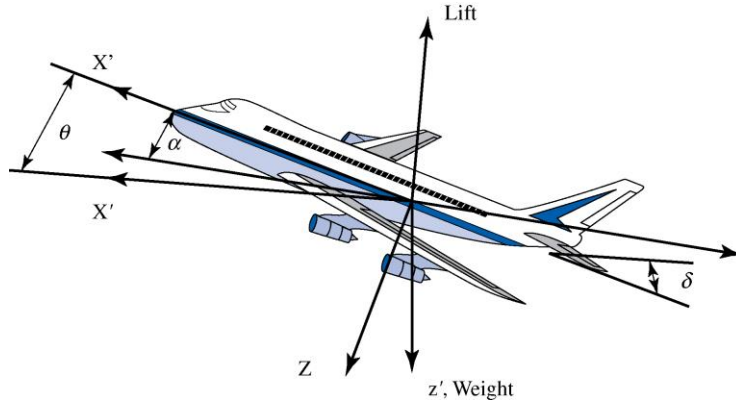


Figure 2P-12

Consider that the airplane is in steady-cruise at constant altitude and velocity, which means the thrust and drag cancel out and the lift and weight balance out each other. To simplify the problem, assume that change in pitch angle does not affect the speed of an aircraft under any circumstance.

Determine the longitudinal equations of motion of the aircraft.

Solution: If the aircraft is at a constant altitude and velocity, and also the change in pitch angle does not change the speed, then from longitudinal equation, the motion in vertical plane can be written as:

$$\begin{cases} \dot{u} = \frac{x}{m} - g \sin \theta - q\omega \\ \dot{\omega} = \frac{z}{m} - g \cos \theta + qu \\ \dot{q} = \frac{M}{I_{yy}} \\ \dot{\theta} = q \end{cases}$$

Where u is axial velocity, ω is vertical velocity, q is pitch rate, and θ is pitch angle.

Converting the Cartesian components with polar inertial components and replace x, y, z by T, D , and L . Then we have:

$$\begin{cases} \dot{V} = \frac{1}{m} [T \cos \alpha - D - mg \sin \gamma] \\ \dot{\gamma} = \frac{1}{mV} [T \sin \alpha + L - mg \cos \gamma] \\ \dot{q} = \frac{M}{I_{yy}} \\ \dot{\theta} = q \end{cases}$$

Where $\alpha = \theta - \gamma$ is an attack angle, V is velocity, and γ is flight path angle.

It should be mentioned that T, D, L and M are function of variables α and V .

Refer to the aircraft dynamics textbooks, the state equations can be written as:

$$\begin{cases} \dot{\alpha} = A_1\alpha + B_1q + C_1\gamma \\ \dot{q} = A_2\alpha + B_2q + C_2\gamma \\ \dot{\theta} = A_3q \end{cases}$$

2-13. Write the torque equations of the rotational systems shown in Fig. 2P-13.

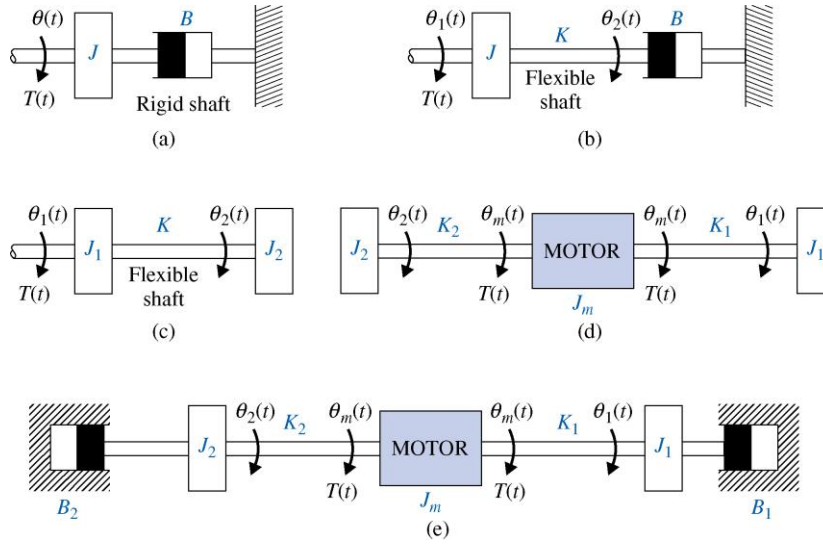


Figure 2P-13

(a) Torque equation:

$$\frac{d^2\theta}{dt^2} = -\frac{B}{J} \frac{d\theta}{dt} + \frac{1}{J} T(t)$$

(b) Torque equations:

$$\frac{d^2\theta_1}{dt^2} = -\frac{K}{J} (\theta_1 - \theta_2) + \frac{1}{J} T \quad K (\theta_1 - \theta_2) = B \frac{d\theta_2}{dt}$$

(c) Torque equations:

$$T(t) = J_1 \frac{d^2\theta_1}{dt^2} + K (\theta_1 - \theta_2) \quad K (\theta_1 - \theta_2) = J_2 \frac{d^2\theta_2}{dt^2}$$

(d) Torque equations:

$$T(t) = J_m \frac{d^2\theta_m}{dt^2} + K_1 (\theta_m - \theta_1) + K_2 (\theta_m - \theta_2) \quad K_1 (\theta_m - \theta_1) = J_1 \frac{d^2\theta_1}{dt^2} \quad K_2 (\theta_m - \theta_2) = J_2 \frac{d^2\theta_2}{dt^2}$$

(e) Torque equations:

$$\frac{d^2\theta_m}{dt^2} = -\frac{K_1}{J_m}(\theta_m - \theta_1) - \frac{K_2}{J_m}(\theta_m - \theta_2) + \frac{1}{J_m}T \quad \frac{d^2\theta_1}{dt^2} = \frac{K_1}{J_1}(\theta_m - \theta_1) - \frac{B_1}{J_1} \frac{d\theta_1}{dt} \quad \frac{d^2\theta_2}{dt^2} = \frac{K_2}{J_2}(\theta_m - \theta_1) - \frac{B_2}{J_2} \frac{d\theta_2}{dt}$$

2-14. Write the torque equations of the gear-train system shown in Fig. 2P-14. The moments of inertia of gears are lumped as J_1 , J_2 , and J_3 . $T_m(t)$ is the applied torque; N_1 , N_2 , N_3 , and N_4 are the number of gear teeth. Assume rigid shafts

(a) Assume that J_1 , J_2 , and J_3 are negligible. Write the torque equations of the system. Find the total inertia the motor sees.

(b) Repeat part (a) with the moments of inertia J_1 , J_2 , and J_3 .

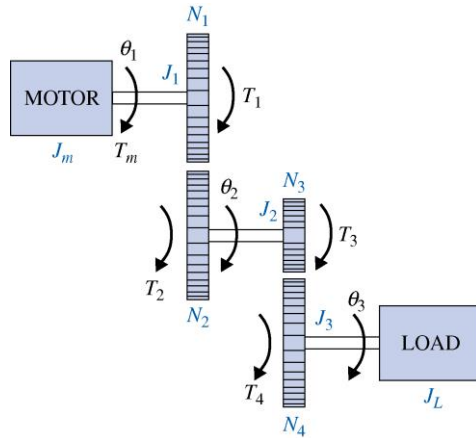


Figure 2P-14

(a)

$$T_m(t) = J_m \frac{d^2\theta_1}{dt^2} + T_1 \quad T_1 = \frac{N_1}{N_2} T_2 \quad T_3 = \frac{N_3}{N_4} T_4 \quad T_4 = J_L \frac{d^2\theta_3}{dt^2} \quad T_2 = T_3 \quad \theta_2 = \frac{N_1}{N_2} \theta_1$$

$$\theta_3 = \frac{N_1 N_3}{N_2 N_4} \theta_1 \quad T_2 = \frac{N_3}{N_4} T_4 = \frac{N_3}{N_4} J_L \frac{d^2\theta_3}{dt^2} \quad T_m = J_m \frac{d^2\theta_1}{dt^2} + \frac{N_1 N_3}{N_2 N_4} T_4 = \left[J_m + \left[\frac{N_1 N_3}{N_2 N_4} \right]^2 J_L \right] \frac{d^2\theta_1}{dt^2}$$

(b)

$$T_m = J_m \frac{d^2 \theta_1}{dt^2} + T_1 \quad T_2 = J_2 \frac{d^2 \theta_2}{dt^2} + T_3 \quad T_4 = (J_3 + J_L) \frac{d^2 \theta_3}{dt^2} \quad T_1 = \frac{N_1}{N_2} T_2 \quad T_3 = \frac{N_3}{N_4} T_4$$

$$\theta_2 = \frac{N_1}{N_2} \theta_1 \quad \theta_3 = \frac{N_1 N_3}{N_2 N_4} \theta_1 \quad T_2 = J_2 \frac{d^2 \theta_2}{dt^2} + \frac{N_3}{N_4} T_4 = J_2 \frac{d^2 \theta_2}{dt^2} + \frac{N_3}{N_4} (J_3 + J_L) \frac{d^2 \theta_3}{dt^2}$$

$$T_m(t) = J_m \frac{d^2 \theta_1}{dt^2} + \frac{N_1}{N_2} \left(J_2 \frac{d^2 \theta_2}{dt^2} + \frac{N_3}{N_4} (J_3 + J_L) \frac{d^2 \theta_3}{dt^2} \right) = \left[J_m + \left(\frac{N_1}{N_2} \right)^2 J_2 + \left(\frac{N_1 N_3}{N_2 N_4} \right)^2 (J_3 + J_L) \right] \frac{d^2 \theta_1}{dt^2}$$

2-15. Fig. 4P-15 shows a motor-load system coupled through a gear train with gear ratio $n = N_1/N_2$. The motor torque is $T_m(t)$, and $T_L(t)$ represents a load torque.

(a) Find the optimum gear ratio n^* such that the load acceleration $\alpha_L = d^2 \theta_L / dt^2$ is maximized.

(b) Repeat part (a) when the load torque is zero.

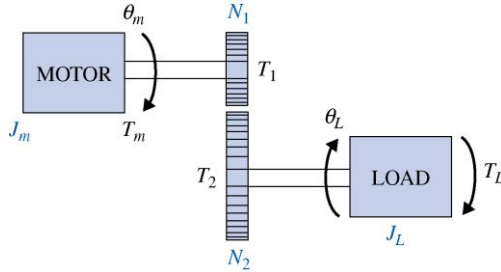


Figure 2P-15

(a)

$$T_m = J_m \frac{d^2 \theta_m}{dt^2} + T_1 \quad T_2 = J_L \frac{d^2 \theta_L}{dt^2} + T_L \quad T_1 = \frac{N_1}{N_2} T_2 = n T_2 \quad \theta_m N_1 = \theta_L N_2$$

$$T_m = J_m \frac{d^2 \theta_m}{dt^2} + n J_L \frac{d^2 \theta_L}{dt^2} + n T_L = \left(\frac{J_m}{n} + n J_L \right) \alpha_L + n T_L \quad \text{Thus, } \alpha_L = \frac{n T_m - n^2 T_L}{J_m + n^2 J_L}$$

$$\text{Set } \frac{\partial \alpha_L}{\partial n} = 0. \quad (T_m - 2n T_L)(J_m + n^2 J_L) - 2n J_L (n T_m - n^2 J_L) = 0$$

$$n^2 + \frac{J_m T_L}{J_L T_m} n - \frac{J_m}{J_L} = 0$$

Or,

$$n^* = -\frac{J_m T_L}{2J_L T_m} + \frac{\sqrt{J_m^2 T_L^2 + 4J_m J_L T_m^2}}{2J_L T_m}$$

Optimal gear ratio:
been chosen.

where the + sign has

(b) When $T_L = 0$, the optimal gear ratio is

$$n^* = \sqrt{J_m / J_L}$$

2-16. Fig. 2P-16 shows the simplified diagram of the printwheel control system of a word processor. The printwheel is controlled by a dc motor through belts and pulleys. Assume that the belts are rigid. The following parameters and variables are defined: $T_m(t)$ is the motor torque; $\theta_m(t)$, the motor displacement; $y(t)$, the linear displacement of the printwheel; J_m , the motor inertia; B_m , the motor viscous-friction coefficient; r , the pulley radius; M , the mass of the printwheel.

Write the differential equation of the system.

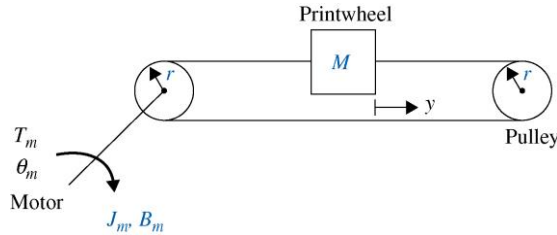


Figure 2P-16

Torque equation about the motor shaft:
rotational displacements:

Relation between linear and

$$T_m = J_m \frac{d^2 \theta_m}{dt^2} + M r^2 \frac{d^2 \theta_m}{dt^2} + B_m \frac{d \theta_m}{dt} \quad y = r \theta_m$$

2-17. Fig. 2P-17 shows the diagram of a printwheel system with belts and pulleys. The belts are modeled as linear springs with spring constants K_1 and K_2 .

Write the differential equations of the system using θ_m and y as the dependent variables.

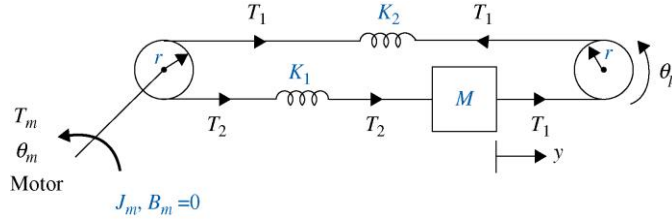


Figure 2P-17

$$T_m = J_m \frac{d^2 \theta_m}{dt^2} + r(T_1 - T_2) \quad T_1 = K_2(r\theta_m - r\theta_p) = K_2(r\theta_m - y) \quad T_2 = K_1(y - r\theta_m)$$

$$T_1 - T_2 = M \frac{d^2 y}{dt^2} \quad \text{Thus, } T_m = J_m \frac{d^2 \theta_m}{dt^2} + r(K_1 + K_2)(r\theta_m - y)$$

$$M \frac{d^2 y}{dt^2} = (K_1 + K_2)(r\theta_m - y)$$

2-18. Classically, the quarter-car model is used in the study of vehicle suspension systems and the resulting dynamic response due to various road inputs. Typically, the inertia, stiffness, and damping characteristics of the system as illustrated in Fig. 2P-18(a) are modeled in a two degree of freedom (2-DOF) system, as shown in (b). Although a 2-DOF system is a more accurate model, it is sufficient for the following analysis to assume a 1-DOF model, as shown in (c).

Find the equations of motion for absolute motion x and the relative motion (bounce) $z=x-y$.

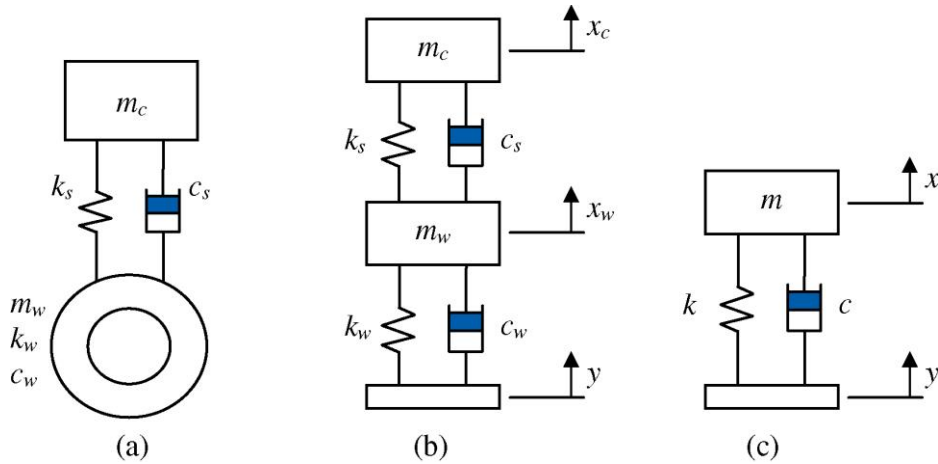


Figure 2P-18 Quarter-car model realization. (a) Quarter car. (b) Two degrees of freedom. (c) One degree of freedom.

Assuming motion is upward, such that $x > y$, the equation of motion of the system is defined as follows:

$$c(\dot{x}(t) - \dot{y}(t)) + k(x(t) - y(t)) = m\ddot{x}(t)$$

or

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = c\dot{y}(t) + ky(t)$$

which can be simplified by substituting the relation $z(t) = x(t) - y(t)$ and non-dimensionalizing the coefficients to the form

$$\ddot{z}(t) + 2\zeta\omega_n\dot{z}(t) + \omega_n^2 z(t) = -\ddot{y}(t)$$

2-19. The schematic diagram of a motor-load system is shown in Fig. 4P-19. The following parameters and variables are defined: $T_m(t)$ is the motor torque; $\omega_m(t)$, the motor velocity; $\theta_m(t)$, the motor displacement; $\omega_L(t)$, the load velocity; $\theta_L(t)$, the load displacement; K , the torsional spring constant; J_m , the motor inertia; B_m , the motor viscous-friction coefficient; and B_L , the load viscous-friction coefficient.

Write the torque equations of the system.

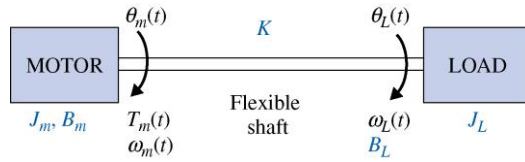


Figure 2P-19

Torque equations:

$$T_m(t) = J_m \frac{d^2\theta_m}{dt^2} + B_m \frac{d\theta_m}{dt} + K(\theta_m - \theta_L) \quad K(\theta_m - \theta_L) = J_L \frac{d^2\theta_L}{dt^2} + B_L \frac{d\theta_L}{dt}$$

2-20 This problem deals with the attitude control of a guided missile. When traveling through the atmosphere, a missile encounters aerodynamic forces that tend to cause instability in the attitude of the missile. The basic concern from the flight-control standpoint is the lateral force of the air, which tends to rotate the missile about its center of gravity. If the missile centerline is not aligned with the direction in which the center of gravity C is traveling, as shown in Fig. 2P-20, with angle θ , which is also called the angle of attack, a side force is produced by the drag of the air through which the missile travels. The total force F_α may be considered to be applied at the center of pressure P . As shown in Fig. 4P-20, this side force has a tendency to cause the missile to tumble end over end, especially if the point P is in front of the center of gravity C . Let the angular acceleration of the missile about the point C , due to the side force, be denoted by α_F . Normally, α_F is directly proportional to the angle of attack θ and is given by

$$\alpha_F = \frac{K_F d_1}{J} \theta$$

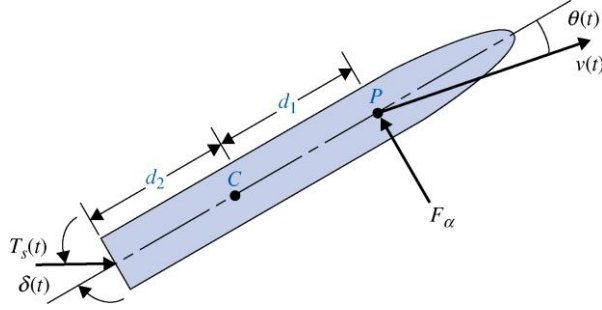


Figure 2P-20

where K_F is a constant that depends on such parameters as dynamic pressure, velocity of the missile, air density, and so on, and

J = missile moment of inertia about C

d_1 = distance between C and P

The main objective of the flight-control system is to provide the stabilizing action to counter the effect of the side force. One of the standard control means is to use gas injection at the tail of the missile to deflect the direction of the rocket engine thrust T_s , as shown in the figure.

(a) Write a torque differential equation to relate among T_s , δ , θ , and the system parameters given. Assume that δ is very small, so that $\sin \delta(t)$ is approximated by $\delta(t)$.

(b) Repeat parts (a) with points C and P interchanged. The d_1 in the expression of α_F should be changed to d_2 .

(a) Torque equation: (About the center of gravity C)

$$J \frac{d^2 \theta}{dt^2} = T_s d_2 \sin \delta + F_a d_1 \quad F_a d_1 = J \alpha_1 = K_F d_1 \theta \quad \sin \delta \cong \delta$$

$$\text{Thus,} \quad J \frac{d^2 \theta}{dt^2} = T_s d_2 \delta + K_F d_1 \theta \quad J \frac{d^2 \theta}{dt^2} - K_F d_1 \theta = T_s d_2 \delta$$

(b) With C and P interchanged, the torque equation about C is:

$$T_s (d_1 + d_2) \delta + F_a d_2 = J \frac{d^2 \theta}{dt^2} \quad T_s (d_1 + d_2) \delta + K_F d_2 \theta = J \frac{d^2 \theta}{dt^2}$$

2-21. Fig. 2P-21(a) shows a well-known “broom-balancing” system in control systems. The objective of the control system is to maintain the broom in the upright

position by means of the force $u(t)$ applied to the car as shown. In practical applications, the system is analogous to a one-dimensional control problem of the balancing of a unicycle or a missile immediately after launching. The free-body diagram of the system is shown in Fig. 2P-21(b), where

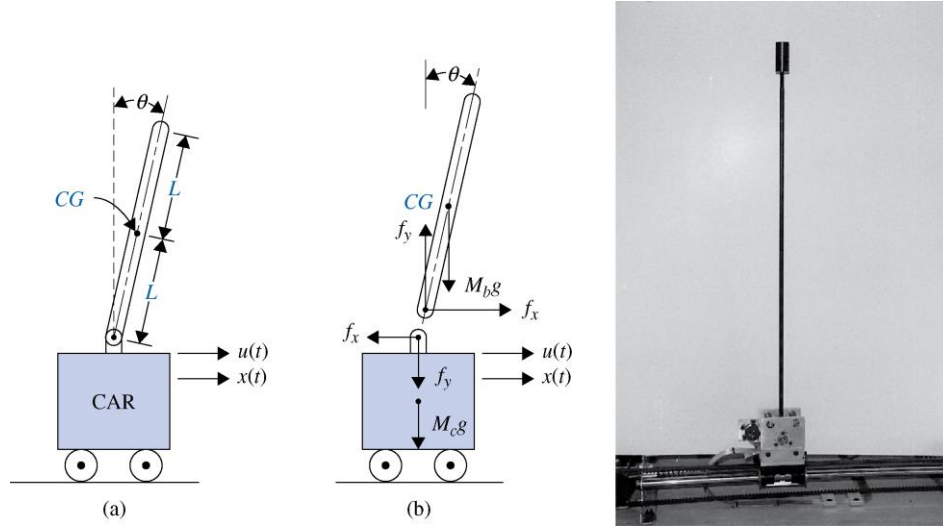


Figure 2P-21

- f_x = force at broom base in horizontal direction
- f_y = force at broom base in vertical direction
- M_b = mass of broom
- g = gravitational acceleration
- M_c = mass of car
- J_b = moment of inertia of broom about center of gravity $CG = M_b L^2/3$

(a) Write the force equations in the x and the y directions at the pivot point of the broom. Write the torque equation about the center of gravity CG of the broom. Write the force equation of the car in the horizontal direction.

(b) Compare your results with those in Problem 2-9.

(a) **Force and torque equations:**

Broom: vertical direction:
$$f_y - M_b g = M_b \frac{d^2(L \cos \theta)}{dt^2}$$

horizontal direction:
$$f_x = M_b \frac{d^2[x(t) + L \sin \theta]}{dt^2}$$

rotation about CG:
$$J_b \frac{d^2 \theta}{dt^2} = f_y L \sin \theta - f_x L \cos \theta$$

Car: horizontal direction: $u(t) = f_x + M_c \frac{d^2 x(t)}{dt^2} J_b = \frac{M_b L^2}{3}$

$$M_b \frac{d^2 [x(t) + L \sin \theta]}{dt^2} + M_c \frac{d^2 x(t)}{dt^2} = u(t)$$

$$J_b \frac{d^2 \theta}{dt^2} - M_b \frac{d^2 (L \cos \theta)}{dt} L \sin \theta + M_b \frac{d^2 [x(t) + L \sin \theta]}{dt^2} L \cos \theta = M_b g$$

Then after simplifications we get:

$$(M_b + M_c) \frac{d^2 x(t)}{dt^2} = u(t) + M_c L \sin \theta \left(\frac{d\theta}{dt} \right)^2 - M_c L \cos \theta \frac{d^2 \theta}{dt^2}$$

$$(J_b + M_c L^2) \frac{d^2 \theta}{dt^2} = M_c g L \sin \theta - M_c L \cos \theta \frac{d^2 x(t)}{dt^2}$$

friction = $B\dot{x}(t)$ can be added to the first equation as in problem 2-9.

- (b) The two problems are similar except for the added inertia J_b in this case. In problem 2-9, particle mass and a massless rod were assumed.

2-22. Most machines and devices have rotating parts. Even a small irregularity in the mass distribution of rotating components can cause vibration, which is called rotating unbalanced. Fig. 2P-22 represents the schematic of a rotating unbalanced mass of m . Assume that the frequency of rotation of the machine is ω .

Derive the equations of motion of the system.

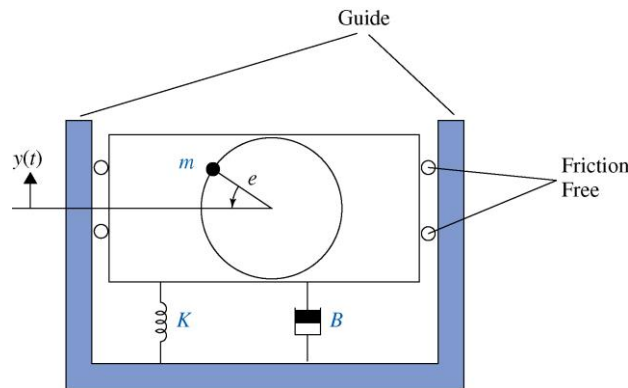
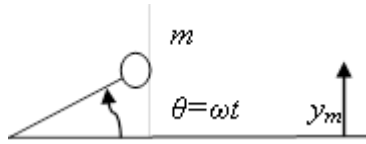


Figure 2P-22

Define θ as the angle between mass m and the horizontal axis (positive in c.c.w. direction). Use Newton's second law:



$$\begin{aligned}
 m(\ddot{y} + \ddot{y}_m) &= -F_m \\
 (M - m)\ddot{y} &= F_m - B\dot{y} - Ky \\
 \ddot{y}_m &= -e\omega^2 \sin \omega t \\
 \Rightarrow \\
 M\ddot{y} + B\dot{y} + Ky &= me\omega^2 \sin \omega t
 \end{aligned}$$

Where M is the Mass of the overall block system. $M-m$ is the mass of the block alone.

2-23. Vibration absorbers are used to protect machines that work at the constant speed from steady-state harmonic disturbance. Fig. 2P-23 shows a simple vibration absorber.

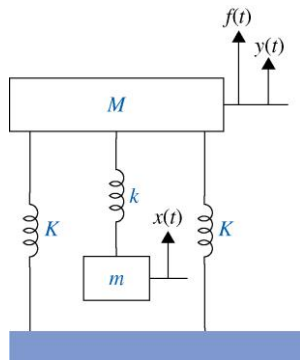


Figure 2P-23

Assuming the harmonic force $F(t) = A \sin(\omega t)$ is the disturbance applied to the mass M , derive the equations of motion of the system.

summation of vertical forces gives:

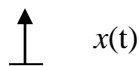
$$\begin{aligned}
 M\ddot{y} &= f(t) - Ky - k(y - x) - Ky \\
 m\ddot{x} &= k(y - x)
 \end{aligned}$$

$$\begin{aligned}
 M\ddot{y} + (2K + k)y - kx &= f(t) \\
 m\ddot{x} - ky + kx &= 0
 \end{aligned}$$

Where $f(t) = A \sin(\omega t)$

2-24. Fig. 2P-24 represents a vibration absorption system.

Assuming the harmonic force $F(t) = A \sin(\omega t)$ is the disturbance applied to the mass M , derive the equations of motion of the system.



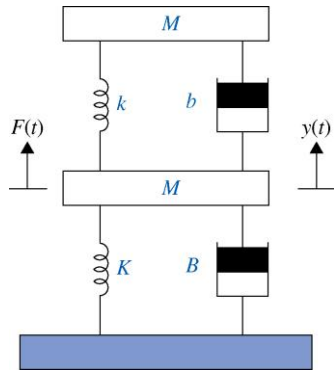


Figure 2P-24

Assume motion is such that both springs are in tension. Summation of vertical forces gives:

$$\begin{cases} M\ddot{y} + (B + b)\dot{y} - b\dot{x} + (K + k)y - kx = F \\ m\ddot{x} - b\dot{y} + b\dot{x} - ky - kx = 0 \end{cases}$$

Where $F(t) = A \sin(\omega t)$

2-25. An accelerometer is a transducer as shown in Fig. 2P-25.

Find the dynamic equation of motion.

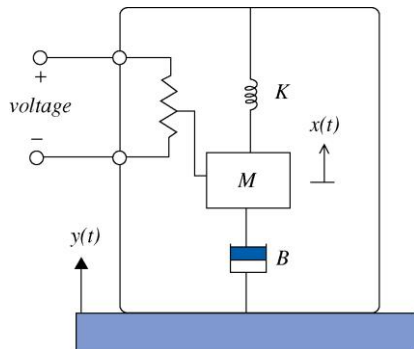


Figure 2P-25

The equation of motion is:

$$M\ddot{x} + B(\dot{x} - \dot{y}) + K(x - y) = 0$$

Considering $z = x - y$ gives:

$$M\ddot{z} + B\dot{z} + Kz = -M\ddot{y}$$

PROBLEMS FOR SECTION 2-2

2-26. Consider the electrical circuits shown in Figs. 4P-26(a) and (b).

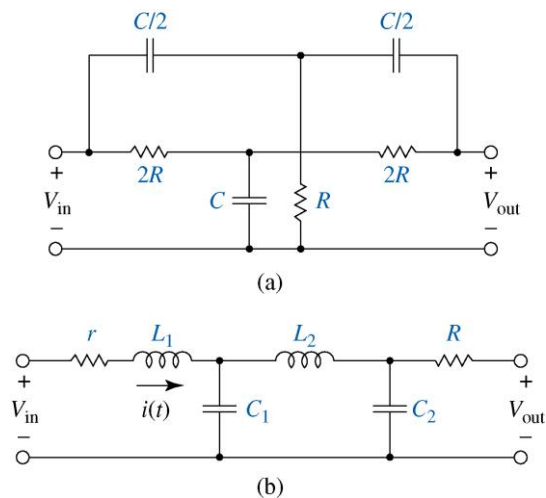


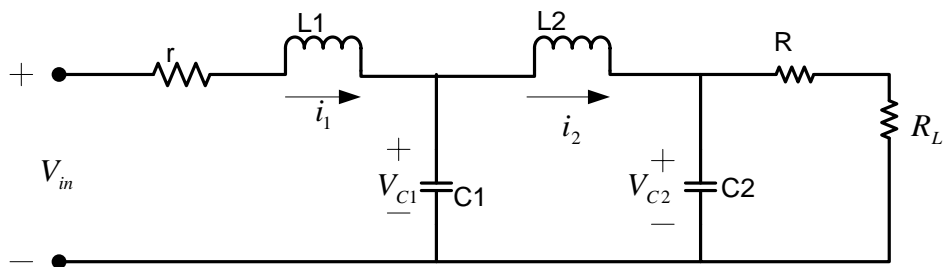
Figure 2P-26

(a) For each circuit find the dynamic equations.

According to the circuit:

$$\begin{cases} \frac{v_{in} - v_1}{2R} + C \frac{d}{dt} v_1 + \frac{v_{out} - v_1}{2R} = 0 \\ \frac{C}{2} \frac{d}{dt} (v_{in} - v_2) - \frac{v_2}{R} + \frac{C}{2} \frac{d}{dt} (v_{out} - v_2) = 0 \\ \frac{C}{2} \frac{d}{dt} (v_2 - v_{out}) + \frac{v_1 - v_{out}}{2R} = 0 \end{cases}$$

Measuring V_{out} requires a load resistor, which means:



Then we have:

$$\begin{cases} L_1 \frac{d}{dt} i_1 = v_{in} - r i_1 - v_{c1} \\ C_1 \frac{d}{dt} v_{c1} = i_1 - i_2 \\ L_2 \frac{d}{dt} i_2 = v_{c1} - v_{c2} \\ C_2 \frac{d}{dt} v_{c2} = i_2 - \frac{v_{c2}}{R + R_L} \end{cases}$$

When

$$v_{out} = \frac{R_L}{R + R_L} v_{c2}$$

If $R_L \gg R$, then $v_{out} = v_{c2}$

2-27. In a strain gauge circuit, the electrical resistance in one or more of the branches of the bridge circuit, shown in Fig. 2P-27, varies with the strain of the surface to which it is rigidly attached to. The change in resistance results in a differential voltage that is related to the strain. The bridge is composed of two voltage dividers, so the differential voltage Δe can be expressed as the difference in e_1 and e_2 .

(a) Find Δe .

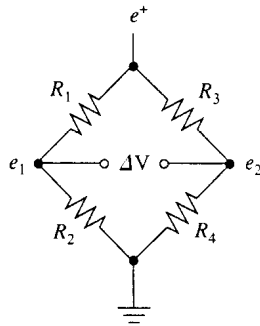


Figure 2P-27

(b) If the resistance R_2 has a fixed value of R_2^* , plus a small increment in resistance, δR , then $R_2 = R_2^* + \delta R$. For equal resistance values ($R_1 = R_3 = R_4 = R_2^* = R$), rewrite the bridge equation (i.e. for Δe).

(a) $\Delta e = e_1 - e_2$

$$\Delta e = \left[\frac{R_2}{R_1 + R_2} - \frac{R_4}{R_3 + R_4} \right] e^+$$

(b) If all four resistors are equal ($R_1 = R_3 = R_4 = R_2^* = R$), then the bridge equation reduces to

$$\Delta e = \frac{\delta R}{2R} e^+$$

The equivalent resistance from e^+ to ground can be calculated by considering two sets of series resistors operated in parallel:

$$R_{eq} = \frac{(R_1 + R_2)(R_3 + R_4)}{(R_1 + R_2) + (R_3 + R_4)}$$

If all of the resistors are equal (with value R), then the equivalent resistance is simply R .

2-28. Fig. 2P-28 shows a circuit made up of two RC circuits. Find the dynamic equations of the system.

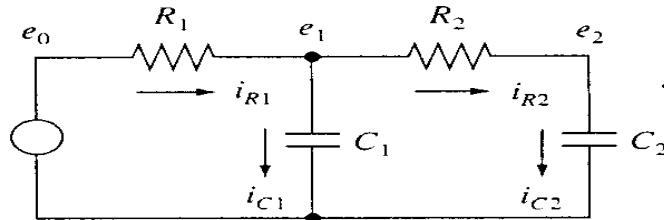


Figure 2P-28

Individual currents are:

$$i_{R1} = \frac{e_0 - e_1}{R_1}$$

$$i_{C1} = C_1 \dot{e}_1 \quad \text{with } e_1(0)$$

$$i_{R2} = \frac{e_1 - e_2}{R_2}$$

$$i_{C2} = C_2 \dot{e}_2 \quad \text{with } e_2(0)$$

The node equations are

$$i_{R1} = i_{C1} + i_{R2}$$

$$i_{R2} = I_{C2}$$

Substitute the current equations into the node equations and rearrange we get:

$$R_1 C_1 \dot{e}_1 + (1 + R_1 / R_2) e_1 = (R_1 / R_2) e_2 + e_0$$

$$R_2 C_2 \dot{e}_2 + e_2 = e_1$$

Since we are interested in e_2 as a function of e_0 , we can substitute the second equation for e_1 into the first and rearrange to obtain

$$R_1 C_1 R_2 C_2 \ddot{e}_2 + (R_1 C_1 + R_1 C_2 + R_2 C_2) \dot{e}_2 + e_2 = e_0$$

2-29. For the Parallel RLC Circuit, shown in Fig. 2P-29, find the dynamic equations of the system.

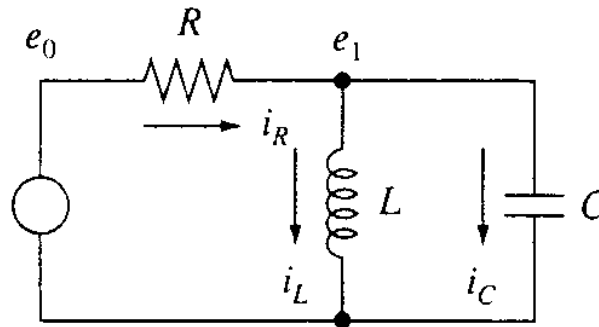


Figure 2P-29

Individual currents are:

$$i_R = \frac{e_0 - e_1}{R}$$

$$i_L = \int \frac{e_1}{L} dt \quad \text{with } i_L(0)$$

$$i_C = C \dot{e}_1 \quad \text{with } e_1(0)$$

The node equation is

$$i_R = i_L + i_C$$

Substituting the component equations into the node equation and rearranging yields the classical differential equation for the voltage e_1 as a function of e_0 :

$$LC \ddot{e}_1 + \frac{L}{R} \dot{e}_1 + e_1 = \frac{L}{R} D e_0$$

PROBLEMS FOR SECTION 2-3

2-30. Hot Oil forging in quenching vat with its cross-sectional view is shown in Fig. 2P-30.

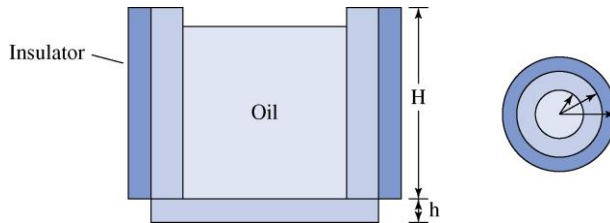


Figure 2P-30

The radii shown in Fig. are r_1 , r_2 , and r_3 from inside to outside. The heat is transferred to the atmosphere from the sides and bottom of the vat and also the surface of the oil with a convective heat coefficient of k_o . Assuming:

k_v = The thermal conductivity of the vat

k_i = The thermal conductivity of the insulator

c_o = The specific heat of the oil

d_o = The density of the oil

c = The specific heat of the forging

m = Mass of the forging

A = The surface area of the forging

h = The thickness of the bottom of the vat

T_a = The ambient temperature

Determine the system model when the temperature of the oil is desired.

Due to insulation, there is no heat flow through the walls. The heat flow through the sides is:

$$\begin{cases} q_{1,2} = \frac{2\pi K_v H}{\ln\left(\frac{r_2}{r_1}\right)} (T_1 - T_2) & (1) \\ q_{2,a} = \frac{2\pi K_i H}{\ln\left(\frac{r_3}{r_2}\right)} (T_2 - T_a) & (2) \end{cases}$$

Where T_1 and T_2 are the temperature at the surface of each cylinder.

As $q_{1,2} = q_{2,a}$, then from equation (1) and (2), we obtain:

$$T_2 = \frac{\ln\left(\frac{r_3}{r_2}\right)}{2\pi K_i H} q_{1,2} + T_a \quad (3)$$

The conduction or convection at:

$$\begin{cases} \text{the surface of the oil: } q_o = C_h(\pi r_1^2)(T_1 - T_a) & (4) \\ \text{the face of forgoing: } q_f = C_h A(T_f - T_1) & (5) \\ \text{the bottom at the vat: } q_v = \frac{K_v}{h}(\pi r_1^2)(T_1 - T_a) & (6) \end{cases}$$

The thermal capacitance dynamics gives:

$$\begin{cases} m_o C_o \frac{d}{dt} T_1 = q_f - q_{1,2} - q_v - q_o & (7) \\ mC \frac{d}{dt} T_f = -q_f & (8) \end{cases}$$

Where $m_o = \pi r_1^2 H d_o$

According to the equation (7) and (8), T_1 and T_f are state variables.

Substituting equation (3), (4), (5) and (6) into equation (7) and (8) gives the model of the system.

2-31. A power supply within an enclosure is shown in Fig. 2P-31. Because the power supply generates lots of heat, a heat sink is usually attached to dissipate the generated heat. Assuming the rate of heat generation within the power supply is known and constant, Q , the heat transfers from the power supply to the enclosure by radiation and conduction, the frame is an ideal insulator, and the heat sink temperature is constant and equal to the atmospheric temperature, determine the model of the system that can give the temperature of the power supply during its operation. Assign any needed parameters.

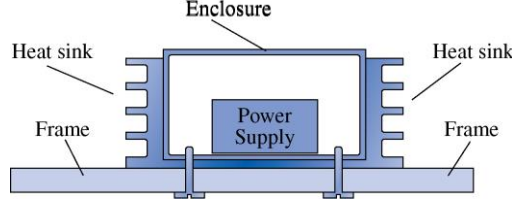


Figure 2P-31

As heat transfer from power supply to enclosure by radiation and conduction, then:

$$C_p \frac{d}{dt} T_p = q_p - q_r - q_c \quad (1)$$

$$q_r = \frac{\sigma(T_p^4 - T_s^4)}{\left[\frac{1 - \varepsilon_1}{\varepsilon_1 A_p} + \frac{1}{A_p F} + \frac{1 - \varepsilon_2}{\varepsilon_2 A_s} \right]} = \frac{\sigma(T_p^4 - T_s^4)}{R_p + \frac{1}{A_p F} + R_s} \quad (2)$$

$$q_c = \left(\frac{K_1 A_1}{\Delta x} \right) (T_p - T_s) = \frac{T_p - T_s}{R_E} \quad (3)$$

Also the enclosure loses heat to the air through its top. So:

$$C_s \frac{d}{dt} T_s = q_r + q_c - q_s - C_t A_t (T_s - T_a) \quad (4)$$

Where

$$q_s = \left(\frac{K_2 A_2}{\Delta x} \right) (T_s - T_s) = \frac{T_s - T_s}{R_s} \quad (5)$$

And C_t is the convective heat transfer coefficient and A_t is the surface area of the enclosure.

The changes if the temperature of heat sink is supposed to be zero, then:

$$C \frac{d}{dt} T_{sink} = q_s - q_s = 0$$

Therefore $q_s = q_s$ where $q_s = C_s A_s (T_s - T_a)$, as a result:

$$\frac{T_s - T_s}{R_s} = C_s A_s (T_s - T_a) \quad (6)$$

According to the equations (1) and (4), T_p and T_s are state variables. The state model of the system is given by substituting equations (2), (3), and (6) into these equations give.

2-32. Fig. 2P-32 shows a heat exchanger system.

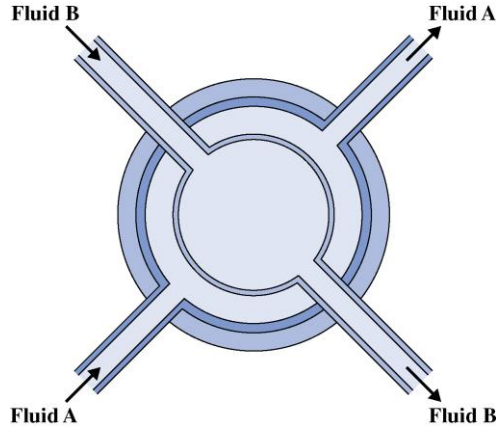


Figure 2P-32

Assuming the simple material transport model represents the rate of heat energy gain for this system, then.

$$(\dot{m}c)(T_2 - T_1) = q_{\text{gained}}$$

where \dot{m} represents the mass flow, T_1 and T_2 are the entering and leaving fluid temperature, and c shows the specific heat of fluid.

If the length of the heat exchanger cylinder is L , derive a model to give the temperature of Fluid B leaving the heat exchanger. Assign any required parameters, such as radii, thermal conductivity coefficients, and the thickness.

If the temperature of fluid B and A at the entrance and exit are supposed to be T_{BN} and T_{BX} , and T_{AN} and T_{AX} , respectively. Then:

$$\begin{cases} q_B = \dot{m}_B C_B (T_{BX} - T_{BN}) & (1) \end{cases}$$

$$\begin{cases} q_A = \dot{m}_A C_A (T_{AX} - T_{AN}) & (2) \end{cases}$$

The thermal fluid capacitance gives:

$$\begin{cases} C_B \frac{d}{dT} T_{Bx} = -q_B - q_{B-A} & (3) \end{cases}$$

$$\begin{cases} C_A \frac{d}{dT} T_{Ax} = -q_A + q_{B-A} & (4) \end{cases}$$

From thermal conductivity:

$$q_{B-A} = \frac{T_{Bx} - T_{Ax}}{\frac{1}{C_i A_i} + \frac{\ln\left(\frac{R_o}{R_i}\right)}{2 \pi K L} + \frac{1}{C_o A_o}} \quad (5)$$

Where C_i and C_o are convective heat transfer coefficient of the inner and outer tube; A_i and A_o are the surface of inner and outer tube; R_i and R_o are the radius of inner and outer tube.

Substituting equations (1), (2), and (5) into equations (3) and (4) gives the state model of the system.

2-33. Vibration can also be exhibited in fluid systems. Fig. 2P-33 shows a U tube manometer.

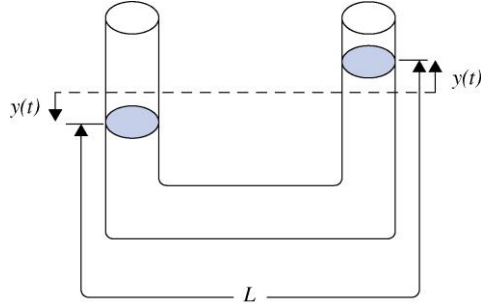


Figure 2P-33

Assume the length of fluid is L , the weight density is μ , and the cross-section area of the tube is A .

- Write the state equation of the system.
- Calculate the natural frequency of oscillation of the fluid.

The total potential energy is:

$$U = \frac{1}{2} \mu A y^2 - \left(-\frac{1}{2} \mu A y^2 \right) = \mu A y^2$$

The total kinetic energy is:

$$T = \frac{AL\mu}{2g} \dot{y}^2$$

Therefore:

$$\frac{AL\mu}{2g} \dot{y}^2 = \mu A y^2$$

$$\frac{L}{2g} \dot{y}^2 = y^2$$

As a result:

$$\dot{y} = \sqrt{\frac{2g}{L}} y$$

So, the natural frequency of the system is calculated by:

$$\omega = \sqrt{\frac{2g}{L}}$$

Also, by assuming $y(t) = Y \sin(\omega t + \theta)$ and substituting into $\frac{L}{2Y} \dot{y}^2 = y^2$ yields the same result when calculated for maximum displacement.

Note: the system at hand is similar to a spring mass system.

2-34. A long pipeline connects a water reservoir to a hydraulic generator system as shown in Fig. 2P-34.

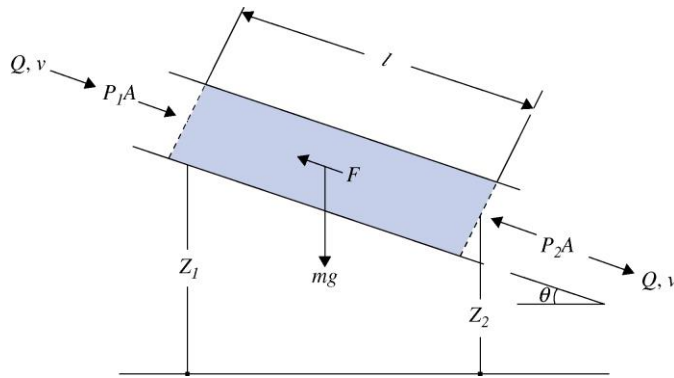


Figure 2P-34

At the end of the pipeline, there is a valve controlled by a speed controller. It may be closed quickly to stop the water flow if the generator loses its load. Determine the dynamic model for the level of the surge tank. Consider the turbine-generator is an energy converter. Assign any required parameters.

If the height of the reservoir, the surge tank and the storage tank are assumed to be H , h_1 and h_2 , then potential energy of reservoir and storage tank are:

$$\begin{cases} P_1 = \rho g H \\ P_t = \rho g h_2 \end{cases}$$

For the pipeline we have:

$$Pl \frac{d}{dt} Q = A(P_1 - P_2) + \rho Ag(z_1 - z_2) - F_f$$

The surge tank dynamics can be written as:

$$P_s = \rho g h_1$$

$$A_s \frac{d}{dt} h_1 = Q_{2-s} (\text{between pipe at point 2 and surge tank})$$

At the turbine generator, we have:

$$(P_{tg} - P_t)Q_{2-v} = I$$

where I is a known input and Q_{2-v} is the fluid flow transfer between point 2 and valve. The behaviour of the valve in this system can be written as:

$$\begin{cases} Q_{2-s} = C_s \text{sgn}(P_2 - P_s) (|P_2 - P_s|)^{\frac{1}{\alpha_s}} \\ Q_{2-v} = C_v \text{sgn}(P_v - P_{tg}) (|P_v - P_{tg}|)^{\frac{1}{\alpha_v}} \end{cases}$$

Regarding Newton's Law:

$$\begin{cases} P_2 = P_v \\ Q = Q_{2-v} + Q_{2-s} \end{cases}$$

According to above equations, it is concluded that Q and h_1 are state variables of the system.

The state equations can be rewritten by substituting P_2 , P_v , P_s and Q_{2-v} from other equations.

2-35. A simplified oil well system is shown in Fig. 2P-35. In this figure, the drive machinery is replaced by the input torque, $T_{in}(t)$. Assuming the pressure in the surrounding rock is fixed at P and the walking beam moves through small angles, determine a model for this system during the upstroke of the pumping rod.

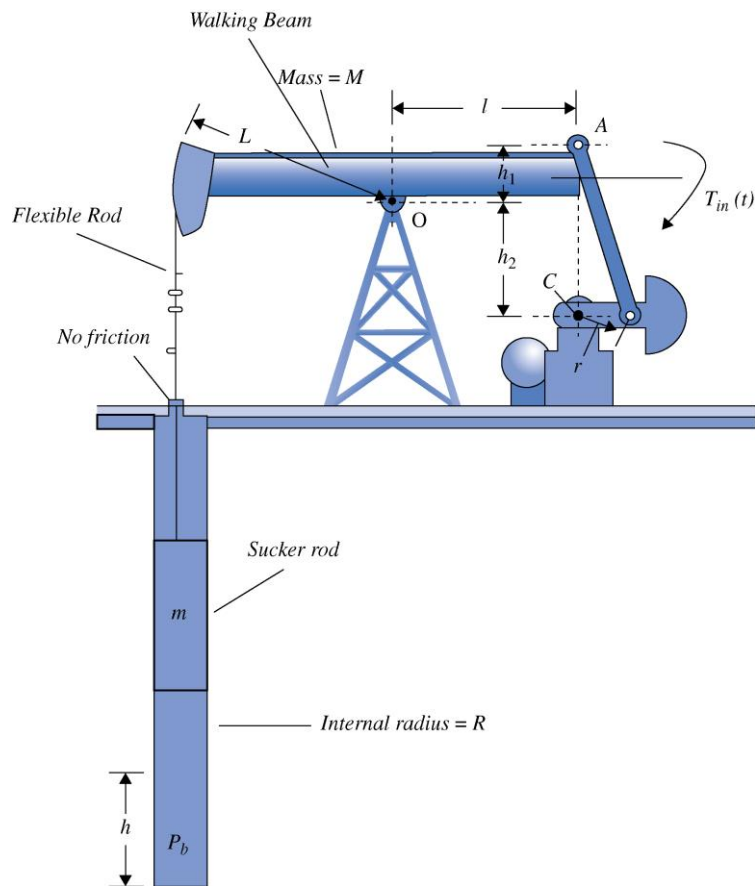
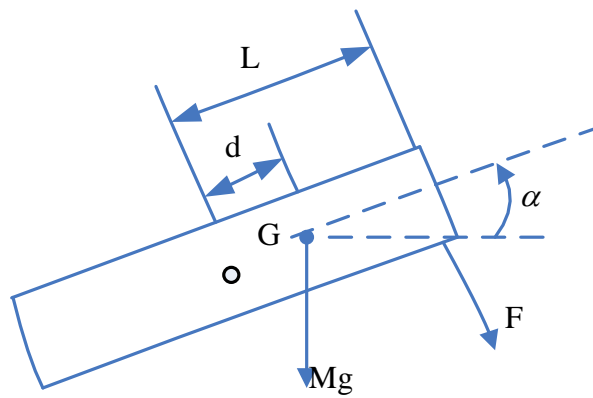


Figure 2P-35



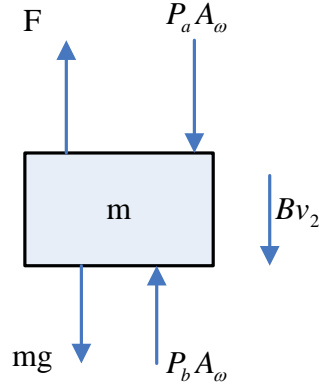
If the beam rotate around small angle of α ($\cos \alpha \cong 1$), then

$$\begin{cases} J \frac{d}{dt} \omega = T_{in} - Mgd - FL \\ F = \frac{AE(L\alpha - y)}{H - y} \end{cases}$$

where A and E are cross sectional area and elasticity of the cable; H is the distance between point O and the bottom of well, and y is the displacement.

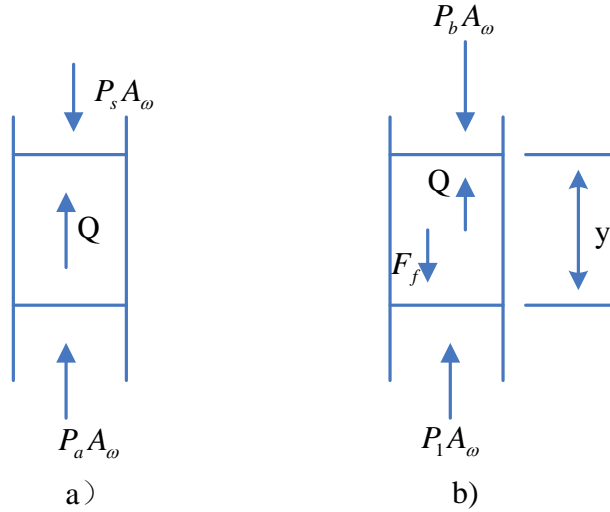
On the other hand, Newton's Law gives:

$$m \frac{d}{dt} v = P_b A_w + F - P_a A_w - Bv_2 - mg$$



where B is the viscous friction coefficient, A_w is the cross sectional area of the well; P_1 and P_2 are pressures above and below the mass m .

The dynamic for the well can be written as two pipes separating by mass m :



$$\begin{cases} \frac{dQ}{dt} = \frac{A_w}{\rho y} (P_1 - P_b) + \frac{A_w g}{y} (0 - y) - \frac{F_f}{\rho y} \\ \frac{dQ_1}{dt} = \frac{A_w}{\rho [H - D - y]} (P_a - P_s) + \frac{8A_w g}{y} (0 - y) - \frac{F_{f1}}{\rho y} \end{cases}$$

Where D is the distance between point O and ground, P_s is the pressure at the surface and known. If the diameter of the well is assumed to be r , the F_f for the laminar flow is

$$F_f = 32 \frac{\mu y Q}{r^2}$$

Therefore:

$$\begin{cases} \frac{dQ}{dt} = \frac{A_w}{\rho y} (P_1 - P_b) - A_w g - 32 \frac{\mu Q}{\rho r^2} \\ \frac{dQ_1}{dt} = \frac{A_w}{\rho [H - D - y]} (P_a - P_s) - A_w g - 32 \frac{\mu Q_1}{\rho r^2} \end{cases}$$

The state variables of the system are ω , v , y , Q , Q_1 .

2-36. Fig. 2P-36 shows a two-tank liquid-level system. Assume that Q_1 and Q_2 are the steady-state inflow rates, and H_1 and H_2 are steady-state heads. If the other quantities shown in Fig. 2P-36 are supposed to be small, derive the state-space model of the system when h_1 and h_2 are outputs of the system and q_{i1} and q_{i2} are the inputs.

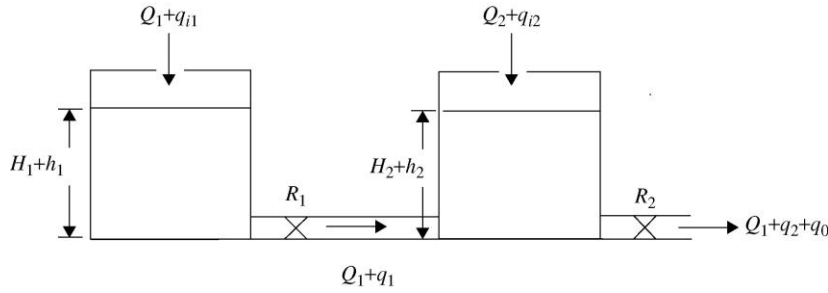


Figure 2P-36

If the capacitances of the tanks are assumed to be C_1 and C_2 respectively, then

$$\begin{cases} C_1 \frac{dh_1}{dt} = (q_{i1} - q_1) \\ C_2 \frac{dh_2}{dt} = (q_1 + q_{i2} - q_o) \\ q_1 = \frac{h_1 - h_2}{R_1} \\ q_o = \frac{h_2}{R_2} \end{cases}$$

Therefore:

$$\begin{cases} \frac{dh_1}{dt} = \frac{1}{C_1} \left(q_{i1} - \frac{h_1 - h_2}{R_1} \right) \\ \frac{dh_2}{dt} = \frac{1}{C_2} \left(\frac{h_1 - h_2}{R_1} + q_{i2} - \frac{h_2}{R_2} \right) \end{cases}$$

As a result:

$$\begin{bmatrix} \frac{dh_1}{dt} \\ \frac{dh_2}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1 C_1} & \frac{1}{R_1 C_1} \\ \frac{1}{R_1 C_2} & -\frac{R_1 + R_2}{R_1 R_2 C_2} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{C_1} & 0 \\ 0 & \frac{1}{C_2} \end{bmatrix} \begin{bmatrix} q_{i1} \\ q_{i2} \end{bmatrix}$$

See Chapter 3 for more information on state space variables.

PROBLEMS FOR SECTION 2-4

2-37. Fig. 2P-37 shows a typical grain scale.

Assign any required parameters.

- Find the free-body diagram.
- Derive a model for the grain scale that determines the waiting time for the reading of the weight of grain after placing on the scale platform.
- Develop an analogous electrical circuit for this system.

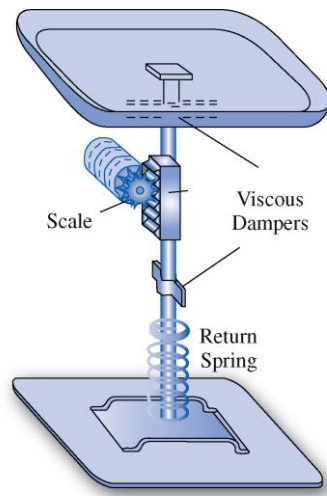
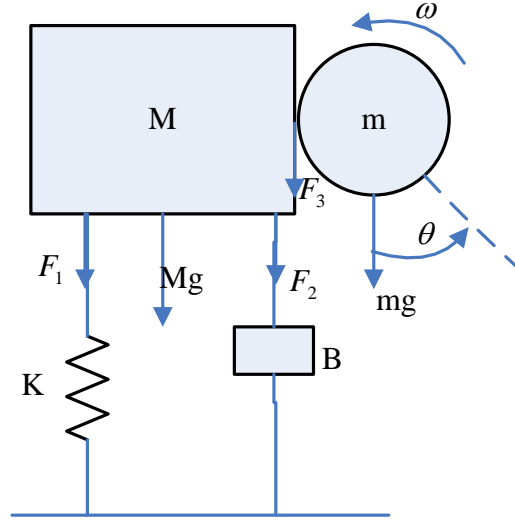


Figure 2P-37

a)



a) The equation of the translational motion is:

$$\left\{ \begin{array}{l} \frac{Mdv}{dt} = Mg - F_1 + F_2 - F_3 \quad (1) \\ F_1 = Ky \\ \frac{dy}{dt} = v \\ F_2 = -Bv \end{array} \right.$$

The equation of rotational motion (by taking a moment about the center of m is:

$$\left\{ \begin{array}{l} J \frac{d\omega}{dt} = F_3 r \\ \frac{d\theta}{dt} = \omega \end{array} \right.$$

where $J = \frac{1}{2}mr^2$

Also, the relation between rotational and translational motion defines:

$$\left\{ \begin{array}{l} v = r \omega \\ y = r \theta \end{array} \right.$$

Therefore, substituting above expression into the first equation gives:

$$F_3 = \left(\frac{m}{2M + m} \right) (Mg - Ky - Bv)$$

The resulted state space equations are:

$$\left\{ \begin{array}{l} \frac{d}{dt} \omega = \left(\frac{2}{r} \right) \left(\frac{Mg - Kr\theta - Br\omega}{2\mu + m} \right) \\ \frac{d}{dt} \theta = \omega \end{array} \right.$$

b) According to generalized elements:

- 1) Viscous friction can be replaced by a resistor where $R = B$
- 2) Spring can be replaced by a capacitor where $C = \frac{1}{k}$
- 3) Mass M and m can be replaced by two inductors where $L_1 = M$ and $L_2 = m$. Then the angular velocity is measured as a voltage of the inductor L_2
- 4) The gear will be replaced by a transformer with the ratio of $N = \frac{1}{r}$
- 5) The term Mg is also replaced by an input voltage of $V_e = Mg$

2-38. Develop an analogous electrical circuit for the mechanical system shown in Figure 2P-38.

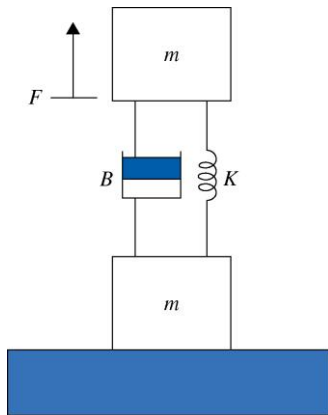
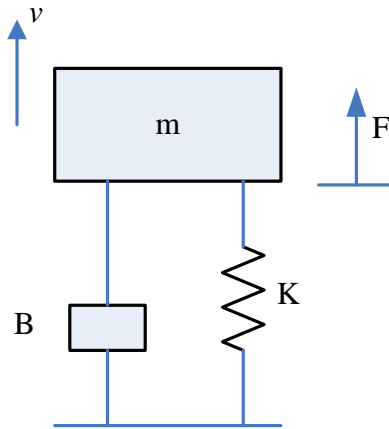


Figure 2P-38

As the base is not moving then the model can be reduced to:



Therefore:

- 1) As $m \frac{dv}{dt} = F$, they can be replaced by an inductor with $L = m$
- 2) Friction B can be replaced by a resistor where $R = B$
- 3) Spring can be replaced by a capacitor where $C = \frac{1}{k}$
- 4) The force F is replaced by a current source where $I_s = F$

2-39. Develop an analogous electrical circuit for the fluid hydraulic system shown in Fig. 2P-39.

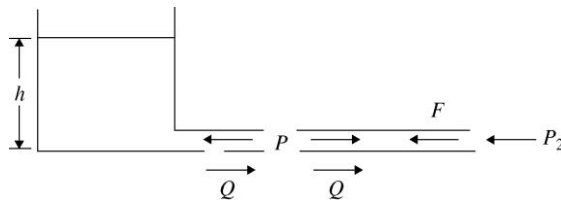
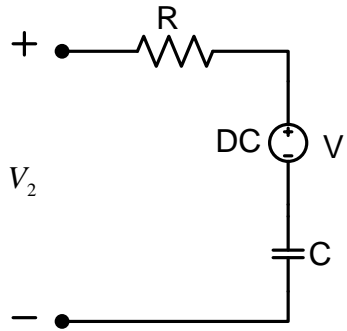


Figure 2P-39



$$R = f_R(Q) = C(|P - P_2|)^{\left(\frac{1}{\alpha}\right)}$$

$$C = \frac{A}{\rho g}$$

$$V = \rho g h$$

PROBLEMS FOR SECTION 2-5

See Chapter 3 for more linearization problems

CHAPTER 3

SOLUTION OF DIFFERENTIAL EQUATIONS OF DYNAMIC SYSTEMS

Problems

Problems for Section 3-2

3-1. Find the poles and zeros of the following functions (including the ones at infinity, if any). Mark the finite poles with \times and the finite zeros with \circ in the s -plane.

(a)
$$G(s) = \frac{10(s+2)}{s^2(s+1)(s+10)}$$

(b)
$$G(s) = \frac{10s(s+1)}{(s+2)(s^2+3s+2)}$$

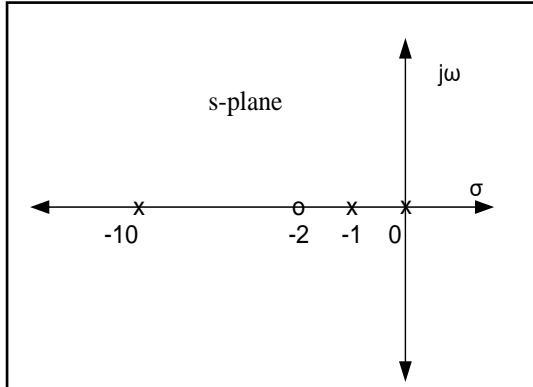
(c)
$$G(s) = \frac{10(s+2)}{s(s^2+2s+2)}$$

(d)
$$G(s) = \frac{e^{-2s}}{10s(s+1)(s+2)}$$

(a) Poles: $s = 0, 0, -1, -10$;

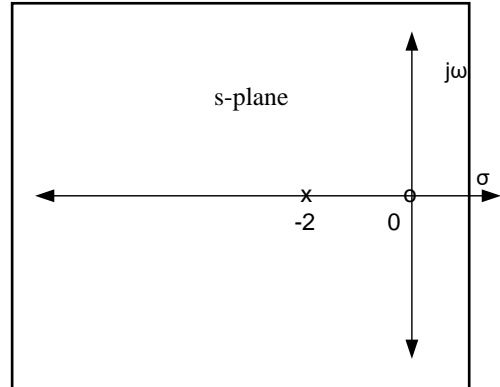
Zeros: $s = -2, \infty, \infty, \infty$.

The pole and zero at $s = -1$ cancel each other.



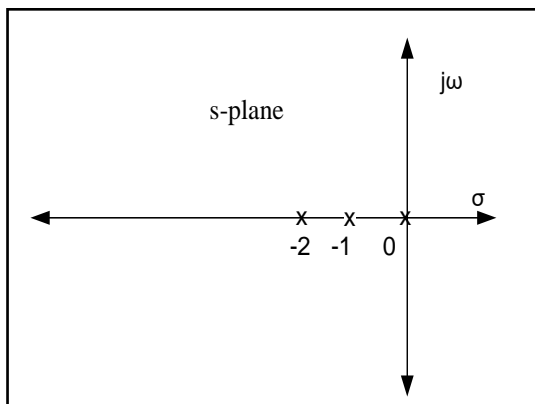
(b) Poles: $s = -2, -2$;

Zeros: $s = 0$.

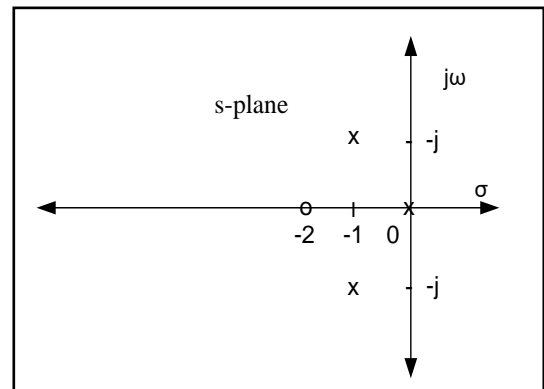


(c) Poles: $s = 0, -1 + j, -1 - j$;

Zeros: $s = -2$.



(d) Poles: $s = 0, -1, -2, \infty$.



3-2. Poles and zeros of a function are given; find the function:

- (a) Simple poles: 0, -2; poles of order 2: -3; zeros: $-1, \infty$
- (b) Simple poles: -1, -4; zeros: 0
- (c) Simple poles: -3, ∞ ; poles of order 2: 0, -1; zeros: $\pm j, \infty$

(a) $G(s) = \frac{(s+1)}{s(s+2)(s+3)^2}$

(b) $G(s) = \frac{s^2}{(s+1)(s+4)}$

(c) $G(s) = \frac{s^2-1}{s^2(s+3)(s+1)^2}$

3-3. Use MATLAB to find the poles and zeros of the functions in Problem 2-1.

MATLAB code:

```
clear all;  
s = tf('s')
```

'Generated transfer function:'

```
Ga=10*(s+2)/(s^2*(s+1)*(s+10))
```

'Poles:'

```
pole(Ga)
```

'Zeros:'

```
zero(Ga)
```

'Generated transfer function:'

```
Gb=10*s*(s+1)/((s+2)*(s^2+3*s+2))
```

'Poles:';

```
pole(Gb)
```

'Zeros:'

```
zero(Gb)
```

'Generated transfer function:'

```
Gc=10*(s+2)/(s*(s^2+2*s+2))
```

'Poles:';

```
pole(Gc)
```

'Zeros:'

```
zero(Gc)
```

'Generated transfer function:'

```
Gd=pade(exp(-2*s),1)/(10*s*(s+1)*(s+2))
'Poles:';
pole(Gd)
'Zeros:'
zero(Gd)
```

Poles and zeros of the above functions:

(a)

```
Poles:  0  0 -10 -1
Zeros: -2
```

(b)

```
Poles: -2.0000 -2.0000 -1.0000
Zeros:  0 -1
```

(c)

```
Poles:  0 -1.0000 + 1.0000i -1.0000 - 1.0000i
Zeros: -2
```

Generated transfer function:

(d)

Using first order Pade approximation for exponential term:

```
Poles:  0 -2.0000 -1.0000 + 0.0000i -1.0000 - 0.0000i
Zeros:  1
```

3-4. Use MATLAB to obtain $\mathcal{L}\{\sin^2 2t\}$. Then, calculate $\mathcal{L}\{\cos^2 2t\}$ when you know $\mathcal{L}\{\sin^2 2t\}$. Verify your answer by calculating $\mathcal{L}\{\cos^2 2t\}$ in MATLAB.

MATLAB code:

```
clear all;
syms t
s=tf('s')
```

```
f1 = (sin(2*t))^2
L1=laplace(f1)
```

```
% f2 = (cos(2*t))^2 = 1-(sin(2*t))^2 ==> L(f2)=1/s-L(f1) ==>
L2= 1/s - 8/s/(s^2+16)
```

```
f3 = (cos(2*t))^2
L3=laplace(f3)
'verified as L2 equals L3'
```

MATLAB solution for

$$L\{\sin^2 2t\} \text{ is : } \frac{8(s^2+16)}{s}$$

Calculating $L\{\cos^2 2t\}$ based on $L\{\sin^2 2t\}$

$$L\{\cos^2 2t\} = s^{\frac{(3+8s)}{(s^4+16s^2)}}$$

$$\text{Verifying } L\{\cos^2 2t\} : \frac{8(s^2+16)}{s}$$

3-5. Find the Laplace transforms of the following functions. Use the theorems on Laplace transforms, if applicable.

(a) $g(t) = 5te^{-5t}u_s(t)$

(b) $g(t) = (t \sin 2t + e^{-2t})u_s(t)$

(c) $g(t) = 2e^{-2t} \sin 2t u_s(t)$

(d) $g(t) = \sin 2t \cos 2t u_s(t)$

(e) $g(t) = \sum_{k=0}^{\infty} e^{-5kT} \delta(t - kT)$ where $\delta(t)$ = unit-impulse function

Solution:

$$\text{Note: } u_s(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

By Laplace transform definition:

$$L\{g(t)u(t)\} = \int_0^{\infty} g(t)e^{-st} dt$$

(a)

$$G(s) = \frac{5}{(s+5)^2}$$

(b)

$$G(s) = \frac{4s}{(s^2+4)} + \frac{1}{s+2}$$

(c)

$$G(s) = \frac{4s}{(s^2+4s+8)}$$

$$g(t) = (\sin 2t \cos 2t)u_s(t) = \frac{\sin 4t}{2}u_s(t)$$

(d)

$$\mathcal{L}[g(t)] = \frac{2}{(s^2+4^2)}$$

$$(e) G(s) = \sum_{k=0}^{\infty} e^{kT(s+5)} = \frac{1}{1 - e^{-T(s+5)}}$$

Note: Section (e) requires assignment of T and a numerical loop calculation

3-6. Use MATLAB to solve Problem 3-5.

MATLAB code:

```
clear all;
syms t u
```

```
f1 = 5*t*exp(-5*t)
L1=laplace(f1)
```

```
f2 = t*sin(2*t)+exp(-2*t)
L2=laplace(f2)
```

```
f3 = 2*exp(-2*t)*sin(2*t)
L3=laplace(f3)
```

```
f4 = sin(2*t)*cos(2*t)
L4=laplace(f4)
```

$f4 = \cos(2t) \sin(2t)$

$$L4 = \frac{2}{s^2 + 16}$$

Section (e) requires assignment of T and a numerical loop calculation

$$(a) g(t) = 5te^{-5t}u_s(t)$$

$$\text{Answer: } \frac{5}{(s+5)^2}$$

$$(b) g(t) = (t\sin 2t + e^{-2t})u_s(t)$$

$$\text{Answer: } 4\left(\frac{s}{(s^2+4)^2} + \frac{1}{(s+2)}\right)$$

$$(c) g(t) = 2e^{-2t}\sin 2t u_s(t)$$

Answer: $\frac{4}{(s^2+4)(s+8)}$

(d) $g(t) = \sin 2t \cos 2t u_s(t)$

Answer: $\frac{2}{(s^2+16)}$

(e) $g(t) = \sum_{k=0}^{\infty} e^{-5kT} \delta(t - kT)$ where $\delta(t)$ = unit-impulse function

Section (e) requires assignment of T and a numerical loop calculation

3-7. Find the Laplace transforms of the functions shown in Fig. 3P-7. First, write a complete expression for $g(t)$, and then take the Laplace transform. Let $gT(t)$ be the description of the function over the basic period and then delay $gT(t)$ appropriately to get $g(t)$. Take the Laplace transform of $g(t)$ to get the following:

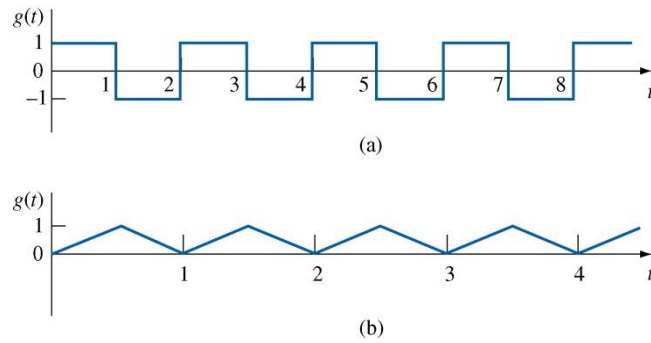


Figure 3P-7

(a)

$$g(t) = u_s(t) - 2u_s(t-1) + 2u_s(t-2) - 2u_s(t-3) + \cdots$$

$$G(s) = \frac{1}{s} \left(1 - 2e^{-s} + 2e^{-2s} - 2e^{-3s} + \cdots \right) = \frac{1 - e^{-s}}{s(1 + e^{-s})}$$

$$g_T(t) = u_s(t) - 2u_s(t-1) + u_s(t-2) \quad 0 \leq t \leq 2$$

$$G_T(s) = \frac{1}{s} \left(1 - 2e^{-s} + e^{-2s} \right) = \frac{1}{s} \left(1 - e^{-s} \right)^2$$

$$g(t) = \sum_{k=0}^{\infty} g_T(t-2k)u_s(t-2k) \quad G(s) = \sum_{k=0}^{\infty} \frac{1}{s} (1 - e^{-s})^2 e^{-2ks} = \frac{1 - e^{-s}}{s(1 + e^{-s})}$$

(b)

$$g(t) = 2tu_s(t) - 4(t-0.5)u_s(t-0.5) + 4(t-1)u_s(t-1) - 4(t-1.5)u_s(t-1.5) + \cdots$$

$$G(s) = \frac{2}{s^2} \left(1 - 2e^{-0.5s} + 2e^{-s} - 2e^{-1.5s} + \cdots \right) = \frac{2(1 - e^{-0.5s})}{s^2(1 + e^{-0.5s})}$$

$$g_T(t) = 2tu_s(t) - 4(t-0.5)u_s(t-0.5) + 2(t-1)u_s(t-1) \quad 0 \leq t \leq 1$$

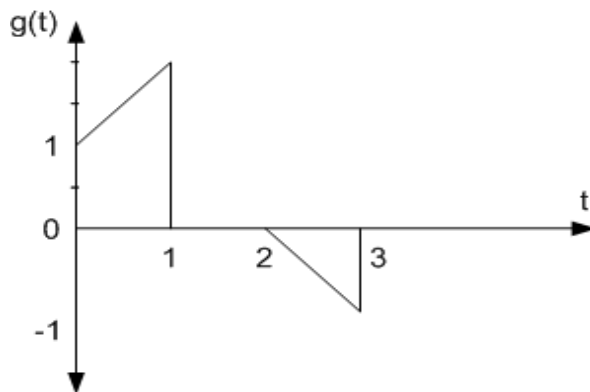
$$G_T(s) = \frac{2}{s^2} \left(1 - 2e^{-0.5s} + e^{-s} \right) = \frac{2}{s^2} \left(1 - e^{-0.5s} \right)^2$$

$$g(t) = \sum_{k=0}^{\infty} g_T(t-k)u_s(t-k) \quad G(s) = \sum_{k=0}^{\infty} \frac{2}{s^2} \left(1 - e^{-0.5s} \right)^2 e^{-ks} = \frac{2(1 - e^{-0.5s})}{s^2(1 + e^{-0.5s})}$$

3-8. Find the Laplace transform of the following function.

$$\left\{ \begin{array}{ll} t+1 & 0 \leq t < 1 \\ 0 & 1 \leq t < 2 \\ 2-t & 2 \leq t < 3 \\ 0 & t \geq 3 \end{array} \right. g(t) =$$

Solution: $g(t) = (t+1)u_s(t) - (t-1)u_s(t-1) - 2u_s(t-1) - (t-2)u_s(t-2) + (t-3)u_s(t-3) + u_s(t-3)$



$$G(s) = \frac{1}{s^2} \left(1 - e^{-s} - e^{-2s} + e^{-3s} \right) + \frac{1}{s} \left(1 - 2e^{-s} + e^{-3s} \right)$$

3-9. Find the Laplace transform of the periodic function in Fig. 3P-9.

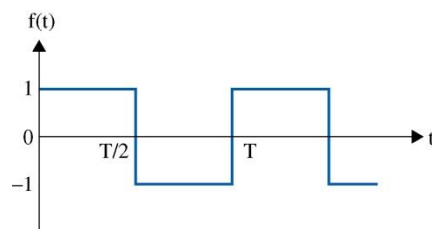


Figure 3P-9

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \int_0^T f(t)e^{-st} dt = \int_0^{T/2} e^{-st} dt + \int_{T/2}^T (-1)e^{-st} dt \\ &= \frac{1 - e^{-Ts/2}}{s} + \frac{e^{-Ts} - e^{-Ts/2}}{s} = \frac{1}{s} \left[1 - e^{-Ts/2} \right]^2 \end{aligned}$$

3-10. Find the Laplace transform of the function in Fig. 3P-10.

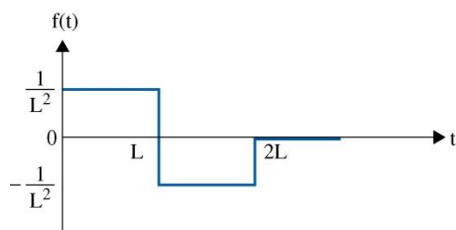


Figure 3P-10

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \int_0^T f(t)e^{-st}dt = \int_0^{\frac{T}{2}} e^{-st}dt + \int_{\frac{T}{2}}^T (-1)e^{-st}dt \\ &= \frac{1 - e^{-\frac{Ts}{2}}}{s} + \frac{e^{-Ts} - e^{-\frac{Ts}{2}}}{s} = \frac{1}{s} \left[1 - e^{-\frac{Ts}{2}} \right]^2\end{aligned}$$

3-11. The following differential equations represent linear time-invariant systems, where $r(t)$ denotes the input and $y(t)$ the output. Find the transfer function $Y(s)/R(s)$ for each of the systems. (Assume zero initial conditions.)

(a) $\frac{d^3 y(t)}{dt^3} + 2\frac{d^2 y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = 3\frac{dr(t)}{dt} + r(t)$

(b) $\frac{d^4 y(t)}{dt^4} + 10\frac{d^2 y(t)}{dt^2} + \frac{dy(t)}{dt} + 5y(t) = 5r(t)$

(c) $\frac{d^3 y(t)}{dt^3} + 10\frac{d^2 y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) + 2\int_0^t y(\tau)d\tau = \frac{dr(t)}{dt} + 2r(t)$

(d) $2\frac{d^2 y(t)}{dt^2} + \frac{dy(t)}{dt} + 5y(t) = r(t) + 2r(t-1)$

(e) $\frac{d^2 y(t+1)}{dt^2} + 4\frac{dy(t+1)}{dt} + 5y(t+1) = \frac{dr(t)}{dt} + 2r(t) + 2\int_{-\infty}^t r(\tau)d\tau$

(f) $\frac{d^3 y(t)}{dt^3} + 2\frac{d^2 y(t)}{dt^2} + \frac{dy(t)}{dt} + 2y(t) + 2\int_{-\infty}^t y(\tau)d\tau = \frac{dr(t-2)}{dt} + 2r(t-2)$

Solution:

(a)

$$\frac{Y(s)}{R(s)} = \frac{3s+1}{s^3 + 2s^2 + 5s + 6}$$

(b)

$$\frac{Y(s)}{R(s)} = \frac{5}{s^4 + 10s^2 + s + 5}$$

(c)

$$\frac{Y(s)}{R(s)} = \frac{s(s+2)}{s^4 + 10s^3 + 2s^2 + s + 2}$$

(d)

$$\frac{Y(s)}{R(s)} = \frac{1 + 2e^{-s}}{2s^2 + s + 5}$$

e) $x(t) = y(t+1)$

$$\Rightarrow \frac{d^2x(t)}{dt^2} + 4\frac{dx(t)}{dt} + 5x(t) = \frac{dr(t)}{dt} + 2r(t) + 2\int_{-\infty}^t r(\tau) d\tau$$

By using Laplace transform, we have:

$$s^2X(s) + 4sX(s) + 5X(s) = sR(s) + 2R(s) + \frac{R(s)}{s}$$

As $X(s) = e^{-s}Y(s)$, then

$$(s^2 + 4s + 5)e^{-s}Y(s) = \frac{s^2 + 2s + 1}{s}R(s)$$

Then:

$$\frac{Y(s)}{R(s)} = \frac{(s+1)^2 e^s}{s(s^2 + 4s + 5)}$$

f) By using Laplace transform we have:

$$\left(s^3 + 2s^2 + s + 2 + \frac{2}{s}\right)Y(s) = se^{-s}R(s) + 2e^{-s}R(s)$$

As a result:

$$\frac{Y(s)}{R(s)} = \frac{s(s+2)e^{-s}}{s^4 + 2s^3 + s^2 + 2s + 2}$$

3-12. Use MATLAB to find $Y(s)/R(s)$ for the differential equations in Problem 2-29.

After taking the Laplace transform, the equation was solved in terms of $Y(s)$, and consecutively was divided by input $R(s)$ to obtain $Y(s)/R(s)$:

MATLAB code:

```
clear all;

syms Ys Rs s

sol1=solve('s^3*Ys+2*s^2*Ys+5*s*Ys+6*Ys=3*s*Rs+Rs','Ys')

Ys_Rs1=sol1/Rs

sol2=solve('s^4*Ys+10*s^2*Ys+s*Ys+5*Ys=5*Rs','Ys')

Ys_Rs2=sol2/Rs

sol3=solve('s^3*Ys+10*s^2*Ys+2*s*Ys+2*Ys/s=s*Rs+2*Rs','Ys')

Ys_Rs3=sol3/Rs

sol4=solve('2*s^2*Ys+s*Ys+5*Ys=2*Rs*exp(-1*s)','Ys')
```

Ys_Rs4=sol4/Rs

%Note: Parts E&F are too complicated with MATLAB, Laplace of integral is not executable in MATLAB.....skipped

MATLAB Answers:

Part (a): $Y(s)/R(s) = (3*s+1)/(5*s+6+s^3+2*s^2);$

Part (b): $Y(s)/R(s) = 5/(10*s^2+s+5+s^4)$

Part (c): $Y(s)/R(s) = (s+2)*s/(2*s^2+2+s^4+10*s^3)$

Part (d): $Y(s)/R(s) = 2*\exp(-s)/(2*s^2+s+5)$

%Note: Parts E&F are too complicated with MATLAB, Laplace of integral is not executable in MATLAB.....skipped

Problems for Section 3-3

3-13. Find the inverse Laplace transforms of the following functions. First, perform partial-fraction expansion on $G(s)$; then, use the Laplace transform table.

(a)
$$G(s) = \frac{1}{s(s+2)(s+3)}$$

(b)
$$G(s) = \frac{10}{(s+1)^2(s+3)}$$

(c)
$$G(s) = \frac{100(s+2)}{s(s^2+4)(s+1)} e^{-s}$$

(d)
$$G(s) = \frac{2(s+1)}{s(s^2+s+2)}$$

(e)
$$G(s) = \frac{1}{(s+1)^3}$$

(f)
$$G(s) = \frac{2(s^2+s+1)}{s(s+1.5)(s^2+5s+5)}$$

(g)
$$G(s) = \frac{2+2se^{-s}+4e^{-2s}}{s^2+3s+2}$$

$$(h) \quad G(s) = \frac{2s+1}{s^3+6s^2+11s+6}$$

$$(i) \quad G(s) = \frac{3s^3+10s^2+8s+5}{s^4+5s^3+7s^2+5s+6}$$

Solution:

(a)

$$G(s) = \frac{1}{3s} - \frac{1}{2(s+2)} + \frac{1}{3(s+3)} \quad g(t) = \frac{1}{3} - \frac{1}{2}e^{-2t} + \frac{1}{3}e^{-3t} \quad t \geq 0$$

(b)

$$G(s) = \frac{-2.5}{s+1} + \frac{5}{(s+1)^2} + \frac{2.5}{s+3} \quad g(t) = -2.5e^{-t} + 5te^{-t} + 2.5e^{-3t} \quad t \geq 0$$

(c)

$$G(s) = \left(\frac{50}{s} - \frac{20}{s+1} - \frac{30s+20}{s^2+4} \right) e^{-s} \quad g(t) = \left[50 - 20e^{-(t-1)} - 30 \cos 2(t-1) - 5 \sin 2(t-1) \right] u_s(t-1)$$

(d)

$$G(s) = \frac{1}{s} - \frac{s-1}{s^2+s+2} = \frac{1}{s} + \frac{1}{s^2+s+2} - \frac{s}{s^2+s+2} \quad \text{Taking the inverse Laplace transform,}$$

$$g(t) = 1 + 1.069e^{-0.5t} \left[\sin 1.323t + \sin(1.323t - 69.3^\circ) \right] = 1 + e^{-0.5t} (1.447 \sin 1.323t - \cos 1.323t) \quad t \geq 0$$

$$(e) \quad g(t) = 0.5t^2 e^{-t} \quad t \geq 0$$

(f) Try using MATLAB

```
>> b=num*2
```

```
b =
```

```
2 2 2
```

```
>> num =
```

```
1 1 1
```

```
>> denom1=[1 1]
```

```
denom1 =
```

```

1 1
>> denom2=[1 5 5]

denom2 =

1 5 5
>> num*2

ans =

2 2 2
>> denom=conv([1 0],conv(denom1,denom2))

denom =

1 6 10 5 0
>> b=num*2

b =

2 2 2
>> a=denom

a =

1 6 10 5 0
>> [r, p, k] = residue(b,a)

r =

-0.9889

2.5889

-2.0000

0.4000

p =

-3.6180

-1.3820

```

$$-1.0000$$

$$0$$

$$k = []$$

If there are no multiple roots, then the number of poles n is

$$\frac{b}{a} = \frac{r_1}{s + p_1} + \frac{r_2}{s + p_2} + \dots + \frac{r_n}{s + p_n} + k$$

In this case, p_1 and k are zero. Hence,

$$G(s) = \frac{0.4}{s} - \frac{0.9889}{s + 3.6180} + \frac{2.5889}{s + 1.3820} - \frac{2}{s + 1}$$

$$g(t) = 0.4 - 0.9889e^{-3.618t} + 1.3820e^{-2.5889t} - 2e^{-t}$$

$$(g) \quad G(s) = \frac{2}{(s+1)(s+2)} + \frac{2e^{-s}}{s+1}$$

$$= \frac{2}{s+1} - \frac{2}{s+2} + \frac{2e^{-s}}{s+1}$$

$$\Rightarrow \mathcal{L}^{-1}\{G(s)\} = 2e^{-t} - 2e^{-2t} + 2e^{-(t-1)}u(t-1)$$

$$(h) \quad G(s) = \frac{2s+1}{(s+1)(s+2)(s+3)} = -\frac{\frac{1}{2}}{s+1} + \frac{3}{s+2} - \frac{5}{2(s+3)}$$

$$\Rightarrow \mathcal{L}^{-1}\{G(s)\} = -\frac{1}{2}e^{-t} + 3e^{-2t} - \frac{5}{2}e^{-3t}$$

$$(i) \quad G(s) = \frac{3s^3+10s^2+8s+5}{s^3+5s^2+7s+6} = \frac{1}{s+2} + \frac{1}{s+3} + \frac{s}{s^2+1}$$

$$\Rightarrow \mathcal{L}^{-1}\{G(s)\} = e^{-2t} + e^{-3t} + \cos t$$

3-14. Use MATLAB to find the inverse Laplace transforms of the functions in Problem 3-13. First, perform partial-fraction expansion on $G(s)$; then, use the inverse Laplace transform.

MATLAB code:

```
clear all;
```

```
syms s
```

```

f1=1/(s*(s+2)*(s+3))
F1=ilaplace(f1)

f2=10/((s+1)^2*(s+3))
F2=ilaplace(f2)

f3=10*(s+2)/(s*(s^2+4)*(s+1))*exp(-s)
F3=ilaplace(f3)

f4=2*(s+1)/(s*(s^2+s+2))
F4=ilaplace(f4)

f5=1/(s+1)^3
F5=ilaplace(f5)

f6=2*(s^2+s+1)/(s*(s+1.5)*(s^2+5*s+5))
F6=ilaplace(f6)

s=tf('s')
f7=(2+2*s*pade(exp(-1*s),1)+4*pade(exp(-2*s),1))/(s^2+3*s+2) %using
Pade command for exponential term
[num,den]=tfdata(f7,'v') %extracting the polynomial values
syms s
f7n=(-2*s^3+6*s+12)/(s^4+6*s^3+13*s^2+12*s+4) %generating symbolic
function for ilaplace
F7=ilaplace(f7n)

f8=(2*s+1)/(s^3+6*s^2+11*s+6)
F8=ilaplace(f8)

```

```
f9 = (3*s^3+10*s^2+8*s+5) / (s^4+5*s^3+7*s^2+5*s+6)
F9 = ilaplace(f9)
```

Solution from MATLAB for the Inverse Laplace transforms:

Part (a):
$$G(s) = \frac{1}{s(s+2)(s+3)}$$

$$G(t) = -1/2 * \exp(-2*t) + 1/3 * \exp(-3*t) + 1/6$$

To simplify:

```
syms t
```

```
digits(3)
```

```
vpa(-1/2*exp(-2*t)+1/3*exp(-3*t)+1/6)
```

```
ans = -.500*exp(-2.*t)+.333*exp(-3.*t)+.167
```

Part (b):
$$G(s) = \frac{10}{(s+1)^2(s+3)}$$

$$G(t) = 5/2 * \exp(-3*t) + 5/2 * \exp(-t) * (-1+2*t)$$

Part (c):
$$G(s) = \frac{100(s+2)}{s(s^2+4)(s+1)} e^{-s}$$

$$G(t) = \text{Step}(t-1) * (-4 * \cos(t-1)^2 + 2 * \sin(t-1) * \cos(t-1) + 4 * \exp(-1/2*t+1/2) * \cosh(1/2*t-1/2) - 4 * \exp(-t+1) - \cos(2*t-2) - 2 * \sin(2*t-2) + 5)$$

Part (d):
$$G(s) = \frac{2(s+1)}{s(s^2+s+2)}$$

$$G(t) = 1 + 1/7 * \exp(-1/2 * t) * (-7 * \cos(1/2 * 7^{1/2} * t) + 3 * 7^{1/2} * \sin(1/2 * 7^{1/2} * t))$$

To simplify:

`syms t`

`digits(3)`

$$\text{vpa}(1 + 1/7 * \exp(-1/2 * t) * (-7 * \cos(1/2 * 7^{1/2} * t) + 3 * 7^{1/2} * \sin(1/2 * 7^{1/2} * t)))$$

$$\text{ans} = 1. + 1.143 * \exp(-.500 * t) * (-7. * \cos(1.32 * t) + 7.95 * \sin(1.32 * t))$$

Part (e):
$$G(s) = \frac{1}{(s+1)^3}$$

$$G(t) = 1/2 * t^2 * \exp(-t)$$

Part (f):
$$G(s) = \frac{2(s^2 + s + 1)}{s(s+1.5)(s^2 + 5s + 5)}$$

$$G(t) = 4/15 + 28/3 * \exp(-3/2 * t) -$$

$$16/5 * \exp(5/2 * t) * (3 * \cosh(1/2 * t * 5^{1/2}) + 5^{1/2} * \sinh(1/2 * t * 5^{1/2}))$$

Part (g):
$$G(s) = \frac{2 + 2se^{-s} + 4e^{-2s}}{s^2 + 3s + 2}$$

$$G(t) = 2 * \exp(-2 * t) * (7 + 8 * t) + 8 * \exp(-t) * (-2 + t)$$

Part (h):
$$G(s) = \frac{2s + 1}{s^3 + 6s^2 + 11s + 6}$$

$$G(t) = -1/2 * \exp(-t) + 3 * \exp(-2 * t) - 5/2 * \exp(-3 * t)$$

Part (i):
$$G(s) = \frac{3s^3 + 10s^2 + 8s + 5}{s^4 + 5s^3 + 7s^2 + 5s + 6}$$

$G(t) = -7 \cdot \exp(-2 \cdot t) + 10 \cdot \exp(-3 \cdot t) -$

$1/10 \cdot \text{ilaplace}(10 \cdot (2 \cdot s) / (s^2 + 1) \cdot s, s, t) + 1/10 \cdot \text{ilaplace}(10 \cdot (2 \cdot s) / (s^2 + 1), s, t) + 1/10 \cdot \sin(t) \cdot (10 + \text{dirac}(t) \cdot (-\exp(-3 \cdot t) + 2 \cdot \exp(-2 \cdot t)))$

3-15. Use MATLAB to find the partial-fraction expansion to the following functions.

(a)
$$G(s) = \frac{10(s+1)}{s^2(s+4)(s+6)}$$

(b)
$$G(s) = \frac{(s+1)}{s(s+2)(s^2+2s+2)}$$

(c)
$$G(s) = \frac{5(s+2)}{s^2(s+1)(s+5)}$$

(d)
$$G(s) = \frac{5e^{-2s}}{(s+1)(s^2+s+1)}$$

(e)
$$G(s) = \frac{100(s^2+s+3)}{s(s^2+5s+3)}$$

(f)
$$G(s) = \frac{1}{s(s^2+1)(s+0.5)^2}$$

(g)
$$G(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2+4)(s^2+2s+2)}$$

(h)
$$G(s) = \frac{2s^4 + 9s^3 + 15s^2 + s + 2}{s^2(s+2)(s+1)^2}$$

MATLAB code:

```
clear all;
s=tf('s')
```

%Part a

```
Eq=10*(s+1)/(s^2*(s+4)*(s+6));
```

```
[num,den]=tfdata(Eq,'v');
```

```
[r,p] = residue(num,den)
```

%Part b

```
Eq=(s+1)/(s*(s+2)*(s^2+2*s+2));
```

```
[num,den]=tfdata(Eq,'v');
```

```
[r,p] = residue(num,den)
```

%Part c

```
Eq=5*(s+2)/(s^2*(s+1)*(s+5));
```

```
[num,den]=tfdata(Eq,'v');
```

```
[r,p] = residue(num,den)
```

%Part d

```
Eq=5*(pade(exp(-2*s),1))/(s^2+s+1); %Pade approximation order 1 used
```

```
[num,den]=tfdata(Eq,'v');
```

```
[r,p] = residue(num,den)
```

%Part e

```
Eq=100*(s^2+s+3)/(s*(s^2+5*s+3));
```

```
[num,den]=tfdata(Eq,'v');
```

```
[r,p] = residue(num,den)
```

%Part f

```
Eq=1/(s*(s^2+1)*(s+0.5)^2);
```

```
[num,den]=tfdata(Eq,'v');
```

```
[r,p] = residue(num,den)
```

%Part g

```
Eq=(2*s^3+s^2+8*s+6)/((s^2+4)*(s^2+2*s+2));
```

```
[num,den]=tfdata(Eq,'v');
```

```
[r,p] = residue(num,den)
```

%Part h

```
Eq=(2*s^4+9*s^3+15*s^2+s+2)/(s^2*(s+2)*(s+1)^2);
```

```
[num,den]=tfdata(Eq,'v');
```

```
[r,p] = residue(num,den)
```

The solutions are presented in the form of two vectors, **r** and **p**, where for each case, the partial fraction expansion is equal to:

$$\frac{b(s)}{a(s)} = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \dots + \frac{r_n}{s-p_n}$$

Following are **r** and **p** vectors for each part:

Part(a):

r=0.6944

-0.9375

0.2431

0.4167

p=-6.0000

-4.0000

0

0

Part(b):

$$r = 0.2500$$

$$-0.2500 - 0.0000i$$

$$-0.2500 + 0.0000i$$

$$0.2500$$

$$p = -2.0000$$

$$-1.0000 + 1.0000i$$

$$-1.0000 - 1.0000i$$

$$0$$

Part(c):

$$r = 0.1500$$

$$1.2500$$

$$-1.4000$$

$$2.0000$$

$$p = -5$$

$$-1$$

$$0$$

$$0$$

Part(d):

$$r = 10.0000$$

$$-5.0000 - 0.0000i$$

$$-5.0000 + 0.0000i$$

$$p = -1.0000$$

$$-0.5000 + 0.8660i$$

$$-0.5000 - 0.8660i$$

Part(e):

$$r = 110.9400$$

$$-110.9400$$

$$100.0000$$

$$p = -4.3028$$

$$-0.6972$$

$$0$$

Part(f):

$$r = 0.2400 + 0.3200i$$

$$0.2400 - 0.3200i$$

$$-4.4800$$

$$-1.6000$$

$$4.0000$$

$$p = -0.0000 + 1.0000i$$

$$-0.0000 - 1.0000i$$

-0.5000

-0.5000

0

Part(g):

$r = -0.1000 + 0.0500i$

$-0.1000 - 0.0500i$

$1.1000 + 0.3000i$

$1.1000 - 0.3000i$

$p = 0.0000 + 2.0000i$

$0.0000 - 2.0000i$

$-1.0000 + 1.0000i$

$-1.0000 - 1.0000i$

Part(h):

$r = 5.0000$

-1.0000

9.0000

-2.0000

1.0000

$p = -2.0000$

-1.0000

-1.0000

0

0

3-16. Use MATLAB to find the inverse Laplace transforms of the functions in 3-15.

MATLAB code:

```
clear all;
```

```
syms s
```

```
%Part a
```

```
Eq=10*(s+1)/(s^2*(s+4)*(s+6));
```

```
ilaplace(Eq)
```

```
%Part b
```

```
Eq=(s+1)/(s*(s+2)*(s^2+2*s+2));
```

```
ilaplace(Eq)
```

```
%Part c
```

```
Eq=5*(s+2)/(s^2*(s+1)*(s+5));
```

```
ilaplace(Eq)
```

```
%Part d
```

```
exp_term=(-s+1)/(s+1) %pade approximation
```

```
Eq=5*exp_term/((s+1)*(s^2+s+1));
```

```
ilaplace(Eq)
```

```
%Part e
```

```
Eq=100*(s^2+s+3)/(s*(s^2+5*s+3));
```

```
ilaplace(Eq)
```

```
%Part f
```

```
Eq=1/(s*(s^2+1)*(s+0.5)^2);
```

```
ilaplace(Eq)
```

```
%Part g
```

```
Eq=(2*s^3+s^2+8*s+6)/((s^2+4)*(s^2+2*s+2));
```

```
ilaplace(Eq)
```

```
%Part h
```

```
Eq=(2*s^4+9*s^3+15*s^2+s+2)/(s^2*(s+2)*(s+1)^2);
```

```
ilaplace(Eq)
```

MATLAB Answers:

Part(a):

$G(t) = -15/16 \exp(-4t) + 25/36 \exp(-6t) + 35/144 + 5/12 t$

To simplify:

```
syms t
```

```
digits(3)
```

```
vpa(-15/16*exp(-4*t)+25/36*exp(-6*t)+35/144+5/12*t)
```

```
ans = -.938*exp(-4.*t)+.694*exp(-6.*t)+.243+.417*tPart(b):
```

$$G(t) = 1/4 \cdot \exp(-2 \cdot t) + 1/4 - 1/2 \cdot \exp(-t) \cdot \cos(t)$$

Part(c):

$$G(t) = 5/4 \cdot \exp(-t) - 7/5 + 3/20 \cdot \exp(-5 \cdot t) + 2 \cdot t$$

Part(d):

$$G(t) = -5 \cdot \exp(-1/2 \cdot t) \cdot (\cos(1/2 \cdot 3^{1/2} \cdot t) + 3^{1/2} \cdot \sin(1/2 \cdot 3^{1/2} \cdot t)) + 5 \cdot (1 + 2 \cdot t) \cdot \exp(-t)$$

Part(e):

$$G(t) = 100 - 800/13 \cdot \exp(-5/2 \cdot t) \cdot 13^{1/2} \cdot \sinh(1/2 \cdot t \cdot 13^{1/2})$$

Part(f):

$$G(t) = 4 + 12/25 \cdot \cos(t) - 16/25 \cdot \sin(t) - 8/25 \cdot \exp(-1/2 \cdot t) \cdot (5 \cdot t + 14)$$

Part(g):

$$G(t) = -1/5 \cdot \cos(2 \cdot t) - 1/10 \cdot \sin(2 \cdot t) + 1/5 \cdot (11 \cdot \cos(t) - 3 \cdot \sin(t)) \cdot \exp(-t)$$

Part(h):

$$G(t) = -2 + t + 5 \cdot \exp(-2 \cdot t) + (-1 + 9 \cdot t) \cdot \exp(-t)$$

Problems for Section 3-4

3-17. Solve the following differential equations by means of the Laplace transform.

(a) $\frac{d^2 f(t)}{dt^2} + 5 \frac{df(t)}{dt} + 4f(t) = e^{-2t} u_s(t)$ Assume zero initial conditions.

$$\mathcal{L} \left[\frac{df(t)}{dt} \right] = sF(s) - f(0)$$

From: $\mathcal{L} \left[\frac{d^n f(t)}{dt^n} \right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \dots - f^{(n-1)}(0)$

$$\mathcal{L} \left\{ \frac{d^2 f(t)}{dt^2} \right\} = s^2 F(s) - sf(0) - \dot{f}(0)$$

$$\mathcal{L} \left\{ \frac{df(t)}{dt} \right\} = sF(s) - f(0)$$

$$\mathcal{L} \{ e^{-2t} \} = \frac{1}{s+2}$$

$$(s^2 + 5s + 4)F(s) = \frac{1}{s+2}$$

$$F(s) = \frac{1}{(s+2)(s^2 + 5s + 4)} = \frac{1}{s^3 + 7s^2 + 14s + 8}$$

$$\mathcal{L}^{-1} [F(s)] = -1/2 e^{-2t} + 1/6 e^{-4t} + 1/3 e^{-t}$$

```
ilaplace(1/(s^3+7*s^2+14*s+8))
```

```
>>
```

```
ans = - 1/2 exp(-2 t) + 1/6 exp(-4 t) + 1/3 exp(-t)
```

$$\begin{cases} \frac{dx_1(t)}{dt} = x_2(t) \\ \frac{dx_2(t)}{dt} = -2x_1(t) - 3x_2(t) + u_s(t) \\ x_1(0) = 1, x_2(0) = 0 \end{cases}$$

(b) $x = x_1$

$$\frac{d^2x(t)}{dt^2} + 3\frac{dx(t)}{dt} + 2x(t) = u_s(t)$$

From

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$$

$$\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1}f(0) - s^{n-2}f^{(1)}(0) - \dots - f^{(n-1)}(0)$$

$$\mathcal{L}\left\{\frac{d^2x(t)}{dt^2}\right\} = s^2X(s) - sx(0) - \dot{x}(0) = s^2X(s) - 1$$

$$\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = sX(s) - x(0) = sX(s) - 1$$

$$\mathcal{L}\{u_s(t)\} = \frac{1}{s}$$

Or

$$(s^2 + 3s + 2)X(s) = \frac{1}{s}$$

$$X(s) = \frac{1}{s(s^2 + 3s + 2)}$$

$$\text{ilaplace}(1/(s^3 + 3s^2 + 2s))$$

$$\text{ans} = 1/2 + 1/2 \exp(-2 t) - \exp(-t)$$

(c)

$$\begin{cases} \frac{d^3 y(t)}{dt^3} + 2 \frac{d^2 y(t)}{dt^2} + \frac{dy(t)}{dt} + 2y(t) = -e^{-t} u_s(t) \\ \frac{d^2 y}{dt^2}(0) = -1 \quad \frac{dy}{dt}(0) = 1 \quad y(0) = 0 \end{cases}$$

From

$$\mathcal{L} \left[\frac{df(t)}{dt} \right] = sF(s) - f(0)$$

$$\mathcal{L} \left[\frac{d^n f(t)}{dt^n} \right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \dots - f^{(n-1)}(0)$$

$$\mathcal{L} \left\{ \frac{d^3 y(t)}{dt^3} \right\} = s^3 Y(s) - s^2 y(0) - s \dot{y}(0) - \ddot{y}(0)$$

$$\mathcal{L} \left\{ \frac{d^3 y(t)}{dt^3} \right\} = s^3 Y(s) - s$$

$$\mathcal{L} \left\{ \frac{d^2 y(t)}{dt^2} \right\} = s^2 Y(s) - sy(0) - \dot{y}(0)$$

$$\mathcal{L} \left\{ \frac{d^2 y(t)}{dt^2} \right\} = s^2 Y(s) - s$$

$$\mathcal{L} \left\{ \frac{dy(t)}{dt} \right\} = sY(s) - y(0)$$

$$\mathcal{L} \left\{ \frac{dy(t)}{dt} \right\} = sY(s)$$

$$\mathcal{L} \{ -e^{-t} u_s(t) \} = -\frac{1}{s+1}$$

$$s^3 Y(s) - s + 2s^2 Y(s) - 2 + sY(s) = -\frac{1}{s+1}$$

$$(s^3 + 2s^2 + s)Y(s) = -\frac{1}{s+1} + (s+2)$$

$$Y(s) = -\frac{s^2 + 3s + 2}{(s+1)(s^3 + 2s^2 + s)}$$

$$y(t) = 2 + (t^2 - 3t - 2) e^{-t}$$

```
>> ilaplace(((s^2+3*s+2))/((s+1)*(s^3+2*s^2+s)))
```

ans =

$$2 + (t^2 - 3t - 2) \exp(-t)$$

3-18. Use MATLAB to find the Laplace transform of the functions in Problem 3-17.

MATLAB code:

```
clear all;

syms t u s x1 x2 Fs

f1 = exp(-2*t)

L1=laplace(f1)/(s^2+5*s+4);

Eq2=solve('s*x1=1+x2','s*x2=-2*x1-3*x2+1','x1','x2')

f2_x1=Eq2.x1

f2_x2=Eq2.x2

f3=solve('(s^3-s+2*s^2+s+2)*Fs=-1+2-(1/(1+s))','Fs')
```

Here is the solution provided by MATLAB:

Part (a): $F(s) = 1/(s+2)/(s^2+5s+4)$

Part (b): $X_1(s) = (4+s)/(2+3s+s^2)$

$$X_2(s) = (s-2)/(2+3s+s^2)$$

Part (c): $F(s) = s/(1+s)/(s^3+2s^2+2)$

3-19. Use MATLAB to solve the following differential equation:

$$\frac{d^2 y}{dt^2} - y = e^t \quad (\text{Assuming zero initial conditions})$$

MATLAB code:

```
clear all;

syms s Fs

f3=solve('s^2*Fs-Fs=1/(s-1)','Fs')
```

Answer from MATLAB: $Y(s) = 1/(s-1)/(s^2-1)$

3-20. A series of a three-reactor tank is arranged as shown in Fig. 3P-20 for chemical reaction.

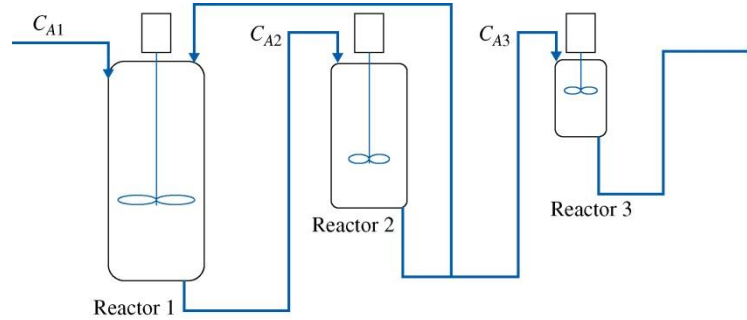


Figure 3P-20

The state equation for each reactor is defined as follows:

$$R1: \frac{dC_{A1}}{dt} = \frac{1}{V_1} [1000 + 100C_{A2} - 1100C_{A1} - k_1V_1C_{A1}]$$

$$R2: \frac{dC_{A2}}{dt} = \frac{1}{V_2} [1100C_{A1} - 1100C_{A2} - k_2V_2C_{A2}]$$

$$R3: \frac{dC_{A3}}{dt} = \frac{1}{V_3} [1000C_{A2} - 1000C_{A3} - k_3V_3C_{A3}]$$

when V_i and k_i represent the volume and the temperature constant of each tank as shown in the following table:

Reactor	V_i	k_i
1	1000	0.1
2	1500	0.2
3	100	0.4

Use MATLAB to solve the differential equations assuming $C_{A1} = C_{A2} = C_{A3} = 0$ at $t = 0$.

MATLAB code:

```
clear all;
syms s CA1 CA2 CA3
v1=1000;
v2=1500;
v3=100;
```

```

k1=0.1

k2=0.2

k3=0.4

f1='s*CA1=1/v1*(1000+100*CA2-1100*CA1-k1*v1*CA1) '
f2='s*CA2=1/v2*(1100*CA1-1100*CA2-k2*v2*CA2) '
f3='s*CA3=1/v3*(1000*CA2-1000*CA3-k3*v3*CA3) '

Sol=solve(f1,f2,f3,'CA1','CA2','CA3')

CA1=Sol.CA1

CA3=Sol.CA2

CA4=Sol.CA3

```

Solution from MATLAB:

```

CA1(s) =
1000*(s*v2+1100+k2*v2)/(1100000+s^2*v1*v2+1100*s*v1+s*v1*k2*v2+1100*s*v2+1100*k2*
v2+k1*v1*s*v2+1100*k1*v1+k1*v1*k2*v2)

CA3(s) =

1100000/(1100000+s^2*v1*v2+1100*s*v1+s*v1*k2*v2+1100*s*v2+1100*k2*v2+k1*v1*s*v2+
1100*k1*v1+k1*v1*k2*v2)

CA4 (s)=

1100000000/(1100000000+1100000*s*v3+1000*s*v1*k2*v2+1100000*s*v1+1000*k1*v1*s*v2
+1000*k1*v1*k2*v2+1100*s*v1*k3*v3+1100*s*v2*k3*v3+1100*k2*v2*s*v3+1100*k2*v2*k3*
v3+1100*k1*v1*s*v3+1100*k1*v1*k3*v3+1100000*k1*v1+1000*s^2*v1*v2+1100000*s*v2+1
100000*k2*v2+1100000*k3*v3+s^3*v1*v2*v3+1100*s^2*v1*v3+1100*s^2*v2*v3+s^2*v1*v2*
k3*v3+s^2*v1*k2*v2*v3+s*v1*k2*v2*k3*v3+k1*v1*s^2*v2*v3+k1*v1*s*v2*k3*v3+k1*v1*k2*
v2*s*v3+k1*v1*k2*v2*k3*v3)

```

Problems for Section 3-5

3-21. Fig. 3P-21 shows a simple model of a vehicle suspension system hitting a bump. If the mass of wheel and its mass moment of inertia are m and J , respectively, then:

(a) Find the equation of the motion.

- (b) Determine the transfer function of the system.
- (c) Calculate its natural frequency.
- (d) Use MATLAB to plot the step response of the system.

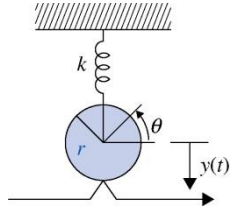


Figure 3P-21

a) Rotational kinetic energy: $T_{rot} = \frac{1}{2}J\dot{\theta}^2$

Translational kinetic energy: $T_T = \frac{1}{2}m\dot{y}^2$

Relation between translational displacement and rotational displacement:

$$y = r\theta$$

$$\dot{y} = r\dot{\theta}$$

$$T_{Rot} = \frac{1}{2} \frac{J}{r^2} \dot{y}^2$$

Potential energy: $U = \frac{1}{2}Ky^2$

From conservation of energy $T_{Rot} + T_T + U = \text{constant}$, then:

$$\frac{1}{2} \frac{J}{r^2} \dot{y}^2 + \frac{1}{2} m \dot{y}^2 + \frac{1}{2} Ky^2 = \text{constant}$$

By differentiating, we have:

$$\frac{J}{r^2} \dot{y}\ddot{y} + m\dot{y}\ddot{y} + Ky\dot{y} = 0$$

$$\dot{y} \left(\frac{J}{r^2} \ddot{y} + m\ddot{y} + Ky \right) = 0$$

Since \dot{y} cannot be zero, then $J \frac{\ddot{y}}{r^2} + m\ddot{y} + Ky = 0$

Alternatively using Newton's law, take a moment about point P , assuming motion is counterclockwise, and as the wheel goes above the bump, y is upwards. Also we assume the system starts from equilibrium (in the vertical direction) where the

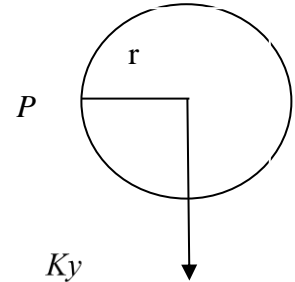
spring force and the weight of the system cancel each other. So mg does not appear in the equations. As the wheel moves up the spring compresses by y measured from the equilibrium.

Assuming positive direction is counterclockwise, we have

$$\sum Mom_p = -Kyr = J\ddot{\theta} + mr\ddot{y}$$

$$J \frac{\ddot{y}}{r^2} + m\ddot{y} + Ky = 0$$

b)



Natural frequency is the coefficient of y divided by the coefficient of \ddot{y}

$$\omega_n = \sqrt{\frac{K}{m + \frac{J}{r^2}}} = r \sqrt{\frac{K}{mr^2 + J}}$$

Time Response Solution: use a coordinate transformation where the new frame is fixed to ground on top of the bump with a height “ h ”. In that case, $x(t)=y(t)+h$. The new equation of the system then becomes:

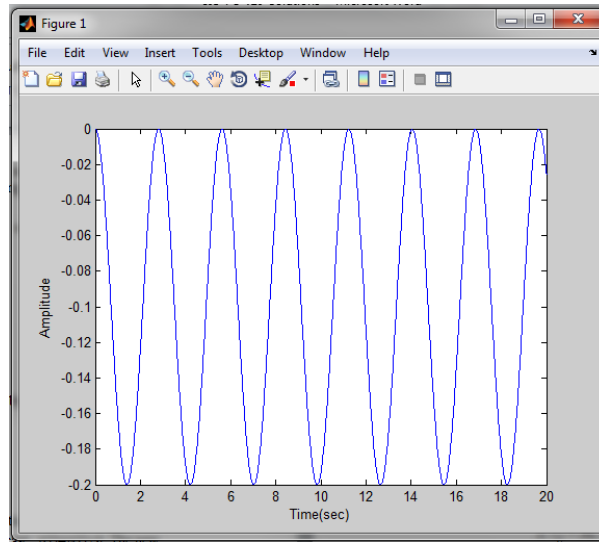
$$J \frac{\ddot{x}}{r^2} + m\ddot{x} + Kx = Kh$$

$$(d) \ G(s) = \frac{Khr^2}{(J + mr^2)s^2 + Kr^2}$$

```
% select values of m, J and K and use r= 1 and h=0.1 (arbitrary)
%Step input
K=10;
m=1;
r=1;
h=-0.1;
J=m*r^2;
ilaplace(K*h*r^2/(J+m*r^2)/(s*(s^2+K*r^2/(J+m*r^2))))
ans =
```

0.1000000000 - 0.1000000000 cos(2.236067977 t)

```
t=0:0.01:20
plot(t,-.1+.1*cos(2.236*t))
xlabel('Time(sec)');
ylabel('Amplitude');
```



3-22. An electromechanical system has the following system equations.

$$L \frac{di}{dt} + RL + K_1 \omega = e(t)$$

$$J \frac{d\omega}{dt} + B\omega - K_2 i = 0$$

For a unit-step applied voltage $e(t)$ and zero initial conditions, find responses $i(t)$ and $\omega(t)$. Assume the following parameter values:

$$L = 1 \text{ H}, J = 1 \text{ kg m}^2, B = 2 \text{ N m s}, R = 1 \text{ } \Omega, K_1 = 1 \text{ V s}, K_2 = 1 \text{ N m / A}.$$

Solution: First find the transfer functions for $i(t)$ and $\omega(t)$:

$$\mathcal{L}\left\{\frac{d\omega(t)}{dt}\right\} = s\Omega(s) - \omega(0), \mathcal{L}\left\{\frac{di(t)}{dt}\right\} = sI(s) - i(0), \mathcal{L}\left\{\frac{de(t)}{dt}\right\} = sE(s) - e(0)$$

Then for zero initial conditions we have:

$$\begin{aligned}
Ls I(s) + RL + K_1 \Omega(s) &= E(s) \\
Js\Omega(s) + B\Omega(s) - K_2 I(s) &= 0
\end{aligned}$$

$$\begin{aligned}
I(s) &= \frac{Js\Omega(s) + B\Omega(s)}{K_2} \\
\Omega(s) &= K_2 \frac{RL + E(s)}{JLs^2 + BLs + K_2 K_1}
\end{aligned}$$

Insert parameter values

$$\begin{aligned}
\Omega(s) &= \frac{1 + E(s)}{s^2 + 2s + 1} \\
\Omega(s) &= \frac{1}{s^2 + 2s + 1} + \frac{1}{s} \frac{1}{s^2 + 2s + 1} \\
I(s) &= (s + 2) \left(\frac{1}{s^2 + 2s + 1} + \frac{1}{s} \frac{1}{s^2 + 2s + 1} \right)
\end{aligned}$$

We can find the time responses through inverse Laplace transforms (e.g. Toolbox 3-4-2), or more easily by using MATLAB simulation (similar to Toolbox 3-4-4).

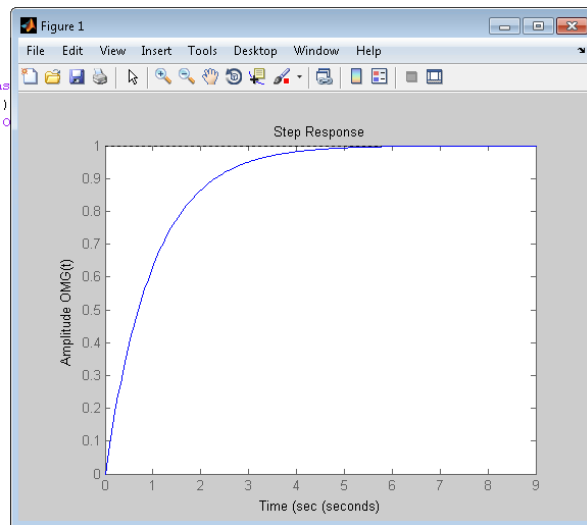
MATLAB code:

$$\Omega(s) = \frac{1}{s^2 + 2s + 1} + \frac{1}{s} \frac{1}{s^2 + 2s + 1} = \frac{1}{s} \frac{s + 1}{s^2 + 2s + 1}$$

```

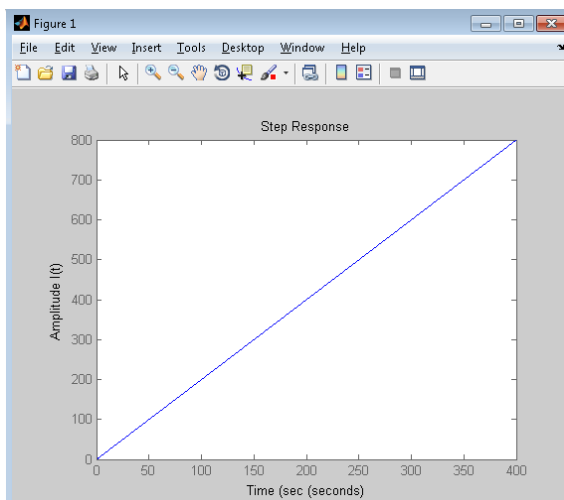
num = [1 1];
den = [1 2 1];
G = tf (num,den);
step(G);
title ('Step Response')
xlabel ('Time (sec)')
ylabel ('Amplitude OMG(t)')

```



$$I(s) = (s+2) \left(\frac{1}{s^2+2s+1} + \frac{1}{s} \frac{1}{s^2+2s+1} \right) = \frac{1}{s} \left(\frac{s^2+3s+2}{s^3+2s^2+s} \right) = \frac{1}{s} \left(\frac{s+2}{s^2+s} \right)$$

```
num = [1 2];
den = [1 1 0];
G = tf (num,den);
step(G);
title ('Step Response')
xlabel ('Time (sec)')
ylabel ('Amplitude I(t)')
```



3-23. Consider the two-degree-of-freedom mechanical system shown in Fig. 3P-23, subjected to two applied forces, $f_1(t)$ and $f_2(t)$, and zero initial conditions. Determine system responses $x_1(t)$ and $x_2(t)$ when

- (a) $f_1(t) = 0, f_2(t) = u_s(t)$
(b) $f_1(t) = u_s(t), f_2(t) = u_s(t)$.

Use the following parameter values:

$$m_1 = m_2 = 1 \text{ kg}, b_1 = 2 \text{ Ns/m}, b_2 = 1 \text{ Ns/m}, k_1 = k_2 = 1 \text{ N/m}.$$

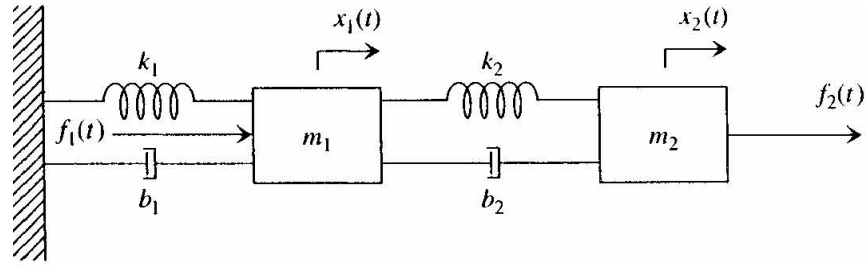
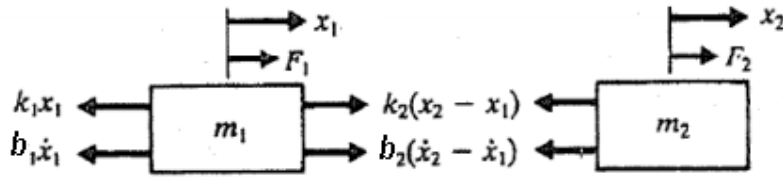


Figure 3P-23

Starting with the Free Body Diagram, we have



The equations of motion using the Newton's Law become:

$$m_1 \ddot{x}_1 = f_1(t) - k_1 x_1 - b_1 \dot{x}_1$$

$$m_2 \ddot{x}_2 = f_2(t) - b_2 (\dot{x}_2 - \dot{x}_1) - k_2 (x_2 - x_1)$$

$$m_1 \ddot{x}_1 + (b_1 + b_2) \dot{x}_1 - b_2 \dot{x}_2 + (k_1 + k_2) x_1 - k_2 x_2 = f_1(t)$$

$$m_2 \ddot{x}_2 - b_2 \dot{x}_1 + b_2 \dot{x}_2 - k_2 x_1 + k_2 x_2 = f_2(t)$$

In Laplace domain with zero initial conditions we get:

$$\mathcal{L} \left\{ \frac{d^2 x_i(t)}{dt^2} \right\} = s^2 X_i(s), \mathcal{L} \left\{ \frac{dx_i(t)}{dt} \right\} = s X_i(s)$$

$$\mathcal{L} \left\{ \frac{df_i(t)}{dt^2} \right\} = s F_i(s)$$

$$i = 1, 2$$

$$m_1 s^2 X_1 + (b_1 + b_2) s X_1 - b_2 s X_2 + (k_1 + k_2) X_1 - k_2 X_2 = F_1(t)$$

$$m_2 s^2 X_2 - b_2 s X_1 + b_2 s X_2 - k_2 X_1 + k_2 X_2 = F_2(t)$$

Solve for X_1 and X_2 transfer functions. In matrix form:

$$\begin{bmatrix} m_1 s^2 + (b_1 + b_2) s + (k_1 + k_2) & -b_2 s - k_2 \\ -b_2 s - k_2 & m_2 s^2 + b_2 s + k_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$\begin{bmatrix} s^2 + 4s + 2 & -2s - 1 \\ -2s - 1 & s^2 + 2s + 2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

Pre-multiply by the inverse

$$A = \begin{bmatrix} s^2 + 4s + 2 & -2s - 1 \\ -2s - 1 & s^2 + 2s + 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{(s^2 + 4s + 2)(s^2 + 2s + 2) - (-2s - 1)^2} \begin{bmatrix} s^2 + 2s + 2 & 2s + 1 \\ 2s + 1 & s^2 + 4s + 2 \end{bmatrix}$$

Solving for $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ we get

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = A^{-1} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \frac{1}{(s^2 + 4s + 2)(s^2 + 2s + 2) - (2s + 1)^2} \begin{bmatrix} s^2 + 2s + 2 & 2s + 1 \\ 2s + 1 & s^2 + 4s + 2 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \frac{\begin{bmatrix} (s^2 + 2s + 2)F_1 - (2s + 1)F_2 \\ (2s + 1)F_1 + (s^2 + 4s + 2)F_2 \end{bmatrix}}{(s^2 + 4s + 2)(s^2 + 2s + 2) - (2s + 1)^2}$$

(a)

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \frac{\begin{bmatrix} \frac{-(2s + 1)}{s} \\ \frac{+(s^2 + 4s + 2)}{s} \end{bmatrix}}{(s^2 + 4s + 2)(s^2 + 2s + 2) - (2s + 1)^2} = \frac{\begin{bmatrix} \frac{-(2s + 1)}{s} \\ \frac{+(s^2 + 4s + 2)}{s} \end{bmatrix}}{s^4 + 6s^3 + 8s^2 + 8s + 1} = \frac{1}{s} \frac{\begin{bmatrix} -(2s + 1) \\ (s^2 + 4s + 2) \end{bmatrix}}{s^4 + 6s^3 + 8s^2 + 8s + 1}$$

We can use MATLAB to find the time responses.

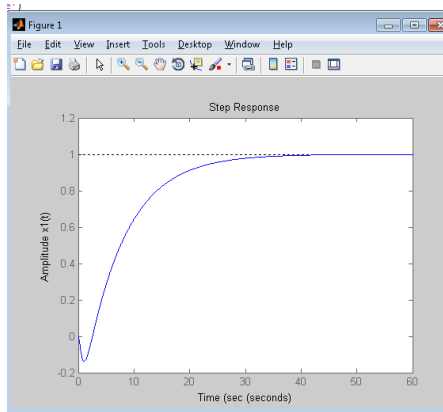
x_1 :

```
num = [-2 1];
den = [1 6 8 1];
```

```

G = tf (num,den);
step(G);
title ('Step Response')
xlabel ('Time (sec)')
ylabel ('Amplitude x1(t)')

```

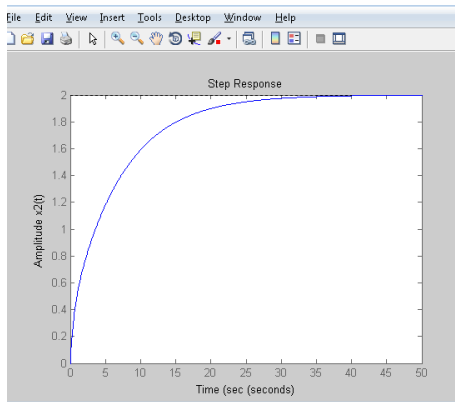


x_2 :

```

num = [1 4 2];
den = [1 6 8 1];
G = tf (num,den);
step(G);
title ('Step Response')
xlabel ('Time (sec)')
ylabel ('Amplitude x2(t)')

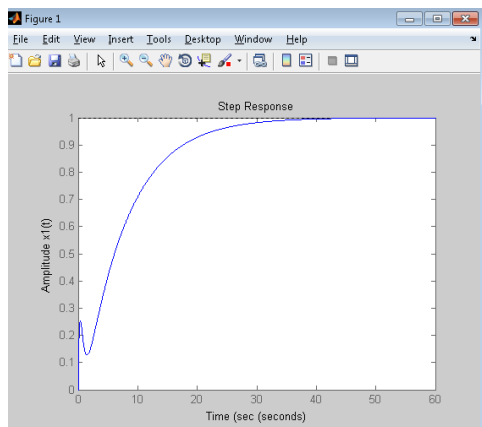
```



$$(b) \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \frac{\begin{bmatrix} \frac{(s^2 + 2s + 2) - (2s + 1)}{s} \\ \frac{(2s + 1) + (s^2 + 4s + 2)}{s} \end{bmatrix}}{s^4 + 6s^3 + 8s^2 + 8s + 1} = \frac{\begin{bmatrix} \frac{s^2 + 1}{s} \\ \frac{s^2 + 6s + 3}{s} \end{bmatrix}}{s^4 + 6s^3 + 8s^2 + 8s + 1}$$

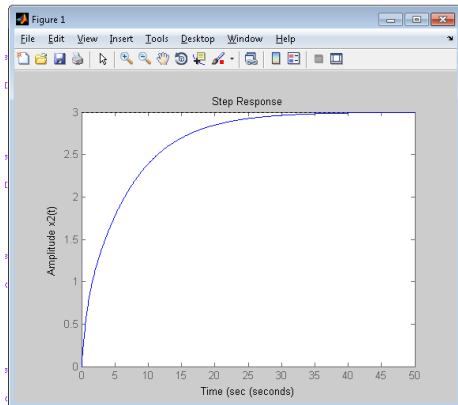
x_1 :

```
num = [2 0 1];
den = [1 6 8 1];
G = tf (num,den);
step(G);
title ('Step Response')
xlabel ('Time (sec)')
ylabel ('Amplitude x1(t)')
```



x_2 :

```
num = [1 6 3];
den = [1 6 8 1];
G = tf (num,den);
step(G);
title ('Step Response')
xlabel ('Time (sec)')
ylabel ('Amplitude x2(t)')
```



Problems for Sections 3-6 and 3-7

3-24. Express the following set of first-order differential equations in the vector-

matrix form of $\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$.

(a)
$$\begin{aligned}\frac{dx_1(t)}{dt} &= -x_1(t) + 2x_2(t) \\ \frac{dx_2(t)}{dt} &= -2x_2(t) + 3x_3(t) + u_1(t) \\ \frac{dx_3(t)}{dt} &= -x_1(t) - 3x_2(t) - x_3(t) + u_2(t)\end{aligned}$$

(b)
$$\begin{aligned}\frac{dx_1(t)}{dt} &= -x_1(t) + 2x_2(t) + 2u_1(t) \\ \frac{dx_2(t)}{dt} &= 2x_1(t) - x_3(t) + u_2(t) \\ \frac{dx_3(t)}{dt} &= 3x_1(t) - 4x_2(t) - x_3(t)\end{aligned}$$

Solution:

a)

$$\mathbf{A} = \begin{bmatrix} -1 & 2 & 0 \\ 0 & -2 & 3 \\ -1 & -3 & -1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

b)

$$\begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \\ \frac{dx_3(t)}{dt} \end{bmatrix} = \begin{bmatrix} -1 & 2 & 0 \\ 2 & 0 & -1 \\ 3 & -4 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

3-25. Given the state equation of the system, convert it to the set of first-order differential equation.

(a) $A = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 1 \\ -1 & -2 & 1 \end{bmatrix} B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$

(b) $A = \begin{bmatrix} 3 & 1 & -2 \\ -1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} B = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

Solution:

a)
$$\frac{dx(t)}{dt} = Ax(t) + Bu(t)$$

b)
$$\begin{cases} \frac{dx_1(t)}{dt} = -x_2(t) + 2x_3(t) - u_2(t) \\ \frac{dx_2(t)}{dt} = x_1(t) + x_3(t) + u_1(t) \\ \frac{dx_3(t)}{dt} = -x_1(t) - 2x_2(t) + x_3(t) \end{cases}$$

$$\begin{cases} \frac{dx_1(t)}{dt} = 3x_1(t) + x_2(t) - 2x_3(t) - u(t) \\ \frac{dx_2(t)}{dt} = -x_1(t) + 2x_2(t) + 2x_3(t) \\ \frac{dx_3(t)}{dt} = x_3(t) + 2u(t) \end{cases}$$

3-26. Consider a train consisting of an engine and a car, as shown in Fig. 4P-6.

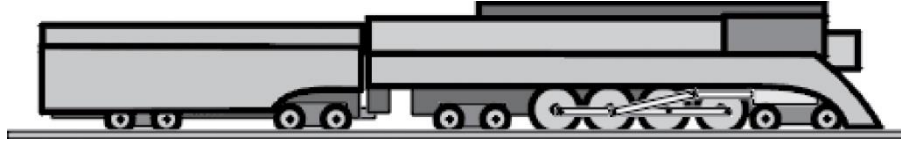


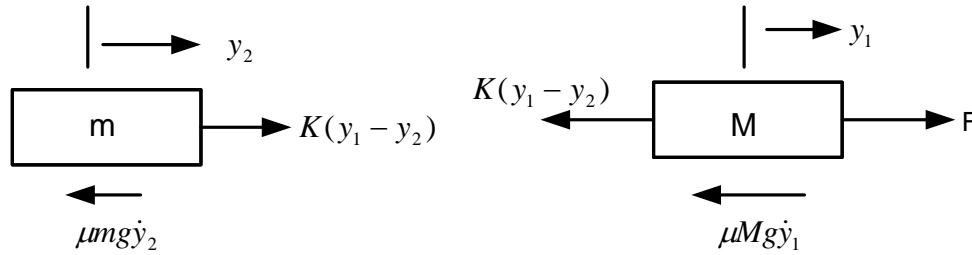
Figure 3P-26

A controller is applied to the train so that it has a smooth start and stop, along with a constant-speed ride. The mass of the engine and the car are M and m , respectively. The two are held together by a spring with the stiffness coefficient of K . F represents the force applied by the engine, and μ represents the coefficient of rolling friction. If the train only travels in one direction:

- Draw the free-body diagram.
- Find the state variables and output equations.
- Find the transfer function.
- Write the state-space of the system.

Solution:

a)



b) From Newton's Law:

$$M\ddot{y}_1 = F - K(y_1 - y_2) - \mu M g \dot{y}_1$$

$$m\ddot{y}_2 = K(y_1 - y_2) - \mu m g \dot{y}_2$$

If y_1 and y_2 are considered as a position and v_1 and v_2 as velocity variables

$$\text{Then: } \begin{cases} \dot{y}_1 = v_1 \\ \dot{y}_2 = v_2 \\ M\dot{v}_1 = F - K(y_1 - y_2) - \mu M g v_1 \\ m\dot{v}_2 = K(y_1 - y_2) - \mu m g v_2 \end{cases}$$

The output equation can be the velocity of the engine, which means $z = v_2$

c)

$$\begin{cases} Ms^2Y_1(s) = F - K(Y_1(s) - Y_2(s)) - \mu M g s Y_1(s) \\ ms^2Y_2(s) = K(Y_1(s) - Y_2(s)) - \mu m g s Y_2(s) \\ Z(s) = V_2(s) = sY_2(s) \end{cases}$$

Obtaining $\frac{Z(s)}{F(s)}$ requires solving above equation with respect to $Y_2(s)$

From the first equation:

$$(Ms^2 + K + \mu Mgs)Y_1(s) = F + KY_2(s)$$

$$Y_1(s) = \frac{F + KY_2(s)}{Ms^2 + \mu Mgs + K}$$

Substituting into the second equation:

$$ms^2Y_2(s) = \frac{KF + K^2Y_2(s)}{Ms^2 + \mu Mgs + K} - KY_2(s) - \mu mgsY_2(s)$$

By solving above equation:

$$\begin{aligned} \frac{Z(s)}{F(s)} &= \frac{sY_2(s)}{F(s)} \\ &= \frac{ms^2 + m\mu gs + 1}{Mms^3 + (2Mm\mu g)s^2 + (Mk + Mm(\mu g)^2 + mk)s + K\mu g(M + m)} \end{aligned}$$

c) If y_1 and y_2 are considered as a position and v_1 and v_2 as velocity variables

$$\text{Then: } \begin{cases} \dot{y}_1 = v_1 \\ \dot{y}_2 = v_2 \\ M\dot{v}_1 = F - K(y_1 - y_2) - \mu Mgv_1 \\ m\dot{v}_2 = F - K(y_1 - y_2) - \mu mgv_2 \end{cases}$$

The output equation can be the velocity of the engine, which means $z = v_2$

$$\begin{aligned} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{K}{m} & \frac{K}{M} & -\mu g & 0 \\ \frac{K}{m} & -\frac{K}{M} & 0 & -\mu g \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M} \\ 0 \end{bmatrix} F \\ Z &= [0 \ 0 \ 0 \ 1] \begin{bmatrix} y_1 \\ y_2 \\ v_1 \\ v_2 \end{bmatrix} + [0]F \end{aligned}$$

3-27. A vehicle towing a trailer through a spring-damper coupling hitch is shown in Fig. 3P-27. The following parameters and variables are defined: M is the mass of the trailer; K_h , the spring constant of the hitch; B_h , the viscous-damping coefficient of the hitch; B_t , the viscous-friction coefficient of the trailer; $y_1(t)$, the displacement of the towing vehicle; $y_2(t)$, the displacement of the trailer; and $f(t)$, the force of the towing vehicle.

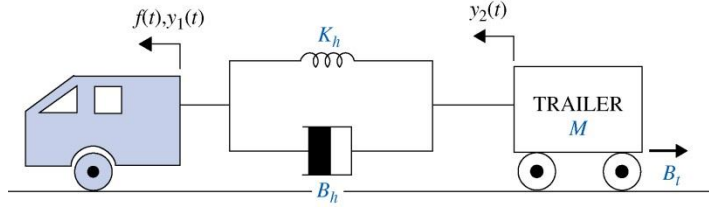


Figure 3P-27

- (a) Write the differential equation of the system.
- (b) Write the state equations by defining the following state variables:
 $x_1(t) = y_1(t) - y_2(t)$ and $x_2(t) = \frac{dy_2(t)}{dt}$.

(a) **Force equations:**

$$f(t) = K_h (y_1 - y_2) + B_h \left(\frac{dy_1}{dt} - \frac{dy_2}{dt} \right) \quad K_h (y_1 - y_2) + B_h \left(\frac{dy_1}{dt} - \frac{dy_2}{dt} \right) = M \frac{d^2 y_2}{dt^2} + B_t \frac{dy_2}{dt}$$

(b) **State variables:** $x_1 = y_1 - y_2$, $x_2 = \frac{dy_2}{dt}$

State equations:

$$\frac{dx_1}{dt} = -\frac{K_h}{B_h} x_1 + \frac{1}{B_h} f(t) \quad \frac{dx_2}{dt} = -\frac{B_t}{M} x_2 + \frac{1}{M} f(t)$$

3-28. Fig. 3P-28 shows a well-known “ball and beam” system in control systems. A ball is located on a beam to roll along the length of the beam. A lever arm is attached to the one end of the beam and a servo gear is attached to the other end of the lever arm. As the servo gear turns by an angle θ , the lever arm goes up and down, and then the angle of the beam is changed by α . The change in angle causes the ball to roll along the beam. A controller is desired to manipulate the ball's position.

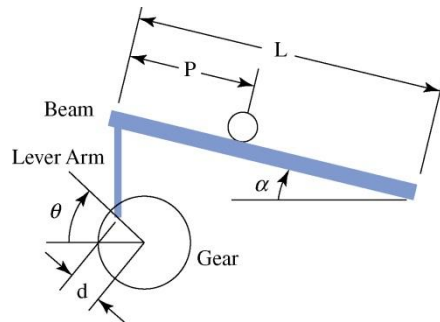


Figure 3P-28

Assuming:

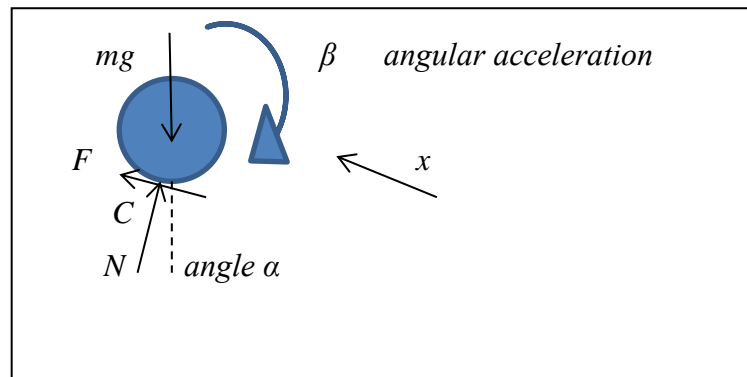
m = mass of the ball

r = radius of the ball
 d = lever arm offset
 g = gravitational acceleration
 L = length of the beam
 J = ball's moment of inertia
 p = ball position coordinate
 α = beam angle coordinate
 θ = servo gear angle

- (a) Determine the dynamic equation of the motion.
- (b) Find the transfer function.
- (c) Write the state space of the system.
- (d) Find the step response of the system by using MATLAB.

Solution:

Considering the FBD of the ball:



- a) For a given α , the acceleration at point C will have two components due to rotation of the beam; that is $a_{Cx} = -p\dot{\alpha}^2$ the centripetal and tangential $a_{Cy} = \ddot{\alpha}p$ accelerations created by rotation of the bar. Also, we assume a case of rolling without slipping. Acceleration of the center of mass of the ball relative to the

rotating axis x, y, z is $a_x = \ddot{p} - p\dot{\alpha}^2$ where $\ddot{p} = -r\beta$ (rolling without slipping and $a_y = p\ddot{\alpha} + 2\dot{\alpha}\dot{p}$

β is the angular acceleration of the ball).

Note, in the case α is fixed, then $a_x = \ddot{p} = -r\beta$, which is in line with the rolling without slipping assumption in a fixed incline case.

From the equation of motion in x direction and by taking a moment about the center of mass of the ball (see a second year dynamics of rigid bodies text in case you need to verify the following formula), we get:

$$\sum F_x = ma_x = m(\ddot{p} - p\dot{\alpha}^2) = F - mg \sin \theta$$

$$\sum M_{c.m.} = J\beta = -\frac{J\ddot{p}}{r} = rF$$

Combining the above we have

$$\left(\frac{J}{r^2} + m\right)\ddot{p} + mg \sin \alpha - mp\dot{\alpha}^2 = 0$$

b) As:

$$\alpha = \frac{d}{L} \theta$$

Then

$$\left(\frac{J}{r^2} + m\right)\ddot{p} + mg \sin \left(\frac{d\dot{\theta}}{L}\right) - mp\frac{d}{L} \dot{\theta}^2 = 0$$

If we linearize the equation about beam angle $\alpha = 0$, then $\sin \alpha \approx \alpha$ and $\sin \theta \approx \theta$

Then:

$$\begin{aligned} \left(\frac{J}{r^2} + m\right)\ddot{p} &= -mg \frac{d}{L} \theta \\ \left(\frac{J}{r^2} + m\right)s^2 P(s) &= -\frac{mgd}{L} \theta(s) \\ \frac{P(s)}{\theta(s)} &= \frac{mgd}{s^2 L \left(\frac{J}{r^2} + m\right)} \end{aligned}$$

c) Considering

$$\begin{cases} \dot{p} = q \\ \dot{q} = \ddot{p} \end{cases}$$

Then the state-space equation is described as:

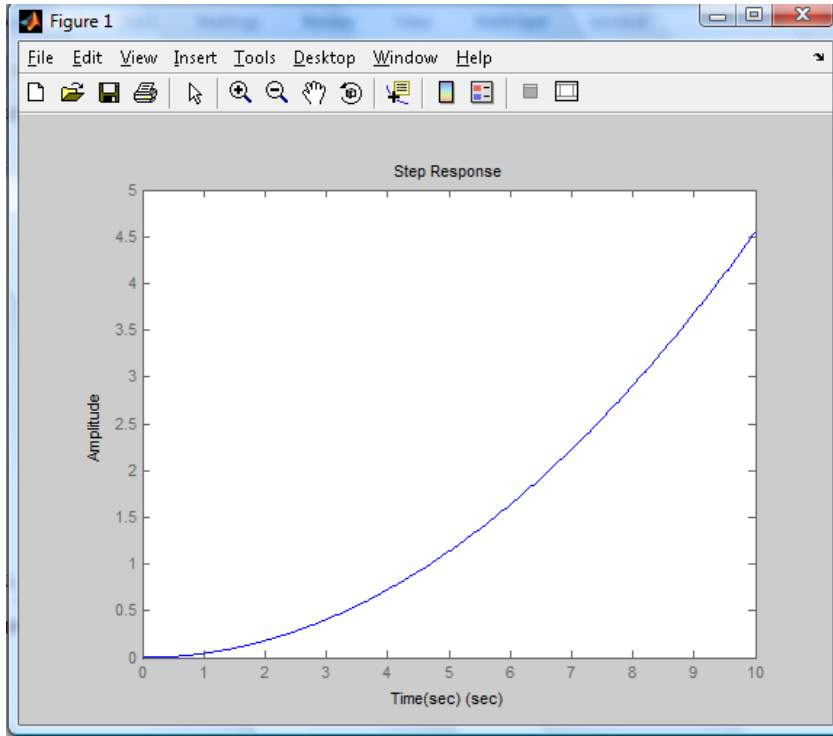
$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{mgd}{L \left(\frac{J}{r^2} + m \right)} \end{bmatrix} \theta$$

$$d) \quad G(s) = \frac{mgd}{(s^2 L (J / r^2 + m))}$$

```
clear all
% select values of m, d, r, and J
%Step input
g=10;
J=10;
M=1;
D=0.5;
R=1;
L=5;
G=tf([M*g*D],[L*(J/R^2+M) 0 0])
step(G,10)
xlabel('Time(sec)');
ylabel('Amplitude');
```

Transfer function:

```
5
-----
55 s^2
```



3-29. Find the transfer function and state-space variables in problem 2-12.

If the aircraft is at a constant altitude and velocity, and also the change in pitch angle does not change the speed, then from longitudinal equation, the motion in vertical plane can be written as:

$$\begin{cases} \dot{u} = \frac{x}{m} - g \sin \theta - q\omega \\ \dot{\omega} = \frac{z}{m} - g \cos \theta + qu \\ \dot{q} = \frac{M}{I_{yy}} \\ \dot{\theta} = q \end{cases}$$

Where u is axial velocity, ω is vertical velocity, q is pitch rate, and θ is pitch angle.

Converting the Cartesian components with polar inertial components and replace x, y, z by T, D , and L . Then we have:

$$\begin{cases} \dot{V} = \frac{1}{m} [T \cos \alpha - D - mg \sin \gamma] \\ \dot{\gamma} = \frac{1}{mV} [T \sin \alpha + L - mg \cos \gamma] \\ \dot{q} = \frac{M}{I_{yy}} \\ \dot{\theta} = q \end{cases}$$

Where $\alpha = \theta - \gamma$ is an attack angle, V is velocity, and γ is flight path angle.

It should be mentioned that T , D , L and M are function of variables α and V .

Refer to the aircraft dynamics textbooks, the state equations can be written as:

$$\begin{cases} \dot{\alpha} = A_1 \alpha + B_1 q + C_1 \gamma \\ \dot{q} = A_2 \alpha + B_2 q + C_2 \gamma \\ \dot{\theta} = A_3 q \end{cases}$$

b) The Laplace transform of the system is:

$$G(s) = \frac{\theta(s)}{\gamma(s)}$$

By using Laplace transform, we have:

$$s\alpha(s) = A_1 \alpha(s) + B_1 q(s) + C_1 \gamma(s) \quad (1)$$

$$sq(s) = A_2 \alpha(s) + B_2 q(s) + C_2 \gamma(s) \quad (2)$$

$$s\theta(s) = A_3 q(s) \quad (3)$$

From equation (1):

$$\alpha(s) = \frac{B_1}{s - A_1} q(s) + \frac{C_1}{s - A_1} \gamma(s)$$

Substituting in equation (2) and solving for $q(s)$:

$$q(s) = \frac{C_3(s - A_1) + A_2 C_1}{s(s - A_1) - B_2(s - A_1) - A_2 B_1} \gamma(s)$$

Substituting above expression in equation (3) gives:

$$\frac{\theta(s)}{\gamma(s)} = \frac{(C_2 s + A_2 C_1 - C_2 A_1) A_3}{s[s^2 - (A_1 + B_2)s - (B_2 A_1 + A_2 B_1)]}$$

If we consider $u = \omega^2 \sin \omega t$, then

$$M\ddot{y} + B\dot{y} + Ky = mlu$$

By using Laplace transform:

$$(Ms^2 + Bs + K)Y(s) = mIU(s) \quad (4)$$

Which gives:

$$\frac{Y(s)}{U(s)} = \frac{ml}{Ms^2 + Bs + K}$$

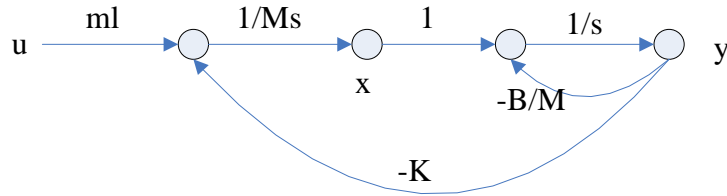
For plotting state flow diagram, equation (4) must be rewritten as:

$$\left(s + \frac{B}{M} + \frac{K}{MS}\right)Y(s) = \frac{ml}{MS}U(s)$$

or

$$\begin{cases} sY(s) = -\frac{B}{M}Y(s) + X(s) \rightarrow Y(s) = -\frac{B}{M}Y(s) + \frac{X(s)}{s} \\ X(s) = -\frac{K}{MS}Y(s) + \frac{ml}{MS}U(s) \end{cases}$$

So, the state flow diagram will plotted as:



3-30. Find the transfer function $Y(s)/T_m(s)$ in problem 2-16.

Recall from (2-16) Torque equation about the motor shaft:

Relation between linear

and rotational displacements:

$$T_m = J_m \frac{d^2\theta_m}{dt^2} + Mr^2 \frac{d^2\theta}{dt^2} + B_m \frac{d\theta_m}{dt} \quad y = r\theta_m$$

Taking the Laplace transform of the equations in part (a), with zero initial conditions, we have

$$T_m(s) = (J_m + Mr^2)s^2\Theta_m(s) + B_ms\Theta_m(s) \quad Y(s) = r\Theta_m(s)$$

Transfer function:

$$\frac{Y(s)}{T_m(s)} = \frac{r}{s[(J_m + Mr^2)s + B_m]}$$

3-31. The schematic diagram of a motor-load system is shown in Fig. 3P-31. The following parameters and variables are defined: $T_m(t)$ is the motor torque; $\omega_m(t)$, the motor velocity; $\theta_m(t)$, the motor displacement; $\omega_L(t)$, the load velocity; $\theta_L(t)$, the load displacement; K , the torsional spring constant; J_m , the motor inertia; B_m , the motor viscous-friction coefficient; and B_L , the load viscous-friction coefficient.

- (a) Write the torque equations of the system.
- (b) Find the transfer functions $\Theta_L(s)/T_m(s)$ and $\Theta_m(s)/T_m(s)$.
- (c) Find the characteristic equation of the system.
- (d) Let $T_m(t) = T_m$ be a constant applied torque; show that $\omega_m = \omega_L = \text{constant}$ in the steady state. Find the steady-state speeds ω_m and ω_L .
- (e) Repeat part (d) when the value of J_L is doubled, but J_m stays the same.

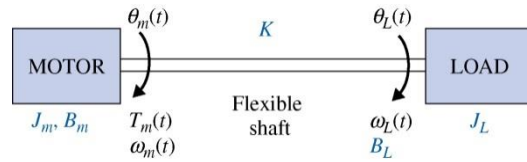
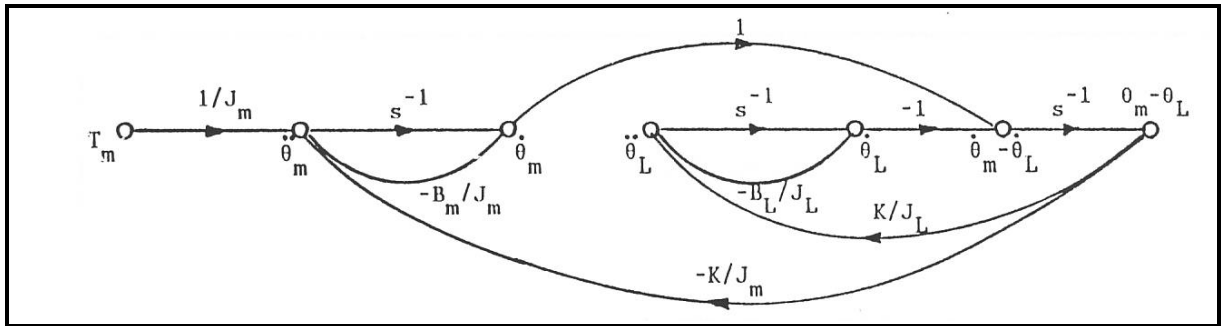


Figure 3P-31

(a) Torque equations:

$$T_m(t) = J_m \frac{d^2 \theta_m}{dt^2} + B_m \frac{d\theta_m}{dt} + K(\theta_m - \theta_L) \quad K(\theta_m - \theta_L) = J_L \frac{d^2 \theta_L}{dt^2} + B_L \frac{d\theta_L}{dt}$$

State diagram:



(b) Transfer functions:

$$\frac{\Theta_L(s)}{T_m(s)} = \frac{K}{\Delta(s)} \quad \frac{\Theta_m(s)}{T_m(s)} = \frac{J_L s^2 + B_L s + K}{\Delta(s)} \quad \Delta(s) = s \left[J_m J_L s^3 + (B_m J_L + B_L J_m) s^2 + (K J_m + K J_L + B_m B_L) s + B_m K \right]$$

(c) Characteristic equation: $\Delta(s) = 0$

(d) Steady-state performance: $T_m(t) = T_m = \text{constant}$. $T_m(s) = \frac{T_m}{s}$.

$$\lim_{t \rightarrow \infty} \omega_m(t) = \lim_{s \rightarrow 0} s \Omega_m(s) = \lim_{s \rightarrow 0} \frac{J_L s^2 + B_L s + K}{J_m J_L s^3 + (B_m J_L + B_L J_m) s^2 + (K J_m + K J_L + B_m B_L) s + B_m K} = \frac{1}{B_m}$$

Thus, in the steady state, $\omega_m = \omega_L$.

(e) The steady-state values of ω_m and ω_L do not depend on J_m and J_L .

3-32. In problem 2-20,

(a) Assume that T_s is a constant torque. Find the transfer function $\Theta(s)/\Delta(s)$, where $\Theta(s)$ and $\Delta(s)$ are the Laplace transforms of $\theta(t)$ and $\delta(t)$, respectively. Assume that $\delta(t)$ is very small.

(b) Repeat part (a) with points C and P interchanged. The d_1 in the expression of α_F should be changed to d_2 .

(Recall) Torque equation: (About the center of gravity C)

$$J \frac{d^2 \theta}{dt^2} = T_s d_2 \sin \delta + F_a d_1 \quad F_a d_1 = J_\alpha \alpha_1 = K_F d_1 \theta \quad \sin \delta \cong \delta$$

$$\text{Thus,} \quad J \frac{d^2 \theta}{dt^2} = T_s d_2 \delta + K_F d_1 \theta \quad J \frac{d^2 \theta}{dt^2} - K_F d_1 \theta = T_s d_2 \delta$$

$$\text{(a)} \quad Js^2 \Theta(s) - K_F d_1 \Theta(s) = T_s d_2 \Delta(s)$$

(b) With C and P interchanged, the torque equation about C is:

$$T_s (d_1 + d_2) \delta + F_a d_2 = J \frac{d^2 \theta}{dt^2} \quad T_s (d_1 + d_2) \delta + K_F d_2 \theta = J \frac{d^2 \theta}{dt^2}$$

$$Js^2 \Theta(s) - K_F d_2 \Theta(s) = T_s (d_1 + d_2) \Delta(s) \quad \frac{\Theta(s)}{\Delta(s)} = \frac{T_s (d_1 + d_2)}{Js^2 - K_F d_2}$$

3-33. In problem 2-21,

(a) Express the equations obtained in earlier as state equations by assigning the state variables as $x_1 = \theta, x_2 = d\theta/dt, x_3 = x$, and $x_4 = dx/dt$. Simplify these equations for small θ by making the approximations $\sin \theta \cong \theta$ and $\cos \theta \cong 1$.

(b) Obtain a small-signal linearized state-equation model for the system in the form of

$$\frac{d\Delta \mathbf{x}(t)}{dt} = \mathbf{A}^* \Delta \mathbf{x}(t) + \mathbf{B}^* \Delta \mathbf{r}(t)$$

at the equilibrium point $x_{01}(t) = 1, x_{02}(t) = 0, x_{03}(t) = 0$, and $x_{04}(t) = 0$.

(Recall) Nonlinear differential equations:

$$\frac{dx(t)}{dt} = v(t) \quad \frac{dv(t)}{dt} = -k(v) - g(x) + f(t) = -Bv(t) + f(t)$$

With $R_a = 0$,

$$\phi(t) = \frac{e(t)}{K_b v(t)} = K_f i_f(t) = K_f i_f(t) = K_f i_a(t) \quad \text{Then, } i_a(t) = \frac{e(t)}{K_b K_f v(t)}$$

$$f(t) = K_i \phi(t) i_a(t) = \frac{K_i e^2(t)}{K_b^2 K_f v^2(t)}. \quad \text{Thus, } \frac{dv(t)}{dt} = -Bv(t) + \frac{K_i}{K_b^2 K_f v^2(t)} e^2(t)$$

(a) State equations: $i_a(t)$ as input.

$$\frac{dx(t)}{dt} = v(t) \quad \frac{dv(t)}{dt} = -Bv(t) + K_i K_f i_a^2(t)$$

(b) State equations: $\phi(t)$ as input.

$$f(t) = K_i K_f i_a^2(t) \quad i_a(t) = i_f(t) = \frac{\phi(t)}{K_f}$$

$$\frac{dx(t)}{dt} = v(t) \quad \frac{dv(t)}{dt} = -Bv(t) + \frac{K_i}{K_f} \phi^2(t)$$

3-34. Vibration absorbers are used to protect machines that work at the constant speed from steady-state harmonic disturbance. Fig. 3P-34 shows a simple vibration absorber.

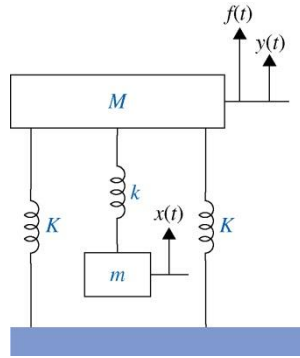


Figure 3P-34

Assuming the harmonic force $F(t) = A \sin(\omega t)$ is the disturbance applied to the mass M :

(a) Derive the state space of the system.

(b) Determine the transfer function of the system.

Solutions:

- a) Assuming the harmonic force $F(t) = A \sin(\omega t)$ is the disturbance applied to the mass M , derive the equations of motion of the system.

summation of vertical forces gives:

$$M\ddot{y} = f(t) - Ky - k(y - x) - Ky$$

$$m\ddot{x} = k(y - x)$$

$$M\ddot{y} + (2K + k)y - kx = f(t)$$

$$m\ddot{x} - ky + kx = 0$$

Where $f(t) = A \sin(\omega t)$

If we consider $\dot{y} = q$ and $\dot{x} = r$, then:

$$\begin{cases} M\dot{q} + (2K + k)y - kx = F \\ m\dot{r} - ky + kx = 0 \end{cases}$$

The state-space model is:

$$\begin{bmatrix} \dot{y} \\ \dot{x} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-2K - k}{M} & k & 0 & 0 \\ \frac{k}{m} & k & 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ x \\ q \\ r \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M} \\ 0 \end{bmatrix} F$$

b) $G(s) = \frac{Y(s)}{X(s)}$

By applying Laplace transform for equations (1) and (2), we obtain:

$$\begin{cases} [Ms^2 + (2K + k)]Y(s) - kX(s) = F(s) \\ (ms^2 + k)X(s) = kY(s) \end{cases}$$

Which gives:

$$X(s) = \frac{k}{ms^2 + k} Y(s)$$

and

$$\left[Ms^2 + (2K + k) - \frac{k^2}{ms^2 + k} \right] Y(s) = F(s)$$

Therefore:

$$\frac{Y(s)}{F(s)} = \frac{ms^2 + k}{Mms^4 + (Mk + 2Km + mk)s^2 + 2Kk}$$

3-35. Fig. 3P-35 represents a damping in the vibration absorption.

Assuming the harmonic force $F(t) = A \sin(\omega t)$ is the disturbance applied to the mass M :

- Derive the state space of the system.
- Determine the transfer function of the system.

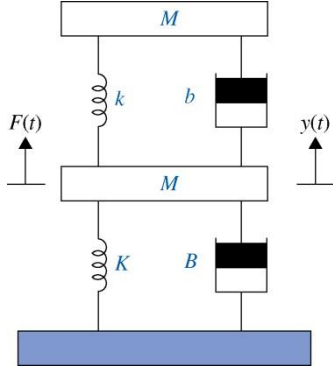


Figure 3P-35

a) Summation of vertical forces gives:

$$\begin{cases} M\ddot{y} + (B + b)\dot{y} - b\dot{x} + (K + k)y - kx = F \\ m\ddot{x} - b\dot{y} + b\dot{x} - ky - kx = 0 \end{cases}$$

Consider $\dot{y} = q$ and $\dot{x} = r$, then

$$\begin{cases} M\dot{q} + (B + b)q - br + (K + k)y - kx = F \\ m\dot{r} - bq + br - ky - kx = 0 \end{cases}$$

So, the state-space model of the system is:

$$\begin{bmatrix} \dot{y} \\ \dot{x} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-(K + k)}{M} & \frac{Kk}{M} & -\frac{B + b}{M} & \frac{b}{M} \\ \frac{k}{M} & \frac{k}{m} & \frac{b}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} y \\ x \\ q \\ r \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M} \\ 0 \end{bmatrix} F$$

b) The Laplace transform of the system is defined by:

$$G(s) = \frac{Y(s)}{X(s)}$$

where

$$\begin{cases} (Ms^2 + (B + b)s + (K + k))Y(s) - (bs + K)X(s) = F(s) \\ (ms^2 + bs - k)X(s) = (bs + k)Y(s) \end{cases}$$

as a result:

$$X(s) = \frac{bs + k}{ms^2 + bs - k} Y(s)$$

Substituting into above equation:

$$\begin{aligned} & [(Ms^2 + (B + b)s + (K + k))(ms^2 + bs + k) - (bs + k)^2]Y(s) \\ & = (ms^2 + bs - k)F(s) \end{aligned}$$

$$\frac{Y(s)}{X(s)} = \frac{ms^2 + bs - k}{[Ms^2 + (B + b)s + (K + k)][ms^2 + bs - k] - (bs + k)^2}$$

3-36. Consider the electrical circuits shown in Figs. 3P-36(a) and (b).

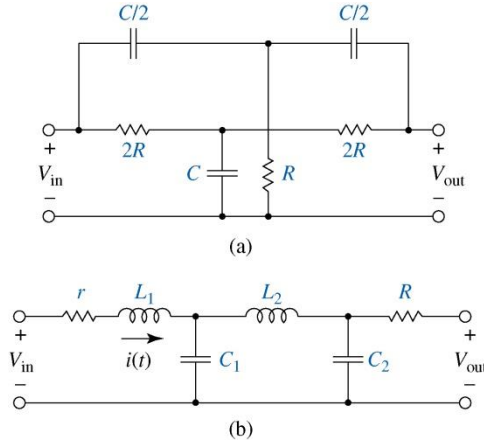


Figure 3P-36

For each circuit:

- Find the dynamic equations and state variables.
- Determine the transfer function.
- Use MATAB to plot the step response of the system.

a) According to the circuit:

$$\begin{cases} \frac{v_{in} - v_1}{2R} + C \frac{d}{dt} v_1 + \frac{v_{out} - v_1}{2R} = 0 \\ \frac{C}{2} \frac{d}{dt} (v_{in} - v_2) - \frac{v_2}{R} + \frac{C}{2} \frac{d}{dt} (v_{out} - v_2) = 0 \\ \frac{C}{2} \frac{d}{dt} (v_2 - v_{out}) + \frac{v_1 - v_{out}}{2R} = 0 \end{cases}$$

By using Laplace transform we have:

$$\begin{cases} \frac{V_{in}(s) - V_1(s)}{2R} + CsV_1(s) + \frac{V_{out}(s) - V_1(s)}{2R} = 0 \\ \frac{Cs}{2}(V_{in}(s) - V_2(s)) - \frac{V_2(s)}{R} + \frac{Cs}{2}(V_{out}(s) - V_2(s)) = 0 \\ \frac{Cs}{2}(V_2(s) - V_{out}(s)) + \frac{V_1(s) - V_{out}(s)}{2R} = 0 \end{cases}$$

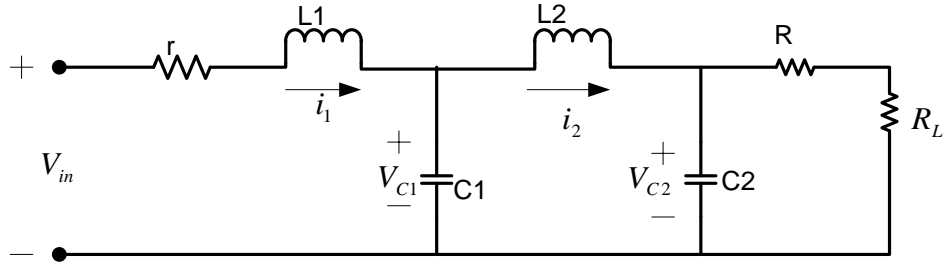
From above equations:

$$\begin{cases} V_1(s) = \frac{1}{2(RCs + 1)}(V_{in}(s) + V_{out}(s)) \\ V_2(s) = \frac{RCS}{2(RCs + 1)}(V_{in}(s) + V_{out}(s)) \end{cases}$$

Substituting $V_1(s)$ and $V_2(s)$ into preceding equations, we obtain:

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{R^2 C^2 s^2 + 1}{R^2 C^2 s^2 + 4RCs + 1}$$

b) Measuring V_{out} requires a load resistor, which means:



Then we have:

$$\begin{cases} L_1 \frac{d}{dt} i_1 = v_{in} - r i_1 - v_{C1} \\ C_1 \frac{d}{dt} v_{C1} = i_1 - i_2 \\ L_2 \frac{d}{dt} i_2 = v_{C1} - v_{C2} \\ C_2 \frac{d}{dt} v_{C2} = i_2 - \frac{v_{C2}}{R + R_L} \end{cases}$$

When

$$v_{out} = \frac{R_L}{R + R_L} v_{C2}$$

If $R_L \gg R$, then $v_{out} = v_{C2}$

By using Laplace transform we have:

$$\begin{cases} L_1 s I_1(s) = V_{in}(s) - r I_1(s) - V_{C1}(s) \\ C_1 s V_{C1}(s) = I_1(s) - I_2(s) \\ L_2 s I_2(s) = V_{C1}(s) - V_{C2}(s) \\ C_2 s V_{C2}(s) = I_2(s) - \frac{V_{C2}(s)}{R + R_L} \end{cases}$$

Therefore:

$$I_2(s) = \frac{C_2(R + R_L) + 1}{R + R_L} V_{C2}(s)$$

$$V_{C1}(s) = \frac{L_2 C_2 s(R + R_L) + s + (R + R_L)}{R + R_L} V_{C2}(s)$$

$$I_1(s) = \frac{L_2 C_2 C_1 s^2(R + R_L) + C_1 s^2 + C_1 s(R + R_L) + C_2(R + R_L) + 1}{R + R_L} V_{C2}$$

$\frac{V_{C2}(s)}{V_{in}(s)}$ can be obtained by substituting above expressions into the first equation of the state variables of the system.

3-37. The following differential equations represent linear time-invariant systems. Write the dynamic equations (state equations and output equations) in vector-matrix form.

(a) $\frac{d^2 y(t)}{dt^2} + 4 \frac{dy(t)}{dt} + y(t) = 5r(t)$

(b) $2 \frac{d^3 y(t)}{dt^3} + 3 \frac{d^2 y(t)}{dt^2} + 5 \frac{dy(t)}{dt} + 2y(t) = r(t)$

(c) $\frac{d^3 y(t)}{dt^3} + 5 \frac{d^2 y(t)}{dt^2} + 3 \frac{dy(t)}{dt} + y(t) + \int_0^t y(\tau) d\tau = r(\tau)$

(d) $\frac{d^4 y(t)}{dt^4} + 1.5 \frac{d^3 y(t)}{dt^3} + 2.5 \frac{dy(t)}{dt} + y(t) = 2r(t)$

(a) **State variables:** $x_1 = y, \quad x_2 = \frac{dy}{dt}$

State equations:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix} r$$

Output equation:

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1$$

(b) **State variables:** $x_1 = y, \quad x_2 = \frac{dy}{dt}, \quad x_3 = \frac{d^2y}{dt^2}$

State equations:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2.5 & -1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.5 \end{bmatrix} r$$

Output equation:

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1$$

(c) **State variables:** $x_1 = \int_0^t y(\tau) d\tau, \quad x_2 = \frac{dx_1}{dt}, \quad x_3 = \frac{dy}{dt}, \quad x_4 = \frac{d^2y}{dt^2}$

State equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -3 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r$$

Output equation:

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1$$

(d) **State variables:** $x_1 = y, \quad x_2 = \frac{dy}{dt}, \quad x_3 = \frac{d^2y}{dt^2}, \quad x_4 = \frac{d^3y}{dt^3}$

State equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -2.5 & 0 & -1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r$$

Output equation:

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1$$

3-38. The following transfer functions show linear time-invariant systems. Write the dynamic equations (state equations and output equations) in vector-matrix form.

(a) $G(s) = \frac{s+3}{s^2+3s+2}$

(b) $G(s) = \frac{6}{s^3+6s^2+11s+6}$

(c) $G(s) = \frac{s+2}{s^2+7s+12}$

$$(d) \quad G(s) = \frac{s^3 + 11s^2 + 35s + 250}{s^2(s^3 + 4s^2 + 39s + 108)}$$

$$a) \quad G(s) = \frac{Y(s)}{U(s)} = \frac{s+3}{s^2+3s+2}$$

$$\Rightarrow (s^2 + 3s + 2)Y(s) = (s + 3)U(s)$$

$$\Rightarrow sY(s) + 3Y(s) = -\frac{2}{s}Y(s) + \frac{3}{s}U(s) + V(s)$$

$$\text{Let } X(s) = -\frac{2}{s}Y(s) + \frac{3}{s}U(s)$$

$$\text{Then } \begin{cases} sY(s) = X(s) + U(s) + 3Y(s) \\ sX(s) = -2Y(s) + 3U(s) \end{cases} \Leftrightarrow \begin{cases} \dot{y} = -3y + x + u \\ \dot{x} = -2y + 3u \end{cases}$$

If $y = x_1$ and $x = x_2$, then

$$\begin{cases} \dot{x}_1 = -3x_1 + x_2 + u \\ \dot{x}_2 = -2x_1 + 3u \end{cases}$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & +1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$b) \quad G(s) = \frac{Y(s)}{U(s)} = \frac{6}{s^3+6s^2+11s+6}$$

$$\Rightarrow Y(s)(s^3 + 6s^2 + 11s + 6) = 6U(s)$$

$$\Rightarrow sY(s) + 6Y(s) = -\frac{6}{s^2}Y(s) - \frac{11}{s}Y(s) + \frac{6}{s^2}U(s)$$

Let $X(s) = -\frac{6}{s^2}Y(s) - \frac{11}{s}Y(s) + \frac{6}{s}U(s)$, therefore $sX(s) = -\frac{6}{s}Y(s) - 11Y(s) + \frac{6}{s}U(s)$ and Let $Z(s) = -\frac{6}{s}Y(s) + \frac{6}{s}U(s)$, then $sZ(s) = -6Y(s) + 6U(s)$. As a result:

$$\begin{cases} sY(s) = -6Y(s) + X(s) \\ sX(s) = -11Y(s) + Z(s) \\ sZ(s) = -6Y(s) + 6U(s) \end{cases}$$

or

$$\begin{cases} \dot{y} = -6y + x \\ \dot{x} = -11y + z \\ \dot{z} = -6y + 6u \end{cases}$$

If $y = x_1$, $x = x_2$ and $z = x_3$, then

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -6 & 1 & 0 \\ -11 & 0 & 1 \\ -6 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$c) G(s) = \frac{Y(s)}{U(s)} = \frac{s+2}{s^2+7s+12}$$

$$\Rightarrow Y(s)(s^2 + 7s + 12) = (s + 2)U(s)$$

$$\Rightarrow sY(s) = -7Y(s) - \frac{12}{s}Y(s) + U(s) + \frac{2}{s}U(s)$$

Let $sX(s) = -\frac{12}{s}Y(s) + \frac{2}{s}U(s)$, then $sX(s) = -12Y(s) + 2U(s)$. As a result:

$$\begin{cases} \dot{y} = -7y + x + u \\ \dot{x} = -12y + 2u \end{cases}$$

Let $y = x_1$ and $x = x_2$, then

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -7 & 1 \\ -12 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$d) G(s) = \frac{Y(s)}{U(s)} = \frac{s^3+11s^2+35s+250}{s^2(s^3+4s^2+39s+108)}$$

$$\Rightarrow (s^3 + 4s + 39s + 108)Y(s) = \left[s + 11 + \frac{35}{s} + \frac{250}{s^2} \right] u(s)$$

$$\Rightarrow sY(s) = -4Y(s) + \frac{39}{s}Y(s) + \frac{108}{s^2}Y(s) + \left[\frac{1}{s} + \frac{11}{s^2} + \frac{35}{s^3} + \frac{250}{s^4} \right] U(s)$$

$$\text{Let } X_2(s) = \frac{39}{s}Y(s) + \frac{108}{s^2}Y(s) + \left[\frac{1}{s} + \frac{11}{s^2} + \frac{35}{s^3} + \frac{250}{s^4} \right] U(s), \text{ then}$$

$$sX_2(s) = 39Y(s) + \frac{108}{s}Y(s) + U(s) + \left[\frac{11}{s^2} + \frac{35}{s^3} + \frac{250}{s^4} \right] U(s)$$

$$\text{Now, let } X_3(s) = \frac{108}{s}Y(s) + \frac{11}{s^2}U(s) + \frac{35}{s^3}U(s) + \frac{250}{s^4}U(s), \text{ therefore}$$

$$\begin{cases} sX_2(s) = 39Y(s) + X_3(s) + U(s) \\ sX_3(s) = 108Y(s) + \frac{11}{s}U(s) + \frac{35}{s^2}U(s) + \frac{250}{s^3}U(s) \end{cases}$$

$$\text{Let } X_4(s) = \frac{11}{s}U(s) + \frac{35}{s^2}U(s) + \frac{250}{s^3}U(s), \text{ then } sX_4(s) = 11U(s) + \frac{35}{s}U(s) + \frac{250}{s^2}U(s)$$

$$\text{Let } X_5(s) = \frac{35}{s}U(s) + \frac{250}{s^2}U(s), \text{ or } sX_5(s) = 35U(s) + \frac{250}{s}U(s)$$

$$\text{Let } X_6(s) = \frac{250}{s}U(s), \text{ then } sX_6(s) = 250U(s). \text{ If } Y(s) = X_1(s), \text{ then:}$$

$$\begin{cases} \dot{x}_1 = -4x_1 + x_2 \\ \dot{x}_2 = 39x_1 + x_2 + u \\ \dot{x}_3 = 108x_1 + x_4 \\ \dot{x}_4 = 11u + x_5 \\ \dot{x}_5 = 35u + 36x_6 \\ \dot{x}_6 = 250u \end{cases}$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} -4 & 1 & 0 & 0 & 0 & 0 \\ 39 & 0 & 1 & 0 & 0 & 0 \\ 108 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 11 \\ 35 \\ 250 \end{bmatrix} u$$

3-39. Repeat Problem 3-38 by using MATLAB.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s + 3}{s^2 + 3s + 2}$$

The state variables are defined as

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= \frac{dy(t)}{dt} \end{aligned}$$

Then the state equations are represented by the vector-matrix equation

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

where $\mathbf{x}(t)$ is the 2×1 state vector, $u(t)$ the scalar input, and

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} & \mathbf{B} &= \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ \mathbf{C} &= \begin{bmatrix} 1 & 0 \end{bmatrix} & \mathbf{D} &= 0 \end{aligned}$$

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} + \mathbf{D}$$

MATLAB

```
>> clear all
```

```
>> syms s
```

```
>> A=[0,1;-2,-3]
```

```
A =
```

```
0    1
```

```
-2   -3
```

```
>> B=[0;1]
```

```
B =
```

```
0
```

```
1
```

```
>> C=[3,1]
```

```
C =
```

```
3    1
```

```
>> s*eye(2)-A
```

```
ans =
```

```
[ s, -1]
```

```
[ 2, s+3]
```

```
>> inv(ans)
```

```
ans =
```

```
[ (s+3)/(s^2+3*s+2),  1/(s^2+3*s+2)]
```

```
[ -2/(s^2+3*s+2),  s/(s^2+3*s+2)]
```

```
>> C*ans*B
```

```
ans =
```

```
3/(s^2+3*s+2)+s/(s^2+3*s+2)
```

Use ACSYS as demonstrated in section 10-19-2

- 1) Activate MATLAB
- 2) Go to the folder containing ACSYS
- 3) Type in Acsys
- 4) Click the “Transfer Function Symbolic” pushbutton
- 5) Enter the transfer function
- 6) Use the “State Space” option as shown below:

Transfer Function Symbolic

Enter Transfer Function: _____

Enter the Numerator and Denominator of the transfer function using a vector of polynomial coefficients, or the numerator or denominator of the transfer function in symbolic form with complex variable 's'. Enter any symbolic variables in the box labeled 'Enter Symbolic Variables.'

ex: For numerator ($s^2 + 3k_p s + k_i^2$):
 enter '[1, 3*kp, ki^2]' in the Numerator box
 and 'kp ki' in the symbolic variables text box.

ex: The following are all equivalent:
 $(s^2 + 7s + 12)$
 '[1 7 12]'
 and $(s+4)(s+3)$.

Enter Symbolic Variables

Numerator

Denominator

Control Panel

Transfer Function Symbolic
 Transfer Function Symbolic
 State Space
 Inverse Laplace - ZPK

You get the next window. Enter the A,B,C, and D values.

State Space Analysis

Inputs:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -3 \end{bmatrix} \quad \begin{bmatrix} 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

State Space Representation:

$$\dot{D}x = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\begin{bmatrix} -2 & -3 \end{bmatrix} \quad \begin{bmatrix} 1 \end{bmatrix}$$

$$y = \begin{bmatrix} 3 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \end{bmatrix} u$$

Determinant of $(sI - A)$:

$$\begin{aligned} &2 \\ &s^2 + 3s + 2 \end{aligned}$$

Characteristic Equation of the Transfer Function:

$$s^2 + 3s + 2$$

The eigen values of A and poles of the Transfer Function are:

$$-1$$

$$-2$$

Inverse of (s*I-A) is:

$$\begin{bmatrix} s+3 & 1 \\ \hline 2 & 2 \end{bmatrix} \\ \begin{bmatrix} s^2 + 3s + 2 & s^2 + 3s + 2 \\ \hline 2 & s \end{bmatrix} \\ \begin{bmatrix} -\frac{1}{2} & \frac{1}{s} \end{bmatrix} \\ \begin{bmatrix} s^2 + 3s + 2 & s^2 + 3s + 2 \end{bmatrix}$$

State transition matrix (phi) of A:

$$\begin{bmatrix} 2\exp(-t) - \exp(-2t) & \exp(-t) - \exp(-2t) \\ -2\exp(-t) + 2\exp(-2t) & -\exp(-t) + 2\exp(-2t) \end{bmatrix}$$

Transfer function between u(t) and y(t) is:

$$\frac{s+3}{s^2 + 3s + 2}$$

No Initial Conditions Specified

States (X) in Laplace Domain:

$$\begin{bmatrix} 1 \\ \hline (s+2)(s+1) \end{bmatrix} \\ \begin{bmatrix} s \end{bmatrix}$$

$$\left[\frac{\quad}{(s+2)(s+1)} \right]$$

Inverse Laplace x(t):

$$\left[\exp(-t) - \exp(-2t) \right]$$

$$\left[-\exp(-t) + 2 \exp(-2t) \right]$$

Output Y(s):

$$\frac{s+3}{(s+2)(s+1)}$$

Inverse Laplace y(t):

$$2 \exp(-t) - \exp(-2t)$$

Use the same procedure for parts b, c and d.

3-40. Find the time response of the following systems:

$$(a) \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$(b) \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -0.5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} u, y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$a) \dot{x} = Ax + Bu \rightarrow sI - A = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix} \text{ and } (sI - A)^{-1} = \frac{1}{s^2+3s+2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

Therefore:

$$\Phi(t) = L^{-1}\{(sI - A)^{-1}\} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$\text{If } x(0) = 0, \text{ then } x(t) = \int_0^t \Phi(t - \tau) Bu(\tau) d\tau = \begin{bmatrix} 0.5 - e^{-t} + 0.5e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

$$b) \Phi(t) = L^{-1}\{(sI - A)^{-1}\}$$

$$= L^{-1}\left\{\frac{1}{s^2+s+0.5} \begin{bmatrix} s & -0.5 \\ 1 & s+1 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} e^{-0.5t}(\cos 0.5t - \sin 0.5t) & e^{-0.5t} \sin 0.5t \\ 2e^{-0.5t} \sin 0.5t & e^{-0.5t}(\cos 0.5t + \sin 0.5t) \end{bmatrix}$$

If $x(0) = 0$, then

$$\begin{aligned} x(t) &= A^{-1}(e^{At} - I)B = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} 0.5e^{-0.5t}(\cos 0.5t - \sin 0.5t) - 0.5 \\ e^{-0.5t} - \sin 0.5t \end{bmatrix} \\ &= \begin{bmatrix} e^{-0.5t} \sin 0.5t \\ -e^{-0.5t}(\cos 0.5t + \sin 0.5t) + 1 \end{bmatrix} \end{aligned}$$

and

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 = e^{-0.5t} \sin 0.5t$$

3-41. Given a system described by the dynamic equations:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \quad y(t) = \mathbf{C}\mathbf{x}(t)$$

$$(a) \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$(b) \quad \mathbf{A} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$(c) \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{C} = [1 \quad 1 \quad 0]$$

- (1) Find the eigenvalues of \mathbf{A} .
 - (2) Find the transfer-function relation between $\mathbf{X}(s)$ and $U(s)$.
 - (3) Find the transfer function $Y(s)/U(s)$.
- (a) (1) **Eigenvalues of A:** $2.325, -0.3376 + j0.5623, -0.3376 - j0.5623$

(2) **Transfer function relation:**

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} U(s) = \frac{1}{\Delta(s)} \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} U(s) = \frac{1}{\Delta(s)} \begin{bmatrix} s^2 + 3s + 2 & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -2s-1 & s^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} U(s) = \frac{1}{\Delta(s)} \begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix} U(s)$$

$$\Delta(s) = s^3 + 3s^2 + 2s + 1$$

(3) **Output transfer function:**

$$\frac{Y(s)}{U(s)} = \mathbf{C}(s) (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} = [1 \quad 0 \quad 0] \frac{1}{\Delta(s)} \begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix} = \frac{1}{s^3 + 3s^2 + 2s + 1}$$

USE ACSYS as illustrated in section 10-19-1

- 1) Activate MATLAB
- 2) Go to the folder containing ACSYS
- 3) Type in Acsys
- 4) Click the “State Space” pushbutton
- 5) Enter the A,B,C, and D values. Note C must be entered here and must have the same number of columns as A. We use [1,1] arbitrarily as it will not affect the eigenvalues.
- 6) Use the “Calculate/Display” menu and find the eigenvalues and other State space calculations.

State Space Tool

Calculate/Display Time Response Accessories

Block Diagram

$$\frac{dx(t)}{dt} = \begin{bmatrix} A \end{bmatrix} x(t) + \begin{bmatrix} B \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} C \end{bmatrix} x(t) + \begin{bmatrix} D \end{bmatrix} u(t)$$

Input Module

Enter coefficient Matrices.
eg. For a 3x3 identity matrix enter [1 0 0;0 1 0;0 0 1]

[1;0;1] is a 3x1 column [1 0 0] is a 1x3 row.

A
[0,1,0;0,0,1;-1,-2,-3]

B
[0;0;1]

C
[1,0,0]

D
0

Initial Conditions
0

Buttons

Reset

Close Window

The A matrix is:

Amat =

0 1 0

0 0 1

-1 -2 -3

Characteristic Polynomial:

ans =

$$s^3 + 3s^2 + 2s + 1$$

Eigenvalues of A = Diagonal Canonical Form of A is:

Abar =

$$\begin{bmatrix} -2.3247 & 0 & 0 \\ 0 & -0.3376 + 0.5623i & 0 \\ 0 & 0 & -0.3376 - 0.5623i \end{bmatrix}$$

Eigen Vectors are

T =

$$\begin{bmatrix} 0.1676 & 0.7868 & 0.7868 \\ -0.3896 & -0.2657 + 0.4424i & -0.2657 - 0.4424i \\ 0.9056 & -0.1591 - 0.2988i & -0.1591 + 0.2988i \end{bmatrix}$$

State-Space Model is:

a =

$$\begin{bmatrix} x1 & x2 & x3 \\ x1 & 0 & 1 & 0 \\ x2 & 0 & 0 & 1 \\ x3 & -1 & -2 & -3 \end{bmatrix}$$

b =

$$\begin{bmatrix} u1 \\ x1 & 0 \\ x2 & 0 \\ x3 & 1 \end{bmatrix}$$

c =

$$\begin{bmatrix} x1 & x2 & x3 \\ y1 & 1 & 0 & 0 \end{bmatrix}$$

d =

$$\begin{bmatrix} u1 \\ y1 & 0 \end{bmatrix}$$

Continuous-time model.

Characteristic Polynomial:

ans =

$$s^3 + 3s^2 + 2s + 1$$

Equivalent Transfer Function Model is:

Transfer function:

$$1.776\text{e-}015 s^2 + 6.661\text{e-}016 s + 1$$

$$s^3 + 3 s^2 + 2 s + 1$$

Pole, Zero Form:

Zero/pole/gain:

$$1.7764\text{e-}015 (s^2 + 0.375s + 5.629\text{e}014)$$

$$(s+2.325) (s^2 + 0.6753s + 0.4302)$$

The numerator is basically equal to 1

Use the same procedure for other parts.

(b) (1) Eigenvalues of A: -1, -1.

(2) Transfer function relation:

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} U(s) = \frac{1}{\Delta(s)} \begin{bmatrix} s+1 & 1 \\ 0 & s+1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} U(s) = \begin{bmatrix} \frac{1}{(s+1)^2} \\ \frac{1}{(s+1)} \end{bmatrix} U(s) \quad \Delta(s) = s^2 + 2s + 1$$

(3) Output transfer function:

$$\frac{Y(s)}{U(s)} = \mathbf{C}(s) (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{(s+1)^2} \\ \frac{1}{s+1} \end{bmatrix} = \frac{1}{(s+1)^2} + \frac{1}{s+1} = \frac{s+2}{(s+1)^2}$$

(c) (1) Eigenvalues of A: 0, -1, -1.

(2) Transfer function relation:

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}U(s) = \frac{1}{\Delta(s)} \begin{bmatrix} s^2 + 2s + 1 & s + 2 & 1 \\ 0 & s(s+2) & s \\ 0 & -s & s^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} U(s) = \frac{1}{\Delta(s)} \begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix} U(s) \quad \Delta(s) = s(s^2 + 2s + 1)$$

(3) Output transfer function:

$$\frac{Y(s)}{U(s)} = \mathbf{C}(s) (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix} = \frac{s+1}{s(s+1)^2} = \frac{1}{s(s+1)}$$

3-42. Given the dynamic equations of a time-invariant system:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \quad y(t) = \mathbf{C}\mathbf{x}(t)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$$

Find the matrices \mathbf{A}_1 and \mathbf{B}_1 so that the state equations are written as

$$\frac{d\bar{\mathbf{x}}(t)}{dt} = \mathbf{A}_1\bar{\mathbf{x}}(t) + \mathbf{B}_1u(t)$$

where

$$\bar{\mathbf{x}}(t) = \begin{bmatrix} x_1(t) \\ y(t) \\ \frac{dy(t)}{dt} \end{bmatrix}$$

We write $\frac{dy}{dt} = \frac{dx_1}{dt} + \frac{dx_2}{dt} = x_2 + x_3$ $\frac{d^2y}{dt^2} = \frac{dx_2}{dt} + \frac{dx_3}{dt} = -x_1 - 2x_2 - 2x_3 + u$

$$\frac{d\bar{\mathbf{x}}}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dy}{dt} \\ \frac{d^2y}{dt^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \quad (1)$$

$$\bar{\mathbf{x}} = \begin{bmatrix} x_1 \\ y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \mathbf{x} \quad \mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \bar{\mathbf{x}} \quad (2)$$

Substitute Eq. (2) into Eq. (1), we have

$$\frac{d\bar{\mathbf{x}}}{dt} = \mathbf{A}_1 \bar{\mathbf{x}} + \mathbf{B}_1 u = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & -2 \end{bmatrix} \bar{\mathbf{x}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

3-43. Fig. 3P-43(a) shows a well-known “broom-balancing” system in control systems. The objective of the control system is to maintain the broom in the upright position by means of the force $u(t)$ applied to the car as shown. In practical applications, the system is analogous to a one-dimensional control problem of the balancing of a unicycle or a missile immediately after launching. The free-body diagram of the system is shown in Fig. 3P-43(b), where

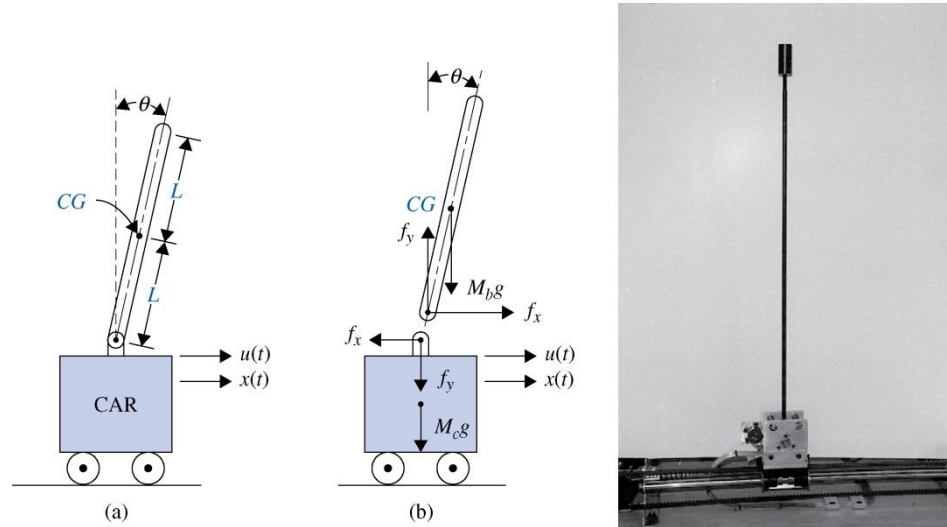


Figure 3P-43

f_x = force at broom base in horizontal direction

f_y = force at broom base in vertical direction

M_b = mass of broom

g = gravitational acceleration

M_c = mass of car

J_b = moment of inertia of broom about center of gravity $CG = M_b L_2/3$

(a) Write the force equations in the x and the y directions at the pivot point of the broom. Write the torque equation about the center of gravity CG of the broom. Write the force equation of the car in the horizontal direction.

(b) Express the equations obtained in part (a) as state equations by assigning the state variables as $x_1 = \theta, x_2 = d\theta/dt, x_3 = x$, and $x_4 = dx/dt$. Simplify these equations for small θ by making the approximations $\sin \theta \cong \theta$ and $\cos \theta \cong 1$.

(c) Obtain a small-signal linearized state-equation model for the system in the form of

$$\frac{d\Delta \mathbf{x}(t)}{dt} = \mathbf{A}^* \Delta \mathbf{x}(t) + \mathbf{B}^* \Delta \mathbf{r}(t)$$

at the equilibrium point $x_{01}(t) = 1, x_{02}(t) = 0, x_{03}(t) = 0$, and $x_{04}(t) = 0$.

(a) Nonlinear differential equations:

$$\frac{dx(t)}{dt} = v(t) \quad \frac{dv(t)}{dt} = -k(v) - g(x) + f(t) = -Bv(t) + f(t)$$

With $R_a = 0$,

$$\phi(t) = \frac{e(t)}{K_b v(t)} = K_f i_f(t) = K_f i_f(t) = K_f i_a(t) \quad \text{Then, } i_a(t) = \frac{e(t)}{K_b K_f v(t)}$$

$$f(t) = K_i \phi(t) i_a(t) = \frac{K_i e^2(t)}{K_b^2 K_f v^2(t)}. \quad \text{Thus, } \frac{dv(t)}{dt} = -Bv(t) + \frac{K_i}{K_b^2 K_f v^2(t)} e^2(t)$$

(b) State equations: $i_a(t)$ as input.

$$\frac{dx(t)}{dt} = v(t) \quad \frac{dv(t)}{dt} = -Bv(t) + K_i K_f i_a^2(t)$$

(c) State equations: $\phi(t)$ as input.

$$f(t) = K_i K_f i_a^2(t) \quad i_a(t) = i_f(t) = \frac{\phi(t)}{K_f}$$

$$\frac{dx(t)}{dt} = v(t) \quad \frac{dv(t)}{dt} = -Bv(t) + \frac{K_i}{K_f} \phi^2(t)$$

3-44. The “broom-balancing” control system described in Problem 3-43 has the following parameters:

$$M_b = 1\text{ kg} \quad M_c = 10\text{ kg} \quad L = 1\text{ m} \quad g = 32.2\text{ ft/sec}^2$$

The small-signal linearized state equation model of the system is

$$\Delta \dot{\mathbf{x}}(t) = \mathbf{A}^* \Delta \mathbf{x}(t) + \mathbf{B}^* \Delta r(t)$$

where

$$\mathbf{A}^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 25.92 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -2.36 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{B}^* = \begin{bmatrix} 0 \\ -0.0732 \\ 0 \\ 0.0976 \end{bmatrix}$$

Find the characteristic equation of \mathbf{A}^* and its roots.

3-45. Fig. 3P-45 shows the schematic diagram of a ball-suspension control system. The steel ball is suspended in the air by the electromagnetic force generated by the electromagnet. The objective of the control is to keep the metal ball suspended at the nominal equilibrium position by controlling the current in the magnet with the voltage $e(t)$. The practical application of this system is the magnetic levitation of trains or magnetic bearings in high-precision control systems. The resistance of the coil is R , and the inductance is $L(y) = L/y(t)$, where L is a constant. The applied voltage $e(t)$ is a constant with amplitude E .

(a) Let E_{eq} be a nominal value of E . Find the nominal values of $y(t)$ and $dy(t)/dt$ at equilibrium.

(b) Define the state variables at $x_1(t) = i(t)$, $x_2(t) = y(t)$, and $x_3(t) = dy(t)/dt$.

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(\mathbf{x}, e)$$

Find the nonlinear state equations in the form of

(c) Linearize the state equations about the equilibrium point and express the linearized state equations as

$$\frac{d\Delta \mathbf{x}(t)}{dt} = \mathbf{A}^* \Delta \mathbf{x}(t) + \mathbf{B}^* \Delta e(t)$$

The force generated by the electromagnet is $Ki^2(t)/y(t)$, where K is a proportional constant, and the gravitational force on the steel ball is Mg .

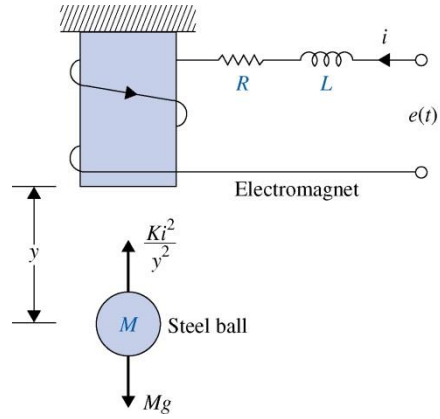


Figure 3P-45

23 (a) Differential equations: $\left[L(y) = \frac{L}{y} \right]$

$$e(t) = Ri(t) + \frac{d[L(y)i(t)]}{dt} = Ri(t) + i(t) \frac{dL(y)}{dy} \frac{dy(t)}{dt} + \frac{L}{y} \frac{di(t)}{dt} = Ri(t) - \frac{L}{y^2} i(t) \frac{dy(t)}{dt} + \frac{L}{y} \frac{di(t)}{dt}$$

$$My(t) = Mg - \frac{Ki^2(t)}{y^2(t)} \quad \text{At equilibrium, } \frac{di(t)}{dt} = 0, \quad \frac{dy(t)}{dt} = 0, \quad \frac{d^2y(t)}{dt^2} = 0$$

$$\text{Thus, } i_{eq} = \frac{E_{eq}}{R} \quad \frac{dy_{eq}}{dt} = 0 \quad y_{eq} = \frac{E_{eq}}{R} \sqrt{\frac{K}{Mg}}$$

(b) Define the state variables as $x_1 = i$, $x_2 = y$, and $x_3 = \frac{dy}{dt}$.

$$\text{Then, } x_{1eq} = \frac{E_{eq}}{R} \quad x_{2eq} = \frac{E_{eq}}{R} \sqrt{\frac{K}{Mg}} \quad x_{3eq} = 0$$

The differential equations are written in state equation form:

$$\frac{dx_1}{dt} = -\frac{R}{L} x_1 x_2 + \frac{x_1 x_3}{x_2} + \frac{x_2}{L} e = f_1 \quad \frac{dx_2}{dt} = x_3 = f_2 \quad \frac{dx_3}{dt} = g - \frac{K}{M} \frac{x_1^2}{x_2^2} = f_3$$

(c) Linearization:

$$\frac{\partial f_1}{\partial x_1} = -\frac{R}{L} x_{2eq} + \frac{x_{3eq}}{x_{2eq}} = -\frac{E_{eq}}{L} \sqrt{\frac{K}{Mg}} \quad \frac{\partial f_1}{\partial x_2} = -\frac{R}{L} x_{1eq} - \frac{x_1 x_3}{x_2^2} + \frac{E_{eq}}{L} = 0 \quad \frac{\partial f_1}{\partial x_3} = \frac{x_{1eq}}{x_{2eq}} = \sqrt{\frac{Mg}{K}}$$

$$\begin{aligned}\frac{\mathcal{F}_1}{\partial e} &= \frac{x_{2eq}}{L} = \frac{1}{L} \sqrt{\frac{K}{Mg}} \frac{E_{eq}}{R} & \frac{\mathcal{F}_2}{\partial x_1} &= 0 & \frac{\mathcal{F}_2}{\partial x_2} &= 0 & \frac{\mathcal{F}_2}{\partial x_3} &= 1 & \frac{\mathcal{F}_2}{\partial e} &= 0 \\ \frac{\mathcal{F}_3}{\partial x_1} &= -\frac{2K}{M} \frac{x_{1eq}}{x_{2eq}^2} = -\frac{2Rg}{E_{eq}} & \frac{\mathcal{F}_3}{\partial x_2} &= \frac{2K}{M} \frac{x_{1eq}^2}{x_{2eq}^3} = \frac{2Rg}{E_{eq}} \sqrt{\frac{Mg}{K}} & \frac{\mathcal{F}_3}{\partial e} &= 0\end{aligned}$$

The linearized state equations about the equilibrium point are written as:

$$\Delta \dot{\mathbf{x}} = \mathbf{A}^* \Delta \mathbf{x} + \mathbf{B}^* \Delta e$$

$$\mathbf{A}^* = \begin{bmatrix} -\frac{E_{eq}}{L} \sqrt{\frac{K}{Mg}} & 0 & \sqrt{\frac{Mg}{K}} \\ 0 & 0 & 0 \\ -\frac{2Rg}{E_{eq}} & \frac{2Rg}{E_{eq}} \sqrt{\frac{Mg}{K}} & 0 \end{bmatrix} \quad \mathbf{B}^* = \begin{bmatrix} \frac{E_{eq}}{RL} \sqrt{\frac{K}{Mg}} \\ 0 \\ 0 \end{bmatrix}$$

3-46. The linearized state equations of the ball-suspension control system described in Problem 3-45 are expressed as

$$\Delta \dot{\mathbf{x}}(t) = \mathbf{A}^* \Delta \mathbf{x}(t) + \mathbf{B}^* \Delta i(t)$$

where

$$\mathbf{A}^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 115.2 & -0.05 & -18.6 & 0 \\ 0 & 0 & 0 & 1 \\ -37.2 & 0 & 37.2 & -0.1 \end{bmatrix} \quad \mathbf{B}^* = \begin{bmatrix} 0 \\ -6.55 \\ 0 \\ -6.55 \end{bmatrix}$$

Let the control current $\Delta i(t)$ be derived from the state feedback $\Delta i(t) = -\mathbf{K} \Delta \mathbf{x}(t)$, where

$$\mathbf{K} = [k_1 \quad k_2 \quad k_3 \quad k_4]$$

(a) Find the elements of \mathbf{K} so that the eigenvalues of $\mathbf{A}^* - \mathbf{B}^* \mathbf{K}$ are at $-1 + j$, $-1 - j$, -10 , and -10 .

(b) Plot the responses of $\Delta x_1(t) = \Delta y_1(t)$ (magnet displacement) and $\Delta x_3(t) = \Delta y_2(t)$ (ball displacement) with the initial condition

$$\Delta \mathbf{x}(0) = \begin{bmatrix} 0.1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(c) Repeat part (b) with the initial condition

$$\Delta \mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 0.1 \\ 0 \end{bmatrix}$$

Comment on the responses of the closed-loop system with the two sets of initial conditions used in (b) and (c).

The solutions using MATLAB

(a) The feedback gains, from k_1 to k_2 :

−6.4840E+01 −5.6067E+00 2.0341E+01 2.2708E+00

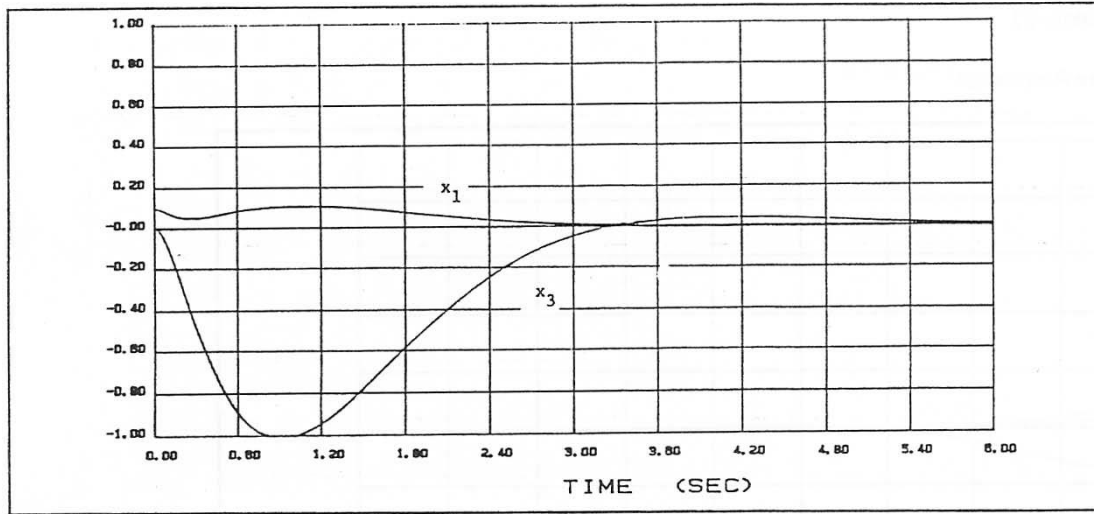
The $\mathbf{A}^* - \mathbf{B}^* \mathbf{K}$ matrix of the closed-loop system

0.0000E+00 1.0000E+00 0.0000E+00 0.0000E+00
 −3.0950E+02 −3.6774E+01 1.1463E+02 1.4874E+01
 0.0000E+00 0.0000E+00 0.0000E+00 1.0000E+00
 −4.6190E+02 −3.6724E+01 1.7043E+02 1.477eE+01

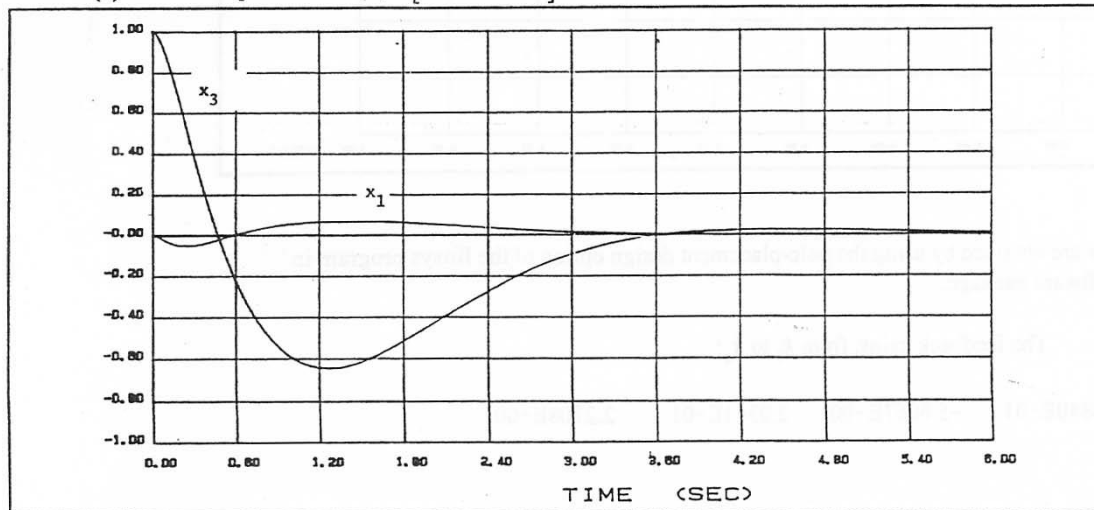
The \mathbf{B} vector

0.0000E+00
 −6.5500E+00
 0.0000E+00
 −6.5500E+00

(b) **Time Responses:** $\Delta \mathbf{x}(0) = [0.1 \ 0 \ 0 \ 0]^T$



(c) Time Responses: $\Delta x(0) = [0 \ 0 \ 0.1 \ 0]^T$

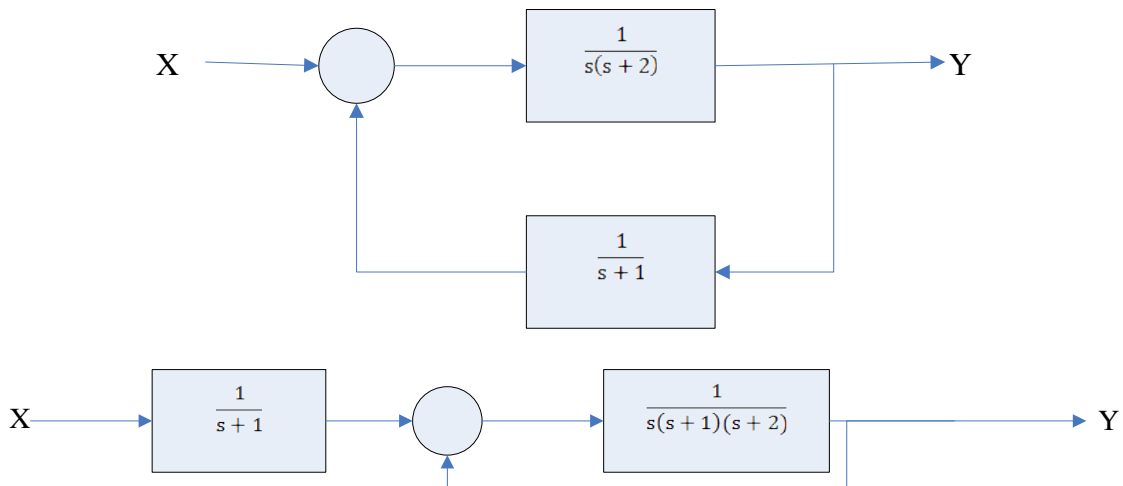


With the initial states $\Delta x(0) = [0.1 \ 0 \ 0 \ 0]^T$, the initial position of Δx_1 or Δy_1 is perturbed downward from its stable equilibrium position. The steel ball is initially pulled toward the magnet, so $\Delta x_3 = \Delta y_2$ is negative at first. Finally, the feedback control pulls both bodies back to the equilibrium position. With the initial states $\Delta x(0) = [0 \ 0 \ 0.1 \ 0]^T$, the initial position of Δx_3 or Δy_2 is perturbed downward from its stable equilibrium. For $t > 0$, the ball is going to be attracted up by the magnet toward the equilibrium position. The magnet will initially be attracted toward the fixed iron plate, and then settles to the stable equilibrium position. Since the steel ball has a small mass, it will move more actively.

Chapter 4

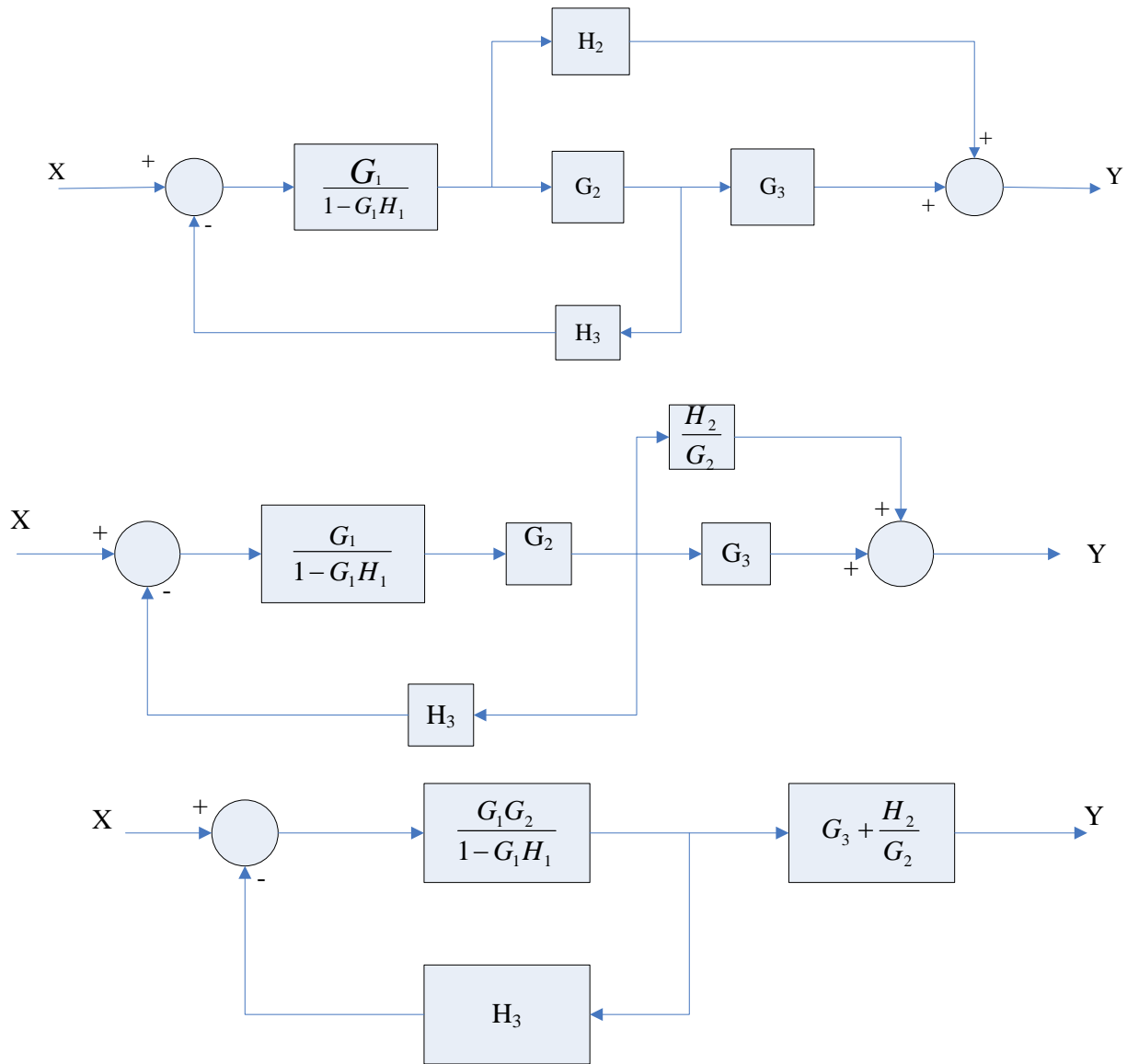
- 4-1) a) $G(s)H(s) = \left[\frac{K_p}{s(s+p)} \right] K_D s = \frac{K_p K_D}{s+p}$
- b) $G(s) = \frac{K_p}{s(s+p)}$
- c) $\frac{E(s)}{R(s)} = \frac{1}{1-G(s)H(s)} = \frac{s+p}{s+p-K_p K_D}$
- d) Feedback ratio = $\frac{G(s)H(s)}{1-G(s)H(s)} = \frac{K_p K_D}{s+p-K_p K_D}$
- e) $\frac{Y(s)}{X(s)} = \frac{G(s)}{1-G(s)H(s)} = \frac{K_p}{s(s+p-K_p K_D)}$

4-2)

Characteristic equation: $s(s+1)(s+2) + 1 = 0$

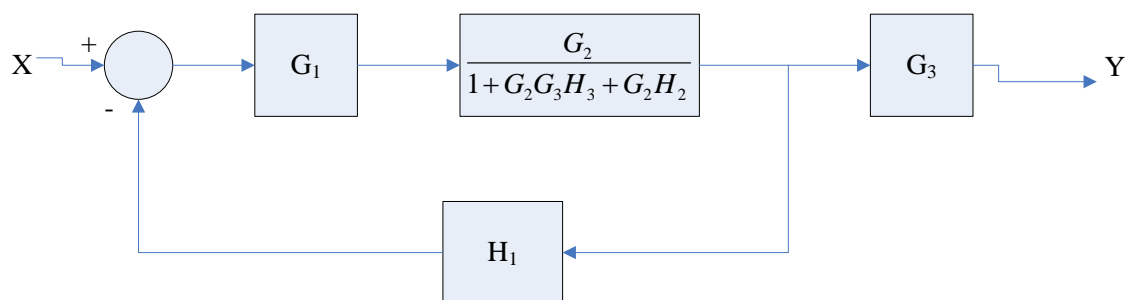
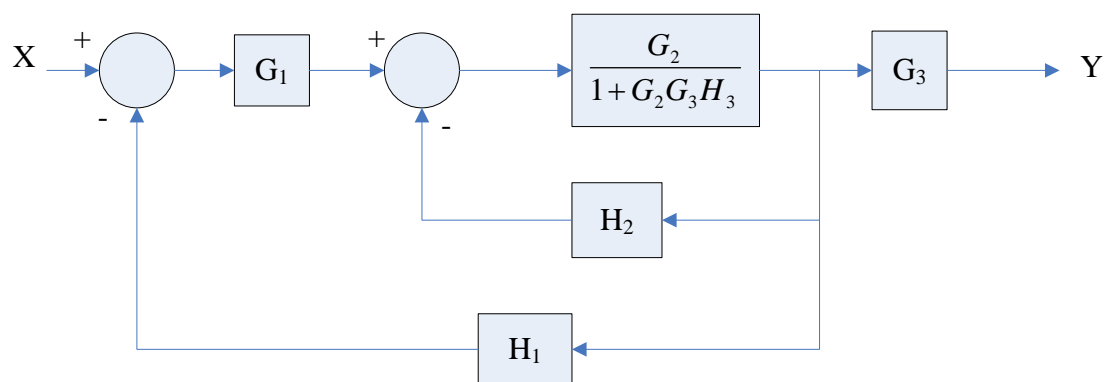
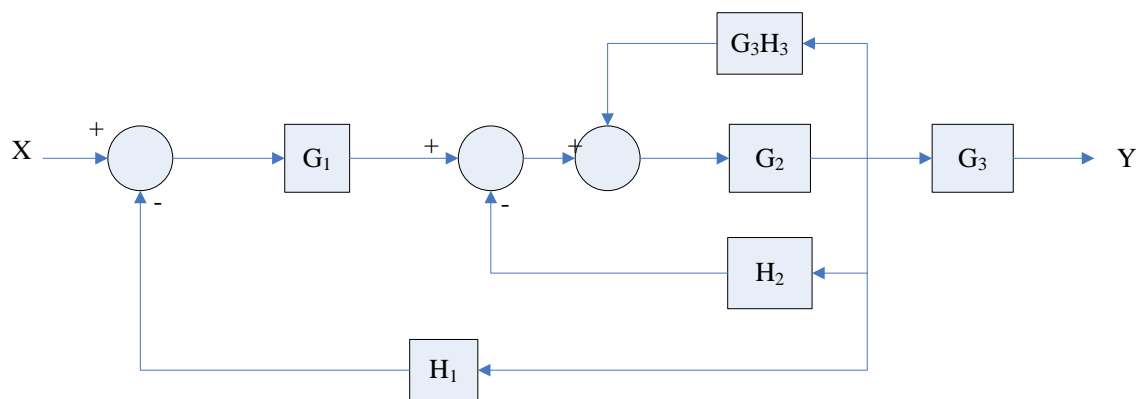
$$\Rightarrow s^3 + 3s^2 + 2s + 1 = 0$$

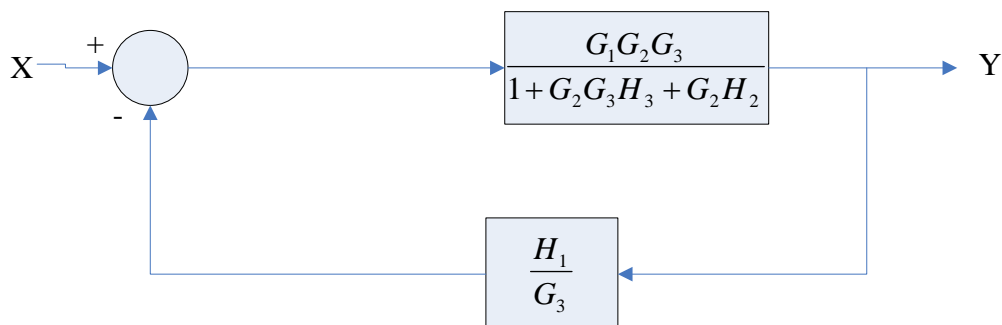
4-3)



$$\frac{Y(s)}{X(s)} = \frac{\frac{G_1G_2}{1-G_1H_1}}{1 + \frac{G_1G_2H_3}{1-G_1H_1}} \left(G_3 + \frac{H_2}{G_2} \right) = \frac{G_1G_2G_3 + G_1H_2}{1 - G_1H_1 + G_1G_2H_3}$$

4-4)





$$\frac{Y(s)}{X(s)} = \frac{G_1 G_2 G_3}{1 + G_1 G_2 H_1 + G_2 H_2 + G_2 G_3 H_3}$$

4-5)

$$\mathbf{Y}(s) = [\mathbf{I} + \mathbf{G}(s)\mathbf{H}(s)]^{-1} \mathbf{G}(s)\mathbf{R}(s) = \mathbf{M}(s)\mathbf{R}(s)$$

$$\mathbf{I} + \mathbf{G}(s)\mathbf{H}(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \frac{2}{s(s+2)} & 10 \\ \frac{5}{s} & \frac{1}{s+1} \end{bmatrix} = \begin{bmatrix} \frac{s^2+2s+2}{s(s+2)} & 10 \\ \frac{5}{s} & \frac{s+2}{s+1} \end{bmatrix}$$

$$[\mathbf{I} + \mathbf{G}(s)\mathbf{H}(s)]^{-1} = \frac{1}{\Delta} \begin{bmatrix} \frac{s+2}{s+1} & -10 \\ -5 & \frac{s^2+2s+2}{s(s+2)} \end{bmatrix} \quad \Delta(s) = \frac{s^2 - 48s - 48}{s(s+1)}$$

$$\mathbf{M}(s) = [\mathbf{I} + \mathbf{G}(s)\mathbf{H}(s)]^{-1} \mathbf{G}(s) = \frac{1}{\Delta} \begin{bmatrix} \frac{s+2}{s+1} & -10 \\ -5 & \frac{s^2+2s+2}{s(s+2)} \end{bmatrix} \begin{bmatrix} \frac{2}{s(s+2)} & 10 \\ \frac{5}{s} & \frac{1}{s+1} \end{bmatrix}$$

$$\mathbf{M}(s) = \frac{1}{\Delta} \begin{bmatrix} \frac{-50s-48}{s(s+1)} & 10 \\ \frac{5}{s} & \frac{-49s^2-148s-98}{s(s+1)(s+2)} \end{bmatrix}$$

4-6) MATLAB

```
syms s
```

```
G=[2/(s*(s+2)),10;5/s,1/(s+1)]
```

```
H=[1,0;0,1]
```

```
A=eye(2)+G*H
```

```
B=inv(A)
```

```
Clp=simplify(B*G)
```

G =

```
[ 2/s/(s+2),      10]
[      5/s,  1/(s+1)]
```

H =

```
1  0
0  1
```

A =

```
[ 1+2/s/(s+2),      10]
[      5/s,  1+1/(s+1)]
```

B =

```
[      s*(s+2)/(s^2-48*s-48),      -10/(s^2-48*s-48)*(s+1)*s]
[      -5/(s^2-48*s-48)*(s+1), (s^2+2*s+2)*(s+1)/(s+2)/(s^2-48*s-48)]
```

Clp =

```
[      -2*(24+25*s)/(s^2-48*s-48),      10/(s^2-48*s-48)*(s+1)*s]
[      5/(s^2-48*s-48)*(s+1), -(49*s^2+148*s+98)/(s+2)/(s^2-48*s-48)]
```

4-7)

(a) Open-loop transfer function:

$$\frac{\Theta_o(s)}{\Theta_e(s)} = \frac{KK_s K_1 K_i N}{s \left[L_a J_t s^2 + (L_a B_t + R_a J_t + K_1 K_2 J_t) s + R_a B_t + K_i K_b + KK_1 K_i K_t + K_1 K_2 B_t \right]}$$

(b) System transfer function:

$$\frac{\Theta_o(s)}{\Theta_r(s)} = \frac{KK_s K_1 K_i N}{\left[L_a J_t s^3 + (L_a B_t + R_a J_t + K_1 K_2 J_t) s^2 + (R_a B_t + K_i K_b + KK_1 K_i K_t + K_1 K_2 B_t) s + KK_s K_1 K_i N \right]}$$

4-8)

(a)

$$\left. \frac{Y(s)}{R(s)} \right|_{N=0} = \frac{\frac{10(s+4)}{s(s+1)}}{1 + \frac{5s}{s(s+1)} + \frac{10(s+2)}{s(s+1)}} = \frac{10(s+4)}{s^2 + 16s + 20}$$

(b)

$$\left. \frac{Y(s)}{E(s)} \right|_{N=0} = \frac{Y(s)/R(s)}{E(s)/R(s)} \bigg|_{N=0} = \frac{\frac{10(s+4)}{s(s+1)}}{1 + \frac{5s}{s(s+1)} - \frac{20}{s(s+1)}} = \frac{10(s+4)}{s^2 + 6s - 20}$$

(c)

$$\left. \frac{Y(s)}{N(s)} \right|_{R=0} = \frac{1}{1 + \frac{5s}{s(s+1)} + \frac{10(s+2)}{s(s+1)}} = \frac{s(s+1)}{s^2 + 16s + 20}$$

(d)

$$Y(s) = \left. \frac{Y(s)}{R(s)} \right|_{N=0} R(s) + \left. \frac{Y(s)}{N(s)} \right|_{R=0} N(s)$$

4-9)

(a)

$$\left. \frac{Y(s)}{R(s)} \right|_{N=0} = \frac{G_1(s)G_2(s)G_3(s) + G_4(s)}{\Delta} \quad \left. \frac{Y(s)}{N(s)} \right|_{R=0} = \frac{1 + G_1(s)G_2(s)H_1(s)}{\Delta}$$

$$\Delta = 1 + G_1(s)G_2(s)H_1(s) + G_2(s)G_3(s)H_2(s) + G_4(s) - G_2(s)G_4(s)H_1(s)H_2(s)$$

$$Y(s) = \left. \frac{Y(s)}{R(s)} \right|_{N=0} R(s) + \left. \frac{Y(s)}{N(s)} \right|_{R=0} N(s)$$

(b) When $1 + G_1(s)G_2(s)H_1(s) = 0$ $Y(s)$ is not affected by $N(s)$.

4-10)

$$\text{Set } \left. \frac{Y(s)}{N(s)} \right|_{R=0} = \frac{1 - \frac{10(s+5)}{s(s+5)(s+10)} G_d(s)}{\Delta} = 0 \quad \text{Then, } G_d(s) = \frac{s(s+10)}{10}$$

4-11)

(a)

$$\left. \frac{Y(s)}{N(s)} \right|_{R=0} = \frac{1 + G(s)H(s)}{\Delta} = 0 \quad H(s) = \frac{-1}{G(s)} = -\frac{s(s+1)(s+2)}{K(s+3)}$$

(b)

$$N = 0. \quad E(s) = \frac{R(s)}{1 + G(s) + G(s)H(s)} = \frac{R(s)}{G(s)} = \frac{s(s+1)(s+2)}{K(s+3)} R(s) \quad R(s) = \frac{1}{s^2}$$

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \frac{s(s+1)(s+2)}{Ks(s+3)} = \frac{2}{3K} = 0.1 \quad K = 6.67$$

4-12)

(a) Controller transfer function:

$$\frac{F(s)}{sE_c(s)} = \frac{100}{s} - \frac{30}{s+6} - \frac{70}{s+10} = \frac{880(s+6.818)}{s(s+6)(s+10)} \quad G_c(s) = \frac{F(s)}{E_c(s)} = \frac{880(s+6.818)}{(s+6)(s+10)}$$

(b) Open-loop transfer function:

$$\frac{V(s)}{E(s)} = \frac{K}{Ms} G_c(s) = \frac{880K(s+6.818)}{30000s(s+6)(s+10)} = \frac{0.0293K(s+6.818)}{s(s+6)(s+10)}$$

(c) System transfer function:

$$\frac{V(s)}{E_r(s)} = \frac{KG_c(s)/Ms}{1 + KK_f G_c(s)/Ms} = \frac{KG_c(s)}{Ms + KK_f G_c(s)} = \frac{0.0293K(s+6.818)}{s^3 + 16s^2 + (0.0044K + 60)s + 0.03K}$$

(d) Steady-state speed: $E_r = 1V$, $E_r(s) = E_r/s = 1/s$

$$\lim_{t \rightarrow \infty} v(t) = \lim_{s \rightarrow 0} sV(s) = \lim_{s \rightarrow 0} \frac{0.0293K(s+6.818)}{s^3 + 16s^2 + (0.0044K + 60)s + 0.03K} = 6.66 \text{ ft/sec}$$

4-13)

syms t

f=100*(1-0.3*exp(-6*t)-0.7*exp(-10*t))

F=laplace(f)

syms s

F=eval(F)

Gc=F*s

M=30000

syms K

Olp=simplify(K*Gc/M/s)

Kt=0.15

Clp= simplify(Olp/(1+Olp*Kt))

s=0

Ess=eval(Clp)

f =

100-30*exp(-6*t)-70*exp(-10*t)

F =

80*(11*s+75)/s/(s+6)/(s+10)

ans =

$$(880*s+6000)/s/(s+6)/(s+10)$$

$$G_c =$$

$$(880*s+6000)/(s+6)/(s+10)$$

$$M =$$

$$30000$$

$$Olp =$$

$$1/375*K*(11*s+75)/s/(s+6)/(s+10)$$

$$K_t =$$

$$0.1500$$

$$Clp =$$

$$20/3*K*(11*s+75)/(2500*s^3+40000*s^2+150000*s+11*K*s+75*K)$$

$$s =$$

$$0$$

$$Ess =$$

$$20/3$$

4-14)

(a) Controller transfer function:

$$\frac{F(s)}{sE_c(s)} = \left(\frac{100}{s} - \frac{30}{s+6} \right) e^{-0.5s} = \frac{70(s+8.5714)}{s(s+6)} e^{-0.5s}$$

$$G_c(s) = \frac{F(s)}{E_c(s)} = \frac{70(s+8.5714)}{s+6} e^{-0.5s}$$

(b) Open-loop transfer function:

$$\frac{V(s)}{E_r(s)} = \frac{K}{Ms} G_c(s) = \frac{70K(s+8.5714)}{30000s(s+6)} e^{-0.5s} = \frac{0.002333K(s+8.5714)}{s(s+6)} e^{-0.5s}$$

(c) System transfer function:

$$\frac{V(s)}{E_r(s)} = \frac{KG_c(s)/Ms}{1 + KG_c(s)/Ms} = \frac{0.002333K(s+8.5714)e^{-0.5s}}{s^2 + 6s + 0.00035K(s+8.5714)e^{-0.5s}}$$

(d) Steady-state speed: $E_r = 1 \text{ V}$, $E_r(s) = E_r / s = 1 / s$

$$\lim_{t \rightarrow \infty} v(t) = \lim_{s \rightarrow 0} sV(s) = \lim_{s \rightarrow 0} \frac{0.002333K(s+8.5714)e^{-0.5s}}{s^2 + 6s + 0.00035K(s+8.5714)e^{-0.5s}} = 6.66 \text{ ft/sec}$$

4-15)

Note: If $\mathcal{L}^{-1}G(s) = g(t)$, then $\mathcal{L}^{-1}\{e^{-as}G(s)\} = u(t-a) \bullet g(t-a)$

```

syms t s
f=100*(1-0.3*exp(-6*(t-0.5)))
F=laplace(f)*exp(-0.5*s)
F=eval(F)
Gc=F*s
M=30000
syms K
Olp=simplify(K*Gc/M/s)
Kt=0.15
Clp= simplify(Olp/(1+Olp*Kt))
s=0
Ess=eval(Clp)
digits (2)
Fsimp=simplify(expand(vpa(F)))
Gcsimp=simplify(expand(vpa(Gc)))
Olpsimp=simplify(expand(vpa(Olp)))
Clpsimp=simplify(expand(vpa(Clp)))

f =
100-30*exp(-6*t+3)

F =
(100/s-30*exp(3)/(s+6))*exp(-1/2*s)

F =
(100/s-2650113767660283/4398046511104/(s+6))*exp(-1/2*s)

Gc =
(100/s-2650113767660283/4398046511104/(s+6))*exp(-1/2*s)*s

M =
30000

Olp =
-1/131941395333120000*K*(2210309116549883*s-2638827906662400)/s/(s+6)*exp(-1/2*s)

Kt =
0.1500

Clp =

```

$$20/3 * K * (2210309116549883 * s - 2638827906662400) * \exp(-1/2 * s) / (-879609302220800000 * s^2 - 5277655813324800000 * s + 2210309116549883 * K * \exp(-1/2 * s) * s - 2638827906662400 * K * \exp(-1/2 * s))$$

$$s = 0$$

$$Ess = 20/3$$

$$F_{simp} = -.10e3 * \exp(-.50 * s) * (5 * s - 6) / (s + 6)$$

$$G_{simp} = -.10e3 * \exp(-.50 * s) * (5 * s - 6) / (s + 6)$$

$$O_{lpsimp} = -.10e-2 * K * \exp(-.50 * s) * (17 * s - 20) / (s + 6)$$

$$C_{lpsimp} = 5 * K * \exp(-.50 * s) * (15 * s - 17) / (-.44e4 * s^2 - .26e5 * s + 11 * K * \exp(-.50 * s) * s - 13 * K * \exp(-.50 * s))$$

4-16)

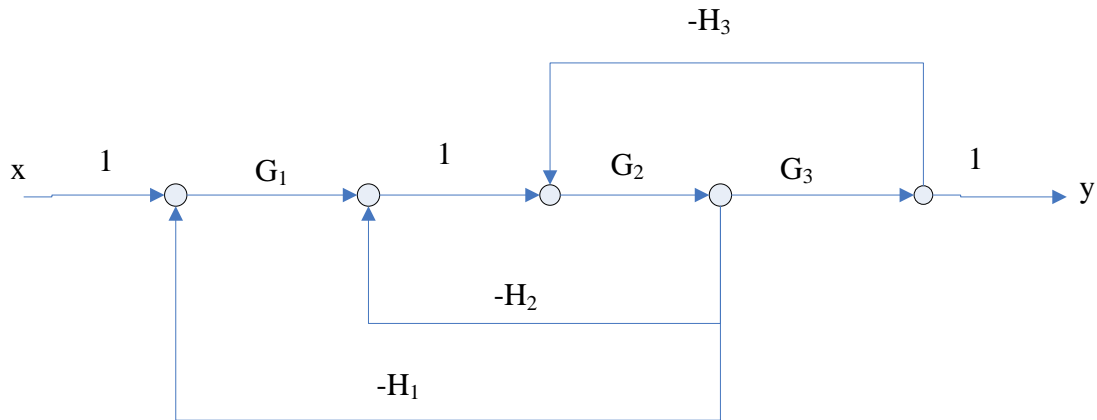
Taking the Laplace transform of the differential equations and expressing in matrix form, the following matrix equations are obtained. All the initial conditions are set to zero.

$$\begin{bmatrix} s(s+2) & 3 \\ 3s+1 & s^2-1 \end{bmatrix} \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ s & 1 \end{bmatrix} \begin{bmatrix} R_1(s) \\ R_2(s) \end{bmatrix} \quad \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} s^2-3s-1 & s^2-4 \\ s^3+2s^2-3s-1 & s^2-s-1 \end{bmatrix} \begin{bmatrix} R_1(s) \\ R_2(s) \end{bmatrix}$$

$$\Delta(s) = s^4 + 2s^3 - s^2 - 11s - 3$$

$$\left. \frac{Y_1(s)}{R_1(s)} \right|_{R_2=0} = \frac{s^2-3s-1}{\Delta} \quad \left. \frac{Y_2(s)}{R_1(s)} \right|_{R_2=0} = \frac{s^3+2s^2-3s-1}{\Delta} \quad \left. \frac{Y_1(s)}{R_2(s)} \right|_{R_1=0} = \frac{s^2-4}{\Delta} \quad \left. \frac{Y_2(s)}{R_2(s)} \right|_{R_1=0} = \frac{s^2-s-1}{\Delta}$$

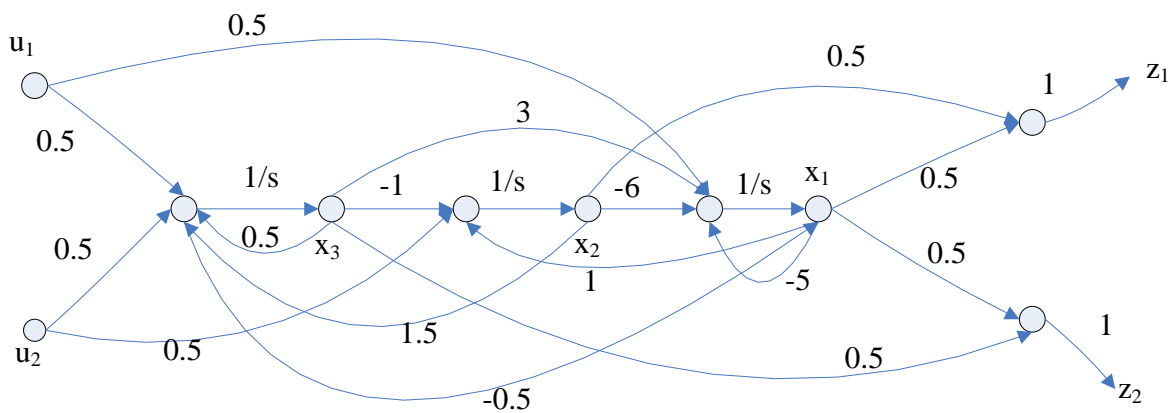
4-17)



4-18)

$$\begin{cases} X_1(s) = \frac{1}{s}[-5X_1(s) - 6X_2(s) + 3X_3(s) + 0.5U_1(s)] \\ X_2(s) = \frac{1}{s}[X_1(s) - X_3(s) + 0.5U_2(s)] \\ X_3(s) = \frac{1}{s}[-0.5X_1(s) + 1.5X_2(s) + 0.5X_3(s) + 0.5U_1(s) + 0.5U_2(s)] \end{cases}$$

$$\begin{cases} Z_1(s) = 0.5X_1(s) + 0.5X_2(s) \\ Z_2(s) = 0.5X_1(s) + 0.5X_3(s) \end{cases}$$



4-19)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{B_1 s + B_0}{s^2 + A_1 s + A_0}$$

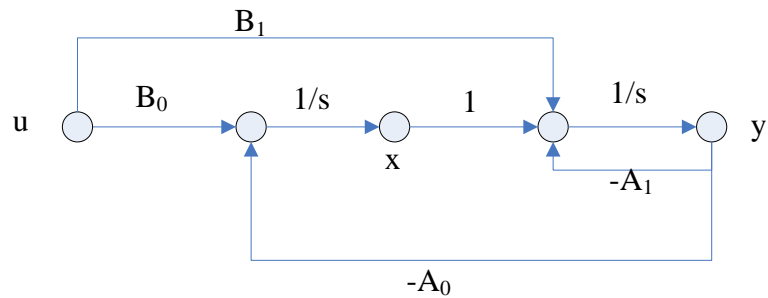
$$\Rightarrow (s + A_1 s + A_0)Y(s) = (B_1 s + B_0)U(s)$$

$$\Rightarrow \left(s + A_1 + \frac{A_0}{s}\right)Y(s) = B_1 U(s) + \frac{B_0}{s}U(s)$$

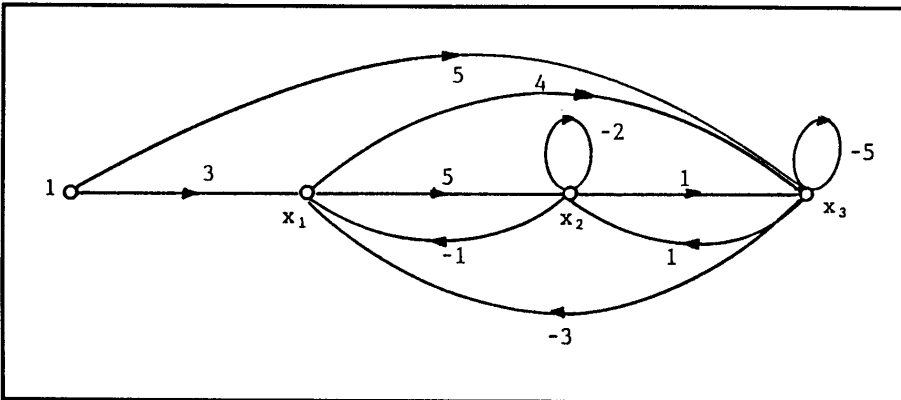
$$\Rightarrow \begin{cases} sY(s) = -A_1 Y(s) + X(s) + B_1 U(s) \\ X(s) = -\frac{A_0}{s}Y(s) + \frac{B_0}{s}U(s) \end{cases}$$

$$\Rightarrow \begin{cases} sY(s) = -A_1 Y(s) + X(s) + B_1 U(s) \\ sX(s) = -A_0 Y(s) + B_0 U(s) \end{cases}$$

$$\Rightarrow \begin{cases} \dot{y} = -A_1 y + x + B_1 u(t) \\ \dot{x} = -A_0 y + B_0 u(t) \end{cases}$$

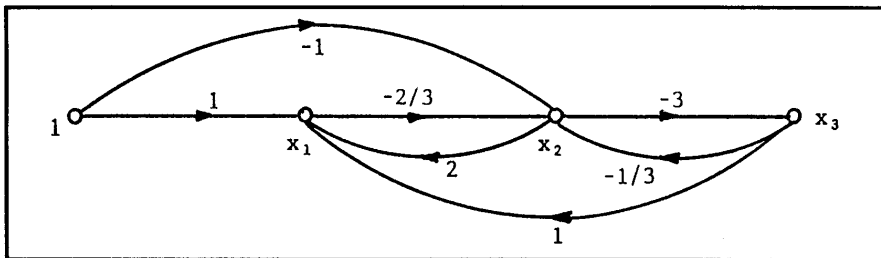
**4-20)**

(a)

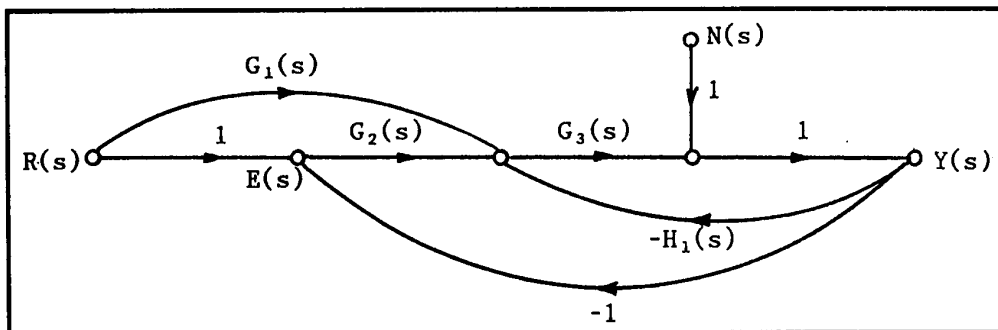


(b) Rewrite the equations as (This is not unique):

$$x_1 = 2x_2 + x_3 + 1 \quad x_2 = (-2/3)x_1 - (1/3)x_3 - 1 \quad x_3 = -3x_2$$



4-21)



$$\left. \frac{Y(s)}{R(s)} \right|_{N=0} = \frac{G_1(s)G_3(s) + G_2(s)G_3(s)}{\Delta} \quad \left. \frac{Y(s)}{N(s)} \right|_{R=0} = \frac{1}{\Delta} \quad \left. \frac{E(s)}{R(s)} \right|_{N=0} = \frac{1 + G_3(s)H_1(s) - G_1(s)G_3(s)}{\Delta}$$

$$\Delta = 1 + G_2(s)G_3(s) + G_3(s)H_1(s)$$

4-22)

(a)

$$\frac{Y_5}{Y_1} = \frac{G_1 G_2 G_3 + G_3 G_4}{\Delta} \quad \frac{Y_2}{Y_1} = \frac{1 + G_3 H_2}{\Delta} \quad \frac{Y_5}{Y_2} = \frac{Y_5 / Y_1}{Y_2 / Y_1} = \frac{G_1 G_2 G_3 + G_3 G_4}{1 + G_3 H_2}$$

$$\Delta = 1 + G_1 H_1 + G_3 H_2 + G_3 G_4 H_3 + G_1 G_2 G_3 H_3 + G_1 G_3 H_1 H_2$$

(b)

$$\frac{Y_5}{Y_1} = \frac{G_1 G_2 G_3 + G_3 G_4}{\Delta} \quad \frac{Y_2}{Y_1} = \frac{1 + G_3 H_2 + H_4}{\Delta} \quad \frac{Y_5}{Y_2} = \frac{Y_5 / Y_1}{Y_2 / Y_1} = \frac{G_1 G_2 G_3 + G_3 G_4}{1 + G_3 H_2 + H_4}$$

$$\Delta = 1 + G_1 H_1 + G_3 H_2 + G_3 G_4 H_3 + G_1 G_2 G_3 H_3 + H_4 + G_1 G_3 H_1 H_2 + G_1 H_1 H_4$$

(c)

$$\frac{Y_5}{Y_1} = \frac{G_1 G_2 G_3 + G_4}{\Delta} \quad \frac{Y_2}{Y_1} = \frac{1 + G_2 G_3 H_3}{\Delta} \quad \frac{Y_5}{Y_2} = \frac{Y_5 / Y_1}{Y_2 / Y_1} = \frac{G_1 G_2 G_3 + G_3 G_4}{1 + G_2 G_3 H_3}$$

$$\Delta = 1 + G_1 H_1 + G_2 G_3 H_3 + G_1 G_2 H_2 - G_2 G_4 H_2 H_3$$

(d)

$$\frac{Y_5}{Y_1} = \frac{G_3 G_4 + G_1 G_2 G_3}{\Delta} \quad \frac{Y_2}{Y_1} = \frac{1 + G_2 H_2}{\Delta} \quad \frac{Y_5}{Y_2} = \frac{Y_5 / Y_1}{Y_2 / Y_1} = \frac{G_3 G_4 + G_1 G_2 G_3}{1 + G_2 H_2}$$

$$\Delta = 1 + G_1 H_1 + G_2 H_2 + G_3 G_4 H_3 + G_1 G_2 G_3 H_3 - G_4 H_1 H_2$$

(e)

$$\frac{Y_5}{Y_1} = \frac{G_1 G_2 G_3 (1 + H_4) + G_4 G_5 (1 + G_2 H_1)}{\Delta} \quad \frac{Y_2}{Y_1} = \frac{1 + G_2 H_1 + G_3 H_2 + H_4 + G_2 H_1 H_4 + G_3 H_2 H_4}{\Delta}$$

$$\frac{Y_5}{Y_2} = \frac{Y_5 / Y_1}{Y_2 / Y_1} = \frac{G_1 G_2 G_3 (1 + H_4) + G_4 G_5 (1 + G_2 H_1)}{1 + G_2 H_1 + G_3 H_2 + H_4 + G_2 H_1 H_4 + G_3 H_2 H_4}$$

$$\Delta = 1 + G_2 H_1 + G_3 H_2 + H_4 + G_4 G_5 H_3 + G_1 G_2 G_3 H_3 + G_2 H_1 H_4 + G_3 H_2 H_4 + G_1 G_2 G_3 H_3 H_4 + G_2 G_4 H_1 H_3$$

4-23)

(a)

$$\frac{Y_7}{Y_1} = \frac{G_1 G_2 G_3 G_4 G_5 + G_5 G_6 (1 + G_2 H_2 + G_3 H_3)}{\Delta}$$

$$\frac{Y_2}{Y_1} = \frac{1 + G_2 H_2 + G_3 H_3 + G_4 G_5 H_4 + H_6 + G_2 G_3 G_4 G_5 H_5 + G_2 H_2 G_4 G_5 H_4 + G_2 H_2 H_6 + G_2 H_3 H_6}{\Delta}$$

$$\Delta = 1 + G_1 H_1 + G_2 H_2 + G_3 H_3 + G_4 G_5 H_4 + H_6 + G_2 G_3 G_4 G_5 H_5 - G_5 G_6 H_1 H_5 - G_5 G_6 H_1 H_2 H_3 H_4 + G_1 G_3 H_1 H_3$$

$$+ G_1 G_4 G_5 H_1 H_4 + G_1 H_1 H_6 + G_2 G_4 G_5 H_2 H_4 + G_2 H_2 H_6 + G_3 H_3 H_6 - G_3 G_5 G_6 H_1 H_3 H_5 + G_1 G_3 H_1 H_3 H_6$$

(b)

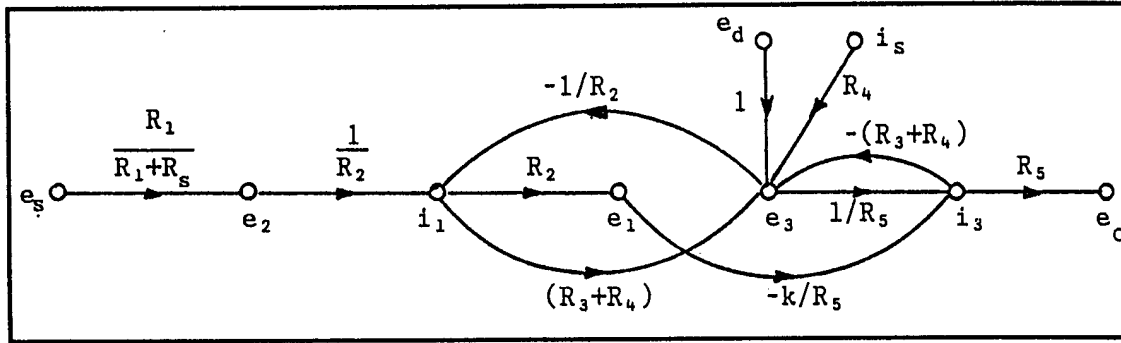
$$\frac{Y_7}{Y_1} = \frac{G_1 G_2 G_3 G_4 G_5 + G_6 (1 + G_3 H_2 + G_4 H_3)}{\Delta} \quad \frac{Y_2}{Y_1} = \frac{1 + G_3 H_2 + G_4 H_3 + G_2 G_3 G_4 G_5 H_4}{\Delta}$$

$$\Delta = 1 + G_1 G_2 H_1 + G_3 H_2 + G_4 H_3 + G_2 G_3 G_4 G_5 H_4 - G_2 G_6 H_1 H_4 + G_1 G_2 G_4 H_1 H_3 - G_2 G_4 G_6 H_1 H_3 H_4$$

4-24)

$$e_2 = \frac{R_1}{R_1 + R_s} e_s \quad i_1 = \frac{e_2 - e_3}{R_2} \quad e_1 = R_2 i_1 \quad e_3 = e_d + R_3(i_1 - i_3) + (i_s + i_1 - i_3)R_4$$

$$i_3 = \frac{e_3 - k e_1}{R_5} \quad e_o = R_5 i_3 \quad \frac{e_o}{e_d} = \frac{1+k}{\Delta} = 0 \quad k = -1$$



4-25)

(a)

$$\frac{Y_3}{Y_1} = \frac{G}{1+GH}$$

(b)

$$\frac{Y_3}{Y_1} = \frac{G}{1+GH}$$

4-26)

(a) The three loops are not in touch.

$$\Delta = 1 + G_1 H_1 + G_2 H_2 + G_3 H_3 + G_1 G_2 H_1 H_2 + G_2 G_3 H_2 H_3 + G_1 G_3 H_1 H_3 + G_1 G_2 G_3 H_1 H_2 H_3$$

(b) The three loops are in touch. $\Delta = 1 + G_1 H_1 + G_2 H_2 + G_3 H_3 + G_1 G_3 H_1 H_3$

4-27)

(a)

$$\left. \frac{Y_6}{Y_1} \right|_{Y_7=0} = \frac{G_1 G_2 G_3 G_4 + G_3 G_4 G_5}{\Delta} \quad \left. \frac{Y_6}{Y_7} \right|_{Y_1=0} = \frac{1 + G_2 H_1}{\Delta}$$

$$\Delta = 1 + G_2 H_1 + G_4 H_2 + G_1 G_2 G_3 G_4 H_3 + G_3 G_4 G_5 H_3 + G_2 G_4 H_1 H_2$$

(b)

$$\left. \frac{Y_6}{Y_1} \right|_{Y_7=0} = \frac{G_1 G_2 G_3 G_4 + G_3 G_4 G_5}{\Delta} \quad \left. \frac{Y_6}{Y_7} \right|_{Y_1=0} = \frac{1 + G_1 H_1 + G_3 H_2 + G_1 G_3 H_1 H_2}{\Delta}$$

$$\Delta = 1 + G_1 H_1 + G_3 H_2 + G_3 G_4 G_5 H_4 + G_1 G_2 G_3 G_4 H_4 + G_1 G_3 H_1 H_2 + G_1 G_3 G_4 H_1 H_3$$

4-28)

(a)

$$\left. \frac{Y_7}{Y_1} \right|_{Y_8=0} = \frac{G_1 G_2 G_3 G_4 G_5 + G_3 G_4 G_5 G_6}{\Delta}$$

$$\Delta = 1 + G_2 H_1 + G_5 H_2 + G_1 G_2 G_3 G_4 G_5 H_3 + G_3 G_4 G_5 G_6 H_3 + G_2 G_5 H_1 H_2$$

(b)

$$\left. \frac{Y_7}{Y_8} \right|_{Y_1=0} = \frac{G_4 G_5 (1 + G_2 H_1)}{\Delta}$$

(c)

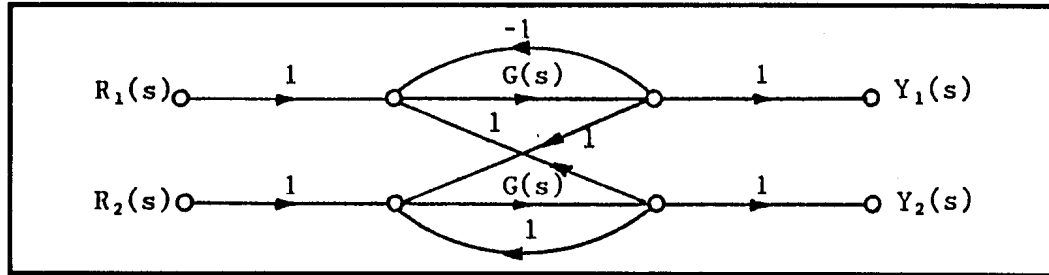
$$\left. \frac{Y_7}{Y_4} \right|_{Y_8=0} = \left. \frac{Y_7 / Y_1}{Y_4 / Y_1} \right|_{Y_8=0} = \frac{G_1 G_2 G_3 G_4 G_5 + G_3 G_4 G_5 G_6}{(G_1 G_2 + G_6)(1 + G_5 H_2)}$$

(d)

$$\left. \frac{Y_7}{Y_4} \right|_{Y_1=0} = \left. \frac{Y_7 / Y_8}{Y_4 / Y_8} \right|_{Y_1=0} = \frac{-G_4 G_5 (1 + G_2 H_1)}{G_4 G_5 H_3 (G_6 + G_1 G_2)}$$

The results in (c) and (d) are different because different inputs are used.

4-29)

(a) Equivalent SFG:

(b) $\Delta = 1 - 2[G(s)]^2$

(c)

$$\left. \frac{Y_1(s)}{R_1(s)} \right|_{R_2=0} = \frac{G(s)[1 - G(s)]}{\Delta}$$

$$\left. \frac{Y_1(s)}{R_2(s)} \right|_{R_1=0} = \frac{[G(s)]^2}{\Delta}$$

$$\left. \frac{Y_2(s)}{R_1(s)} \right|_{R_2=0} = \frac{[G(s)]^2}{\Delta}$$

$$\left. \frac{Y_2(s)}{R_2(s)} \right|_{R_1=0} = \frac{G(s)[1 + G(s)]}{\Delta}$$

(d) Transfer function in matrix form: $Y(s) = G(s)R(s)$

$$G(s) = \frac{1}{\Delta} \begin{bmatrix} G(s)[1 - G(s)] & [G(s)]^2 \\ [G(s)]^2 & G(s)[1 + G(s)] \end{bmatrix}$$

4-30) Use Mason's formula:

(a)

$$\left. \frac{Y(s)}{R(s)} \right|_{N=0} = \frac{G_p(s)[1+G_c(s)H(s)]}{1+G_p(s)H(s)} \quad \left. \frac{Y(s)}{N(s)} \right|_{R=0} = \frac{G_p(s)}{1+G_p(s)H(s)}$$

$$\text{When } G_c(s) = G_p(s) \quad \left. \frac{Y(s)}{R(s)} \right|_{N=0} = G_p(s)$$

(b)

$$G_p(s) = G_c(s) = \frac{100}{(s+1)(s+5)} \quad \left. \frac{Y(s)}{R(s)} \right|_{N=0} = G_p(s) = \frac{100}{(s+1)(s+5)}$$

$$R(s) = \frac{1}{s} \quad Y(s) = \frac{100}{s(s+1)(s+5)} = \frac{20}{s} - \frac{25}{s+1} + \frac{5}{s+5} \quad y(t) = (20 - 25e^{-t} + 5e^{-5t})u_s(t)$$

(c)

$$\left. \frac{Y(s)}{N(s)} \right|_{R=0} = \frac{G_p(s)}{1+G_p(s)H(s)} = \frac{100}{(s+1)(s+5)+100H(s)} \quad N(s) = \frac{1}{s} \quad G(s)|_{R=0} = \frac{100}{s(s+1)(s+5)+100s}$$

$H(s)$ must have a pole at $s=0$, but the system must be stable.

$$H(s) = \frac{K}{s} \quad \Delta = s(s+1)(s+5) + 100K$$

K must be selected so that the system is stable.

4-31) MATLAB

```
syms s K
G=100/(s+1)/(s+5)
g=ilaplace(G/s)
H=K/s
YN=simplify(G/(1+G*H))
Yn=ilaplace(YN/s)

G =
100/(s+1)/(s+5)

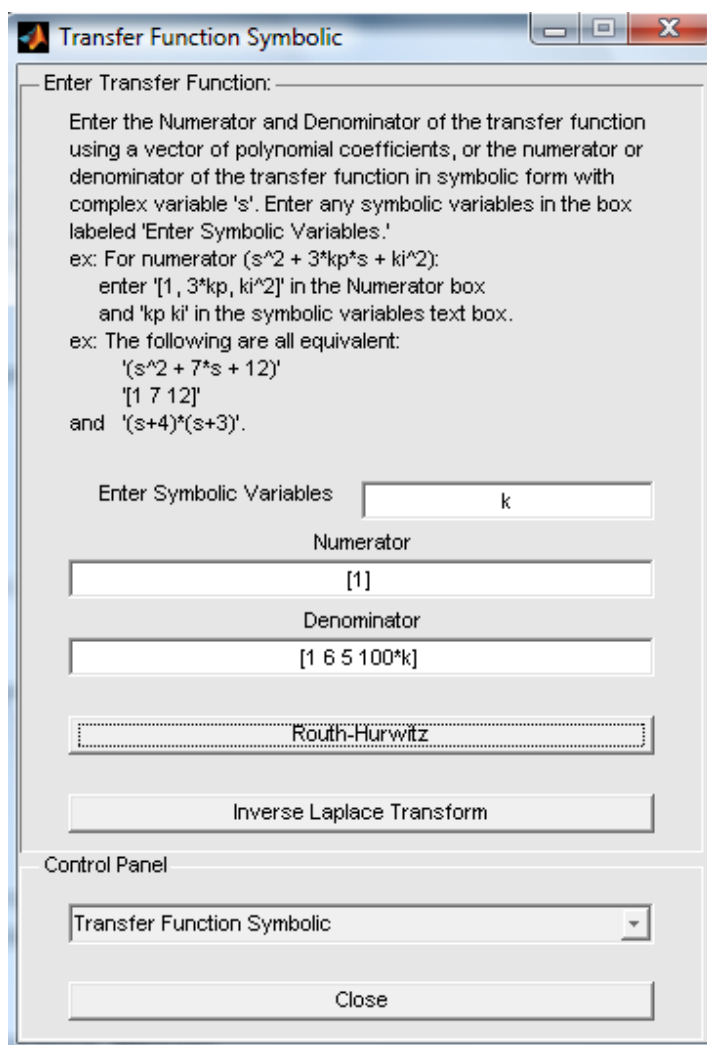
g =
-25*exp(-t)+5*exp(-5*t)+20

H =
K/s
```

YN =

$$100*s/(s^3+6*s^2+5*s+100*K)$$

Apply Routh-Hurwitz within Symbolic tool of ACSYS (see chapter 3)

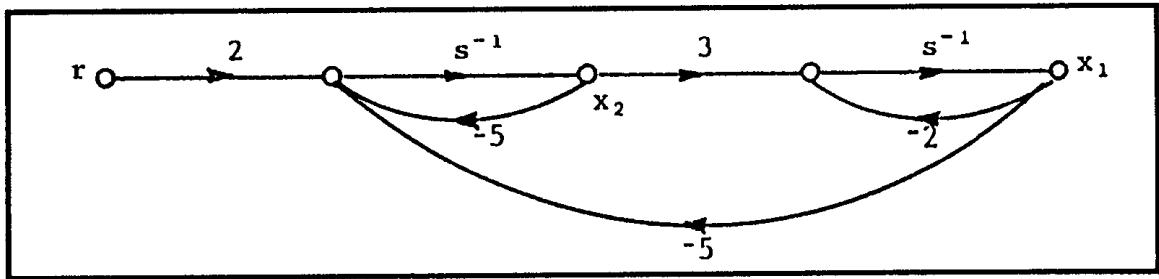


RH =

$$\begin{bmatrix} 1 & 5 \\ 6 & 100*k \\ -50/3*k+5 & 0 \\ 100*k & 0 \end{bmatrix}$$

Stability requires: $0 < k < 3/10$.

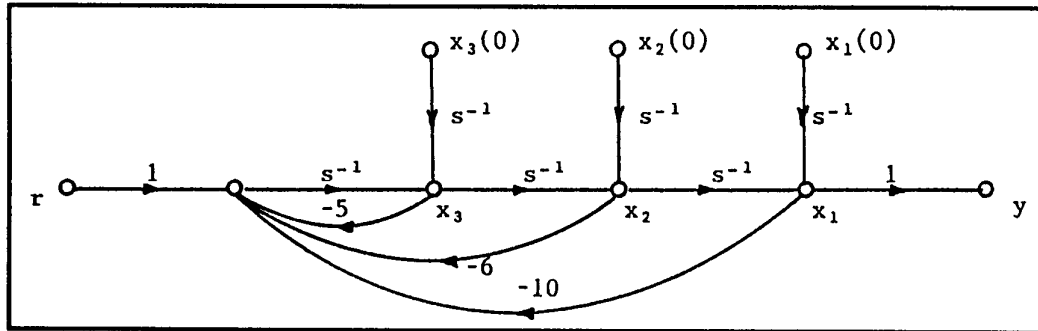
4-32)

(a) State diagram:**(b) Characteristic equation:** $\Delta = 1 + 2s^{-1} + 5s^{-1} + 15s^{-1} + 10s^{-2} = 0$ $s^2 + 7s + 25 = 0$ **(c) Transfer functions:**

$$\frac{X_1(s)}{R(s)} = \frac{6s^{-2}}{\Delta} = \frac{6}{s^2 + 7s + 25} \quad \frac{X_2(s)}{R(s)} = \frac{2s^{-1}(1 + 2s^{-1})}{\Delta} = \frac{2(s+2)}{s^2 + 7s + 25}$$

4-33) MATLAB solutions are in 4-34.**(a)** Write the differential equation as

$$\frac{d^3 y(t)}{dt^3} = r(t) - 5 \frac{d^2 y(t)}{dt^2} - 6 \frac{dy(t)}{dt} - 10y(t)$$

State diagram:**(b) State equations:**

$$\frac{dx_1(t)}{dt} = x_2(t) \quad \frac{dx_2(t)}{dt} = x_3(t) \quad \frac{dx_3(t)}{dt} = -10x_1(t) - 6x_2(t) - 5x_3(t) + r(t)$$

(c) Characteristic equation:

$$\Delta = 1 + 5s^{-1} + 6s^{-2} + 10s^{-3} = 0 \quad s^3 + 5s^2 + 6s + 10 = 0$$

Characteristic equation roots:

$$-4.1337, \quad -0.43313 + j1.4938, \quad -0.43313 - j1.4938$$

(d) Transfer function:

$$\frac{Y(s)}{R(s)} = \frac{s^{-3}}{1 + 5s^{-1} + 6s^{-2} + 10s^{-3}} = \frac{1}{s^3 + 5s^2 + 6s + 10}$$

(e) $R(s) = 1/s$.

$$Y(s) = \frac{1}{s(s^3 + 5s^2 + 6s + 10)} = \frac{0.1}{s} - \frac{0.01519}{s + 4.1337} - \frac{0.08481(s + 0.4331)}{(s + 0.4331)^2 + 2.232} - \frac{0.09953}{(s + 0.4331)^2 + 2.232}$$

$$y(t) = \left[0.1 - 0.01519e^{-4.1337t} - 0.08481e^{-0.4331t} \cos(1.494t) - 0.06662e^{-0.4331t} \sin(1.494t) \right] u_s(t)$$

4-34) MATLAB

```
clear all
p = [1 5 6 10] % Define polynomial s^3+5*s^2+6*s+10=0
roots(p)
G=tf(1,p)
step(G)
```

p =

1 5 6 10

ans =

-4.1337

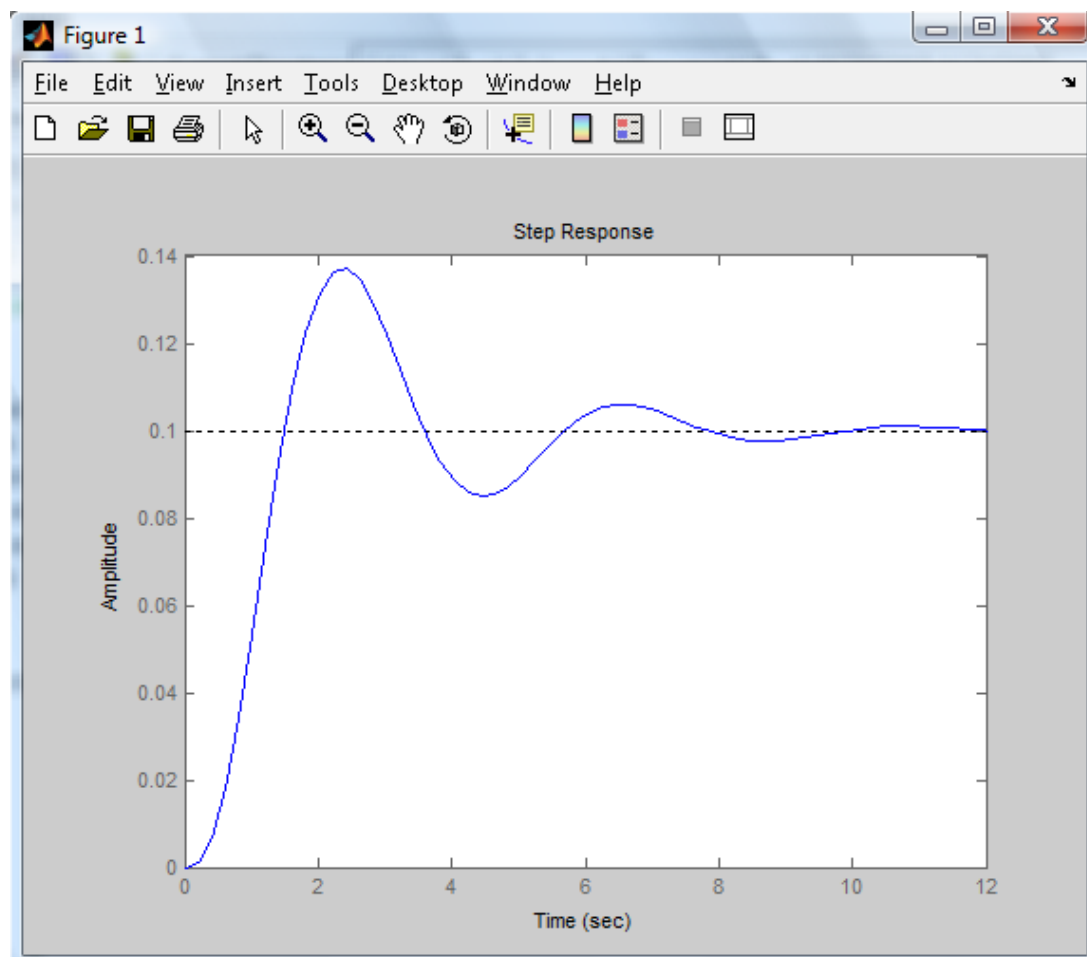
-0.4331 + 1.4938i

-0.4331 - 1.4938i

Transfer function:

1

 $s^3 + 5s^2 + 6s + 10$



Alternatively:

```

clear all
syms s
G=1/( s^3 + 5*s^2 + 6*s + 10)
y=ilaplace(G/s)
s=0
yfv=eval(G)

G =
    1/(s^3+5*s^2+6*s+10)

Y =
    1/10+1/5660*sum((39*_alpha^2-91+160*_alpha)*exp(_alpha*t),_alpha =
    RootOf(_Z^3+5*_Z^2+6*_Z+10))

s =
    0

yfv =
    0.1000

```

Problem finding the inverse Laplace.**Use Toolbox 2-5-1 to find the partial fractions to better find inverse Laplace**

```

clear all
B=[1]
A = [1 5 6 10 0] % Define polynomial s*(s^3+5*s^2+6*s+10)=0
[r,p,k]=residue(B,A)

B =
    1
A =
    1     5     6    10     0

r =
    -0.0152
    -0.0424 + 0.0333i
    -0.0424 - 0.0333i
    0.1000
p =
    -4.1337
    -0.4331 + 1.4938i
    -0.4331 - 1.4938i
    0
k =
    []

```

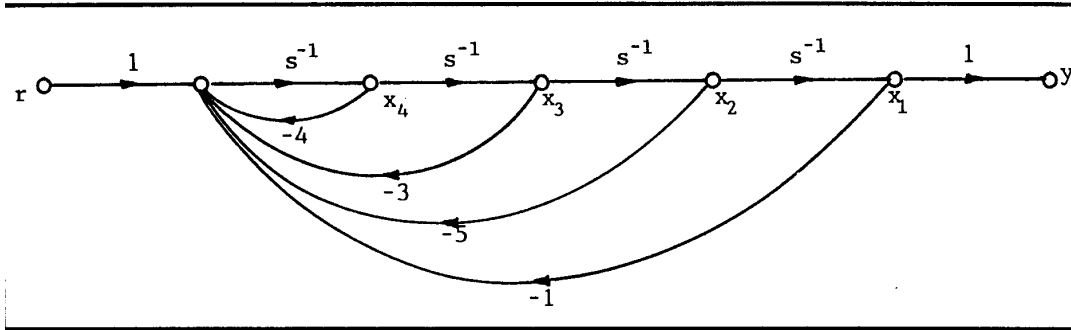
So partial fraction of Y is: $\frac{1}{s} + \frac{-0.0152}{s-4.1337} + \frac{-0.0424 + 0.0333i}{s-0.4331 + 1.4938i} + \frac{-0.0424 - 0.0333i}{s-0.4331 - 1.4938i}$

4-35) MATLAB solutions are in 4-36.

(a) Write the differential equation as

$$\frac{d^4 y(t)}{dt^4} = r(t) - 4 \frac{d^3 y(t)}{dt^3} - 3 \frac{d^2 y(t)}{dt^2} - 5 \frac{dy(t)}{dt} - y(t)$$

State diagram:



(b) State equations:

$$\frac{dx_1(t)}{dt} = x_2(t) \quad \frac{dx_2(t)}{dt} = x_3(t) \quad \frac{dx_3(t)}{dt} = x_4(t) \quad \frac{dx_4(t)}{dt} = -x_1(t) - 5x_2(t) - 3x_3(t) - 4x_4(t) + r(t)$$

(c) Characteristic equation:

$$s^4 + 4s^3 + 3s^2 + 5s + 1 = 0$$

Characteristic equation roots:

$$-3.5286, \quad -0.2212, \quad -0.1251 + j1.125, \quad -0.1251 - j1.125$$

(d) Transfer function:

$$\frac{Y(s)}{R(s)} = \frac{1}{s^4 + 4s^3 + 3s^2 + 5s + 1}$$

(e) $R(s) = 1/s$.

$$Y(s) = \frac{1}{s(s^4 + 4s^3 + 3s^2 + 5s + 1)} = \frac{1}{s} - \frac{1.072}{s + 0.2212} + \frac{0.006668}{s + 3.5286} + \frac{0.06558(s + 0.1251)}{(s + 0.1251)^2 + 1.2656} - \frac{0.2054}{(s + 0.1251)^2 + 1.2656}$$

$$y(t) = [1 - 1.072e^{-0.2212t} + 0.006668e^{-3.5286t} + 0.06558e^{-0.1251t} \cos(1.125t) - 0.1826e^{-0.1251t} \sin(1.125t)]u_s(t)$$

4-36)

```
clear all
p = [1 4 3 5 1] % Define polynomial s^4+4*s^3+3*s^2+5*s+1=0
roots(p)
G=tf(1,p)
step(G)
```

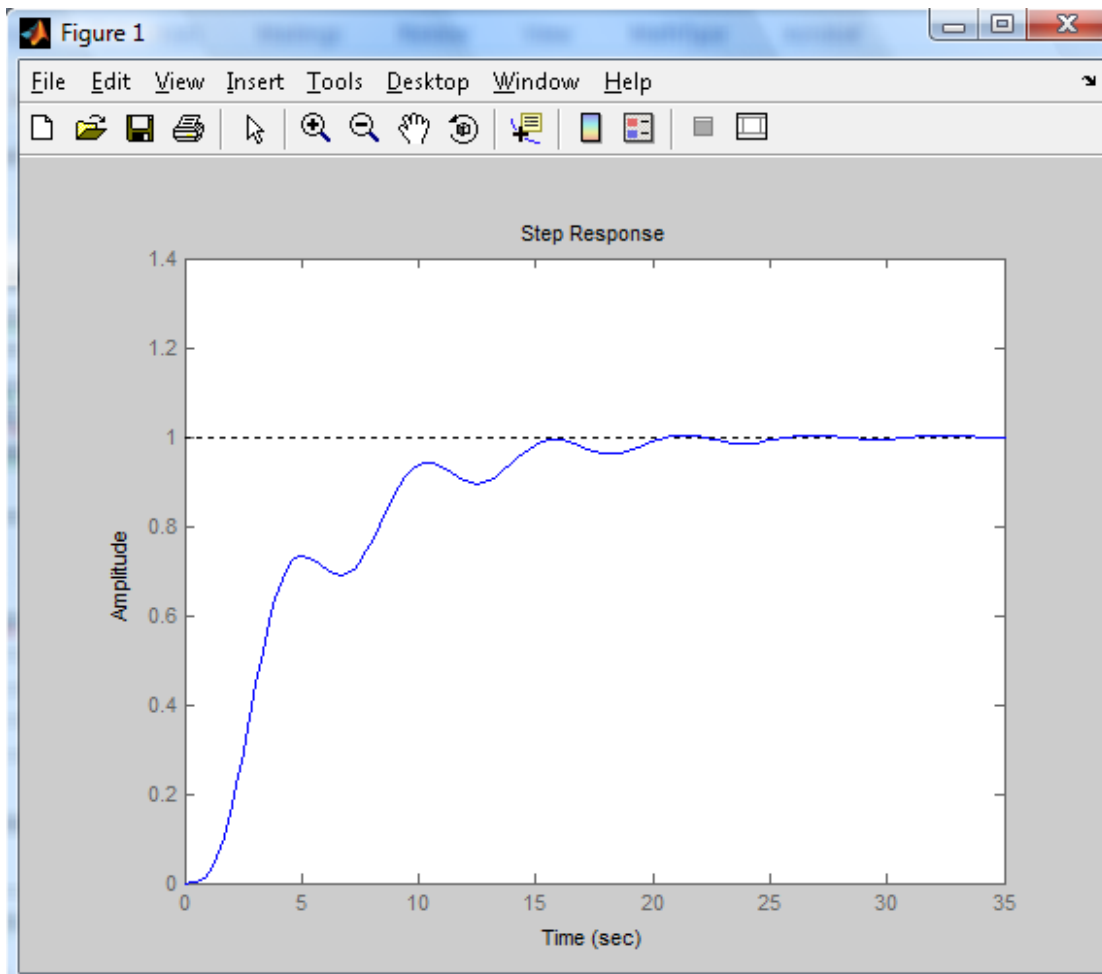
```
p =
    1    4    3    5    1
```

```
ans =
   -3.5286
   -0.1251 + 1.1250i
   -0.1251 - 1.1250i
   -0.2212
```

Transfer function:

1

 $s^4 + 4 s^3 + 3 s^2 + 5 s + 1$



Alternatively:

```

clear all
syms s t
G=1/(s^4+4*s^3+3*s^2+5*s+1)
y=ilaplace(G/s)
s=0
yfv=eval(G)

G =
1/(s^4+4*s^3+3*s^2+5*s+1)

y =
1-
1/14863*sum((3955*_alpha^3+16873+14656*_alpha^2+7281*_alpha)*exp(_alpha*t),_alp
ha = RootOf(_Z^4+4*_Z^3+3*_Z^2+5*_Z+1))

s =
0

yfv =
1

```

Problem finding the inverse Laplace.**Use Toolbox 2-5-1 to find the partial fractions to better find inverse Laplace**

```

clear all
B=[1]
A = [1 4 3 5 1] % Define polynomial s^4+4*s^3+3*s^2+5*s+1=0
[r,p,k]=residue(B,A)

B =
1
A =
1      4      3      5      1

r =
-0.0235
-0.1068 + 0.0255i
-0.1068 - 0.0255i
0.2372

p =
-3.5286
-0.1251 + 1.1250i
-0.1251 - 1.1250i
-0.2212

k =
[]

```

4-37)

(a)

$$\begin{aligned} \left. \frac{Y(s)}{R(s)} \right|_{N=0} &= \frac{G_1 G_2 G_3}{1 + G_1 G_2 H_1 + G_2 G_3 H_2 + G_1 G_2 G_3} = \frac{100(s+1)}{101s^3 + 2122s^2 + 3050s + 1010} \\ \left. \frac{Y(s)}{N(s)} \right|_{R=0} &= \frac{(1 + G_1 G_2 H_1) - G_2 G_3 G_4}{1 + G_1 G_2 H_1 + G_2 G_3 H_2 + G_1 G_2 G_3} = \frac{(101s^3 + 2122s^2 + 2040s) - 10(s+1)G_4}{101s^3 + 2122s^2 + 3050s + 1010} \\ \left. \frac{E(s)}{R(s)} \right|_{N=0} &= \frac{1 + G_2 G_3 H_2}{1 + G_1 G_2 H_1 + G_2 G_3 H_2 + G_1 G_2 G_3} = \frac{s^3 + 22s^2 + 50s + 10}{101s^3 + 2122s^2 + 3050s + 1010} \end{aligned}$$

(b)

$$G_4(s) = \frac{1 + G_1(s)G_2(s)H_1(s)}{G_2(s)G_3(s)} = \frac{101s^3 + 2122s^2 + 2040s}{10(s+1)}$$

(c) **Characteristic equation:** $101s^3 + 2122s^2 + 3050s + 1010 = 0 \quad s^3 + 21.01s^2 + 30.198s + 10 = 0$ **Characteristic equation roots:** $-0.5029, \quad -1.0205, \quad -19.4867$ (d) $R(s) = 1/s$.

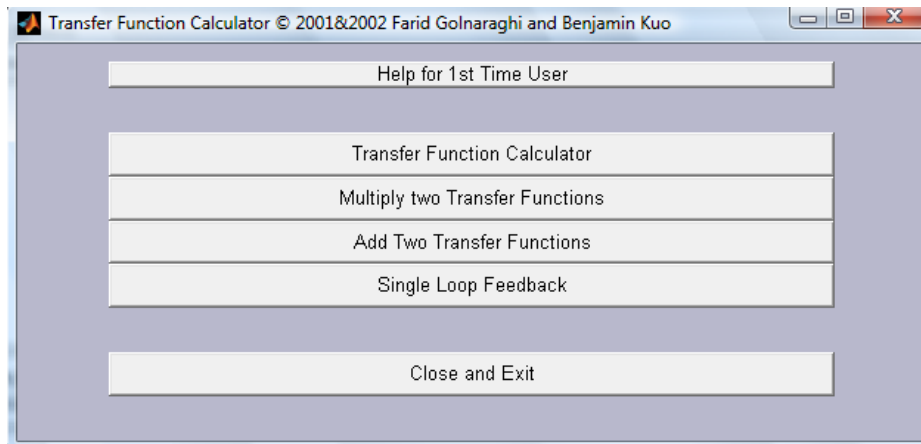
$$E(s) = \frac{s^3 + 22s^2 + 50s + 10}{s(101s^3 + 2122s^2 + 2050s + 1010)} \quad \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) = 0.0099$$

$$\begin{aligned} \text{(e)} \quad Y(s) &= \frac{100(s+1)}{s(101s^3 + 2122s^2 + 3050s + 1010)} = \frac{0.099}{s} + \frac{0.002679}{s+19.49} - \frac{0.002078}{s+1.02} - \frac{0.00996}{s+0.5029} \\ y(t) &= \left(0.099 + 0.002679e^{-19.49t} - 0.002078e^{-1.02t} - 0.00996e^{-0.5029t} \right) u_s(t) \end{aligned}$$

4-38) MATLAB

Use TFcal in ACSYS (go to ACSYS folder and type in TFcal in the MATLAB Command Window).

TFcal



Alternatively use toolboxes 4-4-1 and 4-3-2

```
clear all
syms s
G1=100
G2=(s+1)/(s+2)
G3=10/s/(s+20)
G4=(101*s^3+2122*s^2+2040*s)/10/(s+1)
H1=1
H2=1
simplify(G1*G2*G3/(1+G1*G2*H1+G1*G2*H2+G1*G2*G3))

G1 =
    100

G2 =
    (s + 1)/(s + 2)

G3 =
    10/s/(s + 20)

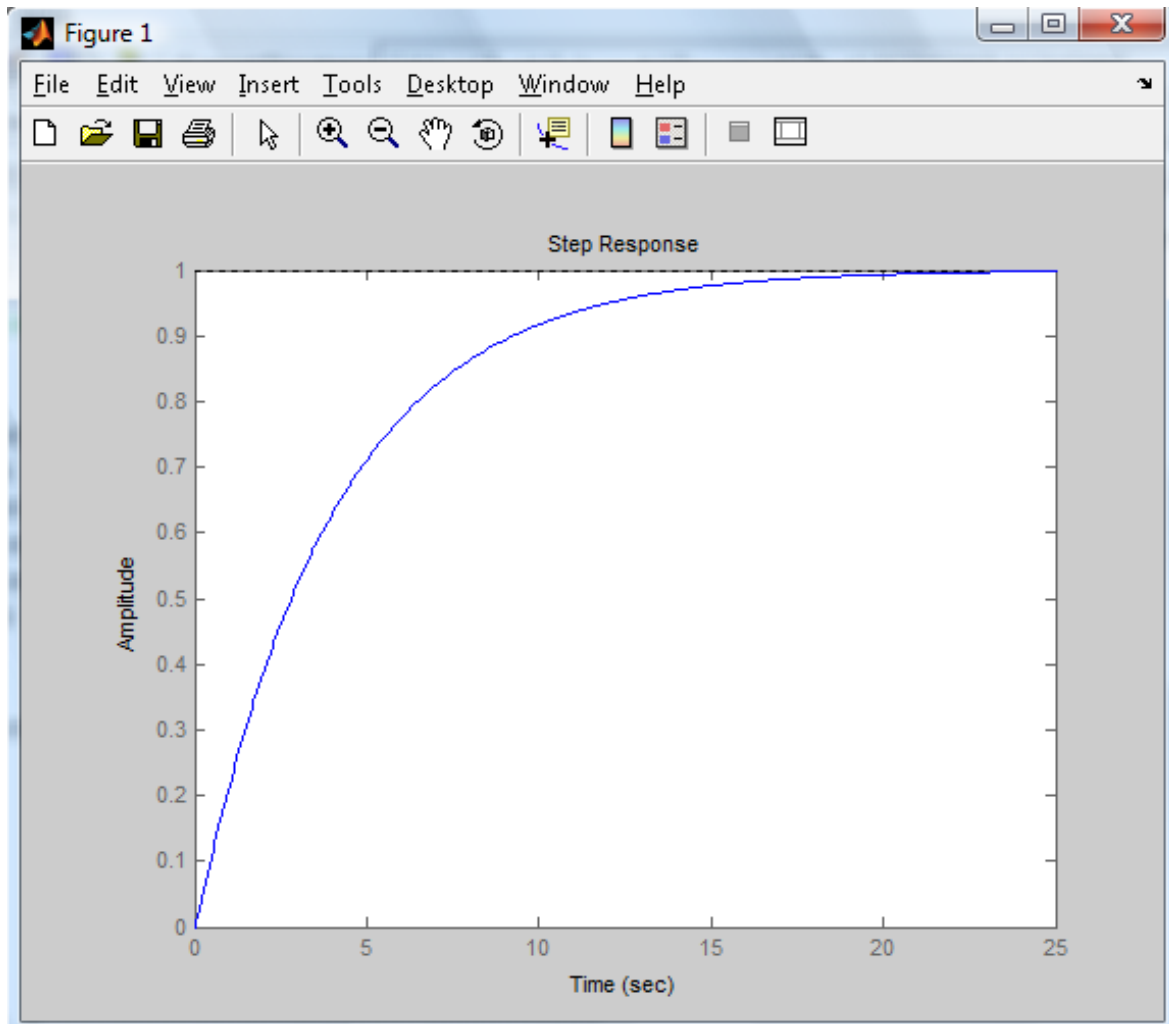
G4 =
    (101/10*s^3+1061/5*s^2+204*s)/(s + 1)

H1 =
    1

H2 =
    1

ans =
    1000*(s + 1)/(201*s^3+4222*s^2+5040*s+1000)

clear all
TF=tf([1000 1000],[201 4222 5040 1000])
step(TF)
```



4-39)

```
clear all
syms s
P1 = 2*s^6+9*s^5+15*s^4+25*s^3+25*s^2+14*s+6 % Define polynomial
P2 = s^6+8*s^5+23*s^4+36*s^3+38*s^2+28*s+16 % Define polynomial
solve(P1, s)
solve(P2, s)
collect(P2-P1)
collect(P2+P1)
collect((P1-P2)*P1)
P1 =
2*s^6+9*s^5+15*s^4+25*s^3+25*s^2+14*s+6

P2 =
s^6+8*s^5+23*s^4+36*s^3+38*s^2+28*s+16

ans =
```

```

-1
-3
i*2^(1/2)
-i*2^(1/2)
-1/4+1/4*i*7^(1/2)
-1/4-1/4*i*7^(1/2)

ans =
-2
-4
i
-i
-1+i
-1-i

ans =

-s^6-s^5+8*s^4+11*s^3+13*s^2+14*s+10

ans =

3*s^6+17*s^5+38*s^4+61*s^3+63*s^2+42*s+22

ans =

-60+2*s^12+11*s^11+8*s^10-54*s^9-195*s^8-471*s^7-796*s^6-1006*s^5-1027*s^4-
848*s^3-524*s^2-224*s

```

Alternative:

```

clear all
P1 = [2 9 15 25 25 14 6] % Define polynomial
roots(P1)
P2 = [1 8 23 36 38 28 16] % Define polynomial
roots(P2)

```

```

P1 =
    2     9    15    25    25    14     6

```

```

ans =
-3.0000
-0.0000 + 1.4142i
-0.0000 - 1.4142i
-1.0000
-0.2500 + 0.6614i
-0.2500 - 0.6614i

```

```
P2 =
      1      8      23      36      38      28      16
```

```
ans =
```

```
-4.0000
-2.0000
-1.0000 + 1.0000i
-1.0000 - 1.0000i
 0.0000 + 1.0000i
 0.0000 - 1.0000i
```

4-40)

```
clear all
syms s
P6 = (s+1)*(s^2+2)*(s+3)*(2*s^2+s+1) % Define polynomial
P7 = (s^2+1)*(s+2)*(s+4)*(s^2+s+1) % Define polynomial
digits(2)
vpa(solve(P6, s))
vpa(solve(P7, s))
collect(P6)
collect(P7)
```

```
P6 =
(s+3)*(s+1)*(2*s^2+s+1)*(s^2+2)
```

```
P7 =
(s^2+1)*(s+2)*(s+4)*(s^2+s+1)
```

```
ans =  -1.
      -3.
      1.4*i
     -1.4*i
    -.25+.65*i
    -.25-.65*i
```

```
ans =  -2.
      -4.
       i
     -1.*i
    -.50+.85*i
    -.50-.85*i
```

```
ans =
2*s^6+9*s^5+15*s^4+25*s^3+25*s^2+14*s+6
```

```
ans =
8+s^6+7*s^5+16*s^4+21*s^3+23*s^2+14*s
```

4-41)**Use Toolbox 2-5-1 to find the partial fractions**

```
clear all
B= conv(conv(conv([1 1],[1 0 2]),[1 4]),[1 10])
A= conv(conv(conv([1 0],[1 2]),[1 2 5]),[2 1 4])
[r,p,k]=residue(B,A)

B =
    1    15    56    70   108    80

A =
    2     9    26    45    46    40     0

r =
-1.0600 - 1.7467i
-1.0600 + 1.7467i
 0.9600
-0.1700 + 0.7262i
-0.1700 - 0.7262i
 2.0000

p =
-1.0000 + 2.0000i
-1.0000 - 2.0000i
-2.0000
-0.2500 + 1.3919i
-0.2500 - 1.3919i
 0

k =
[]
```

4-42) Use toolbox 4-3-2

```
clear all
B= conv(conv(conv([1 1],[1 0 2]),[1 4]),[1 10])
A= conv(conv(conv([1 0],[1 2]),[1 2 5]),[2 1 4])
G1=tf(B,A)
YR1=G1/(1+G1)
pole(YR1)
```

B =

1 15 56 70 108 80

A =

2 9 26 45 46 40 0

Transfer function:

$s^5 + 15s^4 + 56s^3 + 70s^2 + 108s + 80$

 $2s^6 + 9s^5 + 26s^4 + 45s^3 + 46s^2 + 40s$

Transfer function:

$2s^{11} + 39s^{10} + 273s^9 + 1079s^8 + 3023s^7 + 6202s^6 + 9854s^5 + 12400s^4$
 $+ 11368s^3 + 8000s^2 + 3200s$

 $4s^{12} + 38s^{11} + 224s^{10} + 921s^9 + 2749s^8 + 6351s^7 + 11339s^6 + 16074s^5$
 $+ 18116s^4 + 15048s^3 + 9600s^2 + 3200s$

ans =

0

-0.7852 + 3.2346i

-0.7852 - 3.2346i

-2.5822

-1.0000 + 2.0000i

-1.0000 - 2.0000i

-2.0000

-0.0340 + 1.3390i

-0.0340 - 1.3390i

-0.2500 + 1.3919i

-0.2500 - 1.3919i

-0.7794

```
C= [1 12 47 60]
D= [4 28 83 135 126 62 12]
G2=tf(D,C)
YR2=G2/(1+G2)
pole(YR2)
C =
```

1 12 47 60

$$D = \begin{matrix} & 4 & 28 & 83 & 135 & 126 & 62 & 12 \\ & & & & & & & \end{matrix}$$

Transfer function:

$$\frac{4 s^6 + 28 s^5 + 83 s^4 + 135 s^3 + 126 s^2 + 62 s + 12}{s^3 + 12 s^2 + 47 s + 60}$$

Transfer function:

$$\frac{4 s^9 + 76 s^8 + 607 s^7 + 2687 s^6 + 7327 s^5 + 12899 s^4 + 14778 s^3 + 10618 s^2 + 4284 s + 720}{s^9 + 76 s^8 + 607 s^7 + 2688 s^6 + 7351 s^5 + 13137 s^4 + 16026 s^3 + 14267 s^2 + 9924 s + 4320}$$

ans =

$$\begin{aligned} & -5.0000 \\ & -4.0000 \\ & 0.0716 + 0.9974i \\ & 0.0716 - 0.9974i \\ & -1.4265 + 1.3355i \\ & -1.4265 - 1.3355i \\ & -3.0000 \\ & -2.1451 + 0.3366i \\ & -2.1451 - 0.3366i \end{aligned}$$

4-43) Use Toolbox 4-3-1

$$\begin{aligned} G3 &= G1 + G2 \\ G4 &= G1 - G2 \\ G5 &= G4 / G3 \\ G6 &= G4 / (G1 * G2) \end{aligned}$$

$$\begin{aligned} G3 &= G1 + G2 \\ G4 &= G1 - G2 \\ G5 &= G4 / G3 \\ G6 &= G4 / (G1 * G2) \end{aligned}$$

Transfer function:

$$\frac{8 s^{12} + 92 s^{11} + 522 s^{10} + 1925 s^9 + 5070 s^8 + 9978 s^7 + 15154 s^6 + 18427 s^5 + 18778 s^4 + 16458 s^3 + 13268 s^2 + 10720 s + 4800}{s^{12} + 12 s^{11} + 47 s^{10} + 60 s^9 + 126 s^8 + 83 s^7 + 28 s^6 + 4 s^5 + 1 s^4 + 1 s^3 + 1 s^2 + 1 s + 1}$$

$$\frac{2s^9 + 33s^8 + 228s^7 + 900s^6 + 2348s^5 + 4267s^4 + 5342s^3 + 4640s^2 + 2400s}{\dots}$$

Transfer function:

$$\frac{-8s^{12} - 92s^{11} - 522s^{10} - 1925s^9 - 5068s^8 - 9924s^7 - 14588s^6 - 15413s^5 - 9818s^4 - 406s^3 + 7204s^2 + 9760s + 4800}{\dots}$$

$$\frac{2s^9 + 33s^8 + 228s^7 + 900s^6 + 2348s^5 + 4267s^4 + 5342s^3 + 4640s^2 + 2400s}{\dots}$$

Transfer function:

$$\frac{-16s^{21} - 448s^{20} - 5904s^{19} - 49252s^{18} - 294261s^{17} - 1.346e006s^{16} - 4.906e006s^{15} - 1.461e007s^{14} - 3.613e007s^{13} - 7.482e007s^{12} - 1.3e008s^{11} - 1.883e008s^{10} - 2.234e008s^9 - 2.078e008s^8 - 1.339e008s^7 - 2.674e007s^6 + 6.595e007s^5 + 1.051e008s^4 + 8.822e007s^3 + 4.57e007s^2}{\dots}$$

$$1.152e007s$$

$$\frac{16s^{21} + 448s^{20} + 5904s^{19} + 49252s^{18} + 294265s^{17} + 1.346e006s^{16} + 4.909e006s^{15} + 1.465e007s^{14} + 3.643e007s^{13} + 7.648e007s^{12} + 1.369e008s^{11} + 2.105e008s^{10} + 2.803e008s^9 + 3.26e008s^8 + 3.343e008s^7 + 3.054e008s^6 + 2.493e008s^5 + 1.788e008s^4 + 1.072e008s^3 + 4.8e007s^2}{\dots}$$

$$1.152e007 \text{ s} \quad +$$

Transfer function:

$$\begin{aligned} & -16 \text{ s}^{21} - 448 \text{ s}^{20} - 5904 \text{ s}^{19} - 49252 \text{ s}^{18} - 294261 \text{ s}^{17} - 1.346e006 \text{ s}^{16} \\ & - 4.906e006 \text{ s}^{15} - 1.461e007 \text{ s}^{14} - 3.613e007 \text{ s}^{13} - 7.482e007 \text{ s}^{12} \\ & - 1.3e008 \text{ s}^{11} - 1.883e008 \text{ s}^{10} - 2.234e008 \text{ s}^9 - 2.078e008 \text{ s}^8 - \\ & 1.339e008 \text{ s}^7 \\ & - 2.674e007 \text{ s}^6 + 6.595e007 \text{ s}^5 + 1.051e008 \text{ s}^4 + 8.822e007 \text{ s}^3 + \\ & 4.57e007 \text{ s}^2 \end{aligned}$$

$$1.152e007 \text{ s} \quad +$$

$$\begin{aligned} & 8 \text{ s}^{20} + 308 \text{ s}^{19} + 5270 \text{ s}^{18} + 54111 \text{ s}^{17} + 379254 \text{ s}^{16} + 1.955e006 \text{ s}^{15} \\ & + 7.778e006 \text{ s}^{14} + 2.471e007 \text{ s}^{13} + 6.416e007 \text{ s}^{12} + 1.383e008 \text{ s}^{11} \\ & + 2.504e008 \text{ s}^{10} + 3.822e008 \text{ s}^9 + 4.919e008 \text{ s}^8 + 5.305e008 \text{ s}^7 + \\ & 4.73e008 \text{ s}^6 \\ & + 3.404e008 \text{ s}^5 + 1.899e008 \text{ s}^4 + 7.643e007 \text{ s}^3 + 1.947e007 \text{ s}^2 + \\ & 2.304e006 \text{ s} \end{aligned}$$

CHAPTER 5

STABILITY OF LINEAR CONTROL SYSTEMS

Problems

5-1. Without using the Routh-Hurwitz criterion, determine if the following systems are asymptotically stable, marginally stable, or unstable. In each case, the closed-loop system transfer function is given.

(a) $M(s) = \frac{10(s+2)}{s^3 + 3s^2 + 5s}$

(b) $M(s) = \frac{s-1}{(s+5)(s^2+2)}$

(c) $M(s) = \frac{K}{s^3 + 5s + 5}$

(d) $M(s) = \frac{100(s-1)}{(s+5)(s^2+2s+2)}$

(e) $M(s) = \frac{100}{s^3 - 2s^2 + 3s + 10}$

(f) $M(s) = \frac{10(s+12.5)}{s^4 + 3s^3 + 50s^2 + s + 10^6}$

(a) Poles are at $s = 0, -1.5 + j1.6583, -1.5 - j1.6583$

One poles at $s = 0$. **Marginally stable.**

(b) Poles are at $s = -5, -j\sqrt{2}, j\sqrt{2}$
Marginally stable.

Two poles on $j\omega$ axis.

(c) Poles are at $s = -0.8688, 0.4344 + j2.3593, 0.4344 - j2.3593$

Two poles in RHP. **Unstable.**

(d) Poles are at $s = -5, -1 + j, -1 - j$

All poles in the LHP. **Stable.**

(e) Poles are at $s = -1.3387, 1.6634 + j2.164, 1.6634 - j2.164$ Two poles in RHP. **Unstable.**

(f) Poles are at $s = -22.8487 \pm j22.6376, 21.3487 \pm j22.6023$ Two poles in RHP. **Unstable.**

5-2. Use the ROOTS command in MATLAB to solve Problem 5-1.

Find the Characteristic equations and then use the roots command.

(a)

```
p = [ 1 3 5 0]
```

```
sr = roots(p)
```

```
p =
```

```
1    3    5    0
```

```
sr =
```

```
0
```

```
-1.5000 + 1.6583i
```

```
-1.5000 - 1.6583i
```

(b)

```
p = conv([1 5],[1 0 2])
```

```
sr = roots(p)
```

```
p =
```

```
1    5    2   10
```

```
sr =
```

```
-5.0000
```

```
0.0000 + 1.4142i
```

```
0.0000 - 1.4142i
```

(c)

```
>> roots([1 5 5])
```

```
ans =
```

```
-3.6180
```

```
-1.3820
```

(d)

```
roots(conv([1 5],[1 2 2]))
```

ans =

-5.0000

-1.0000 + 1.0000i

-1.0000 - 1.0000i

(e) roots([1 -2 3 10])

ans =

1.6694 + 2.1640i

1.6694 - 2.1640i

-1.3387

(f) roots([1 3 50 1 10^6])

-22.8487 +22.6376i

-22.8487 -22.6376i

21.3487 +22.6023i

21.3487 -22.6023i

Alternatively Problem 5-2

MATLAB code:

```
% Question 5-34,
```

```
clear all;
```

```
s=tf('s')
```

```
%Part a
```

```
Eq=10*(s+2)/(s^3+3*s^2+5*s);
```

```
[num,den]=tfdata(Eq,'v');
```

```
roots(den)
```

```
%Part b
```

```
Eq=(s-1)/((s+5)*(s^2+2));
```

```
[num,den]=tfdata(Eq,'v');
```

```
roots(den)
```

```
%Part c
```

```
Eq=1/(s^3+5*s+5);
[num,den]=tfdata(Eq,'v');
roots(den)
```

%Part d

```
Eq=100*(s-1)/((s+5)*(s^2+2*s+2));
[num,den]=tfdata(Eq,'v');
roots(den)
```

%Part e

```
Eq=100/(s^3-2*s^2+3*s+10);
[num,den]=tfdata(Eq,'v');
roots(den)
```

%Part f

```
Eq=10*(s+12.5)/(s^4+3*s^3+50*s^2+s+10^6);
[num,den]=tfdata(Eq,'v');
roots(den)
```

MATLAB answer:

Part(a)

0
-1.5000 + 1.6583i
-1.5000 - 1.6583i

Part(b)

-5.0000
-0.0000 + 1.4142i
-0.0000 - 1.4142i

Part(c)

0.4344 + 2.3593i
0.4344 - 2.3593i
-0.8688

Part(d)

-5.0000

-1.0000 + 1.0000i

-1.0000 - 1.0000i

Part(e)

1.6694 + 2.1640i

1.6694 - 2.1640i

-1.3387

Part(f)

-22.8487 +22.6376i

-22.8487 -22.6376i

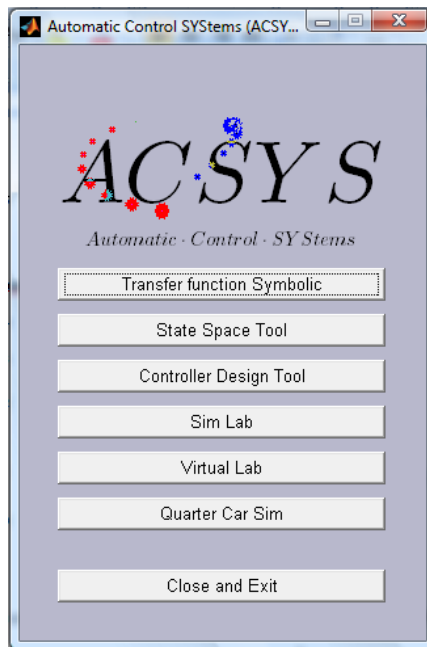
21.3487 +22.6023i

21.3487 -22.6023i

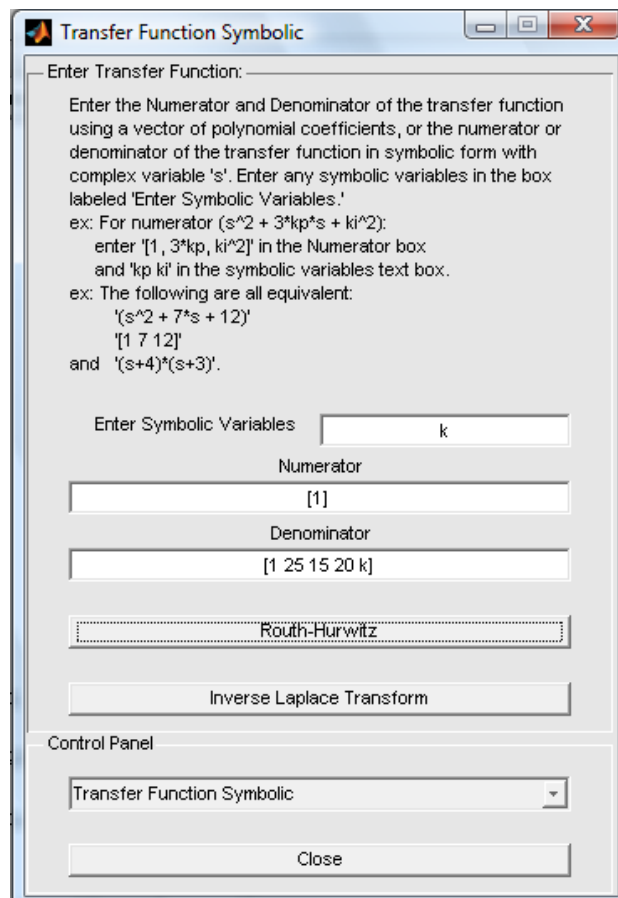
Alternative using ACSYS:

1. Activate MATLAB
2. Go to the directory containing the ACSYS software.
3. Type in

Acsys



4. Then press the “transfer function Symbolic button



5. Enter the characteristic equation in the denominator and press the “Routh-Hurwitz” push-button.

RH =

```
[      1,      15,      k]
[      25,      20,      0]
[    71/5,      k,      0]
[-125/71*k+20,      0,      0]
[      k,      0,      0]
```

6. Find the values of K to make the system unstable..

Using ACSYS toolbar under “Transfer Function Symbolic”, the Routh-Hurwitz option can be used to generate RH matrix based on denominator polynomial. The system is stable if and only if the first column of this matrix contains NO negative values.

Alternative MATLAB code: to calculate the number of right hand side poles

```
%Part a
```

```
den_a=[1 25 10 450]
roots(den_a)
```

```
%Part b
```

```
den_b=[1 25 10 50]
roots(den_b)
```

```
%Part c
```

```
den_c=[1 25 250 10]
roots(den_c)
```

```
%Part d
```

```
den_d=[2 10 5.5 5.5 10]
roots(den_d)
```

```
%Part e
```

```
den_e=[1 2 8 15 20 16 16]
roots(den_e)
```

```
%Part f
```

```
den_f=[1 2 10 20 5]
roots(den_f)
```

```
%Part g
den_g=[1 2 8 12 20 16 16 0 0]
roots(den_g)
```

using ACSYS, the denominator polynomial can be inserted, and by clicking on the “Routh-Hurwitz” button, the R-H chart can be observed in the main MATLAB command window:

Part(a): for the transfer function in part (a), this chart is:

RH chart =

[1, 10]

[25, 450]

[-8, 0]

[450, 0]

Unstable system due to -8 on the 3rd row.

2 complex conjugate poles on right hand side. All the poles are:

-25.3075

$0.1537 + 4.2140i$ and $0.1537 - 4.2140i$

Transfer Function Symbolic

Enter Transfer Function:

Enter the Numerator and Denominator of the transfer function using a vector of polynomial coefficients, or the numerator or denominator of the transfer function in symbolic form with complex variable 's'. Enter any symbolic variables in the box labeled 'Enter Symbolic Variables.'

ex: For numerator ($s^2 + 3*kp*s + ki^2$):
 enter '[1, 3*kp, ki^2]' in the Numerator box.
 and 'kp ki' in the symbolic variables text box.

ex: The following are all equivalent:
 '(s^2 + 7*s + 12)'
 '[1 7 12]'
 and '(s+4)*(s+3)'.

Enter Symbolic Variables:

Numerator:

Denominator:

Control Panel

Part (b):

RH chart:

[1, 10]

[25, 50]

[8, 0]

[50, 0]

Stable system >> No right hand side pole

Part (c):

RH chart:

[1, 250]

[25, 10]

[1248/5, 0]

[10, 0]

Stable system >> No right hand side pole

Part (d):

RH chart:

[2, 11/2, 10]

[10, 11/2, 0]

[22/5, 10, 0]

[-379/22, 0, 0]

[10, 0, 0]

Unstable system due to -379/22 on the 4th row.

2 complex conjugate poles on right hand side. All the poles are:

-4.4660

-1.1116

0.2888 + 0.9611i

0.2888 - 0.9611i

Part (e):

RH chart:

[1, 8, 20, 16]

[2, 15, 16, 0]

[1/2, 12, 16, 0]

[-33, -48, 0, 0]

[124/11, 16, 0, 0]
 [-36/31, 0, 0, 0]
 [16, 0, 0, 0]

Unstable system due to -33 and -36/31 on the 4th and 6th row.

4 complex conjugate poles on right hand side. All the poles are:

$$0.1776 + 2.3520i$$

$$0.1776 - 2.3520i$$

$$-1.2224 + 0.8169i$$

$$-1.2224 - 0.8169i$$

$$0.0447 + 1.1526i$$

$$0.0447 - 1.1526i$$

Part (f):

RH chart:

[1, 10, 5]
 [2, 20, 0]
 [eps, 5, 0]
 [(-10+20*eps)/eps, 0, 0]
 [5, 0, 0]

Unstable system due to ((-10+20*eps)/eps) on the 4th.

2 complex conjugate poles slightly on right hand side. All the poles are:

$$0.0390 + 3.1052i$$

$$0.0390 - 3.1052i$$

$$-1.7881$$

$$-0.2900$$

Part (g):

RH chart:

[1, 8, 20, 16, 0]
[2, 12, 16, 0, 0]
[2, 12, 16, 0, 0]
[12, 48, 32, 0, 0]
[4, 32/3, 0, 0, 0]
[16, 32, 0, 0, 0]
[8/3, 0, 0, 0, 0]
[32, 0, 0, 0, 0]
[0, 0, 0, 0, 0]

Stable system >> No right hand side pole

6 poles wt zero real part:

0
0
0.0000 + 2.0000i
0.0000 - 2.0000i
-1.0000 + 1.0000i
-1.0000 - 1.0000i
0.0000 + 1.4142i
0.0000 - 1.4142i

5-3. Using the Routh-Hurwitz criterion, determine the stability of the closed-loop system that has the following characteristic equations. Determine the number of roots of each equation that are in the right-half s -plane and on the $j\omega$ -axis.

- (a) $s^3 + 25s^2 + 10s + 450 = 0$
- (b) $s^3 + 25s^2 + 10s + 50 = 0$
- (c) $s^3 + 25s^2 + 250s + 10 = 0$
- (d) $2s^4 + 10s^3 + 5.5s^2 + 5.5s + 10 = 0$
- (e) $s^6 + 2s^5 + 8s^4 + 15s^3 + 20s^2 + 16s + 16 = 0$
- (f) $s^4 + 2s^3 + 10s^2 + 20s + 5 = 0$
- (g) $s^8 + 2s^7 + 8s^6 + 12s^5 + 20s^4 + 16s^3 + 16s^2 = 0$

(a) $s^3 + 25s^2 + 10s + 450 = 0$

Roots: $-25.31, 0.1537 + j4.214, 0.1537 - 4.214$

Routh Tabulation:

$$\begin{array}{ccc} s^3 & 1 & 10 \\ s^2 & 25 & 450 \\ s^1 & \frac{250-450}{25} = -8 & 0 \\ s^0 & 450 & \end{array}$$

Two sign changes in the first column. Two roots in

RHP.

(b) $s^3 + 25s^2 + 10s + 50 = 0$

Roots: $-24.6769, -0.1616 + j1.4142, -0.1616 - j1.4142$

Routh Tabulation:

$$\begin{array}{ccc} s^3 & 1 & 10 \\ s^2 & 25 & 50 \\ s^1 & \frac{250-50}{25} = 8 & 0 \\ s^0 & 50 & \end{array}$$

No sign changes in the first column. No roots in

RHP.

(c) $s^3 + 25s^2 + 250s + 10 = 0$

Roots: $-0.0402, -12.48 + j9.6566, -j9.6566$

Routh Tabulation:

$$\begin{array}{ccc} s^3 & 1 & 250 \\ s^2 & 25 & 10 \\ s^1 & \frac{6250-10}{25} = 249.6 & 0 \\ s^0 & 10 & \end{array}$$

No sign changes in the first column. No roots in

RHP.

(d) $2s^4 + 10s^3 + 5.5s^2 + 5.5s + 10 = 0$
 $-4.466, -1.116, 0.2888 + j0.9611, 0.2888 - j0.9611$

Roots:

Routh Tabulation:

s^4	2	5.5	10
s^3	10	5.5	
s^2	$\frac{55-11}{10} = 4.4$	10	
s^1	$\frac{24.2-100}{4.4} = -75.8$		
s^0	10		

Two sign changes in the first column. Two roots in RHP.

(e) $s^6 + 2s^5 + 8s^4 + 15s^3 + 20s^2 + 16s + 16 = 0$ **Roots:**
 $-1.222 \pm j0.8169, 0.0447 \pm j1.153, 0.1776 \pm j2.352$

Routh Tabulation:

s^6	1	8	20	16
s^5	2	15	16	
s^4	$\frac{16-15}{2} = 0.5$	$\frac{40-16}{2} = 12$		
s^3	-33	-48		
s^2	$\frac{-396+24}{-33} = 11.27$	16		
s^1	$\frac{-541.1+528}{11.27} = -1.16$	0		
s^0	0			

Four sign changes in the first column. Four roots in RHP.

(f) $s^4 + 2s^3 + 10s^2 + 20s + 5 = 0$ **Roots:**
 $-0.29, -1.788, 0.039 + j3.105, 0.039 - j3.105$

Routh Tabulation:

s^4	1	10	5
s^3	2	20	
s^2	$\frac{20-20}{2} = 0$	5	
s^1	ε	5	

Replace 0 in last row by ε

$$s^1 \quad \frac{20\varepsilon - 10}{\varepsilon} \cong -\frac{10}{\varepsilon}$$

Two sign changes in first column. Two roots in

$$s^0 \quad 5$$

RHP.

(g)

s^8	1	8	20	16	0
s^7	2	12	16	0	0
s^6	2	12	16	0	0
s^5	0	0	0	0	0

$$A(s) = 2s^6 + 12s^5 + 16s^4$$

$$\frac{dA(s)}{ds} = 12s^5 + 60s^4 + 64s^3$$

s^5	12	60	64	0
s^4	2	$\frac{16}{3}$	0	0
s^3	28	64	0	0
s^2	0.759	0	0	0
s^1	28	0		
s^0	0			

5-4. Use MATLAB to solve Problem 5-3.

Use MATLAB roots command

a) roots([1 25 10 450])

ans =

-25.3075

0.1537 + 4.2140i

0.1537 - 4.2140i

b) roots([1 25 10 50])

ans =

$$-24.6769$$

$$-0.1616 + 1.4142i$$

$$-0.1616 - 1.4142i$$

c) $\text{roots}([1 \ 25 \ 250 \ 10])$

$$\text{ans} =$$

$$-12.4799 + 9.6566i$$

$$-12.4799 - 9.6566i$$

$$-0.0402$$

d) $\text{roots}([2 \ 10 \ 5.5 \ 5.5 \ 10])$

$$\text{ans} =$$

$$-4.4660$$

$$-1.1116$$

$$0.2888 + 0.9611i$$

$$0.2888 - 0.9611i$$

e) $\text{roots}([1 \ 2 \ 8 \ 15 \ 20 \ 16 \ 16])$

$$\text{ans} =$$

$$0.1776 + 2.3520i$$

$$0.1776 - 2.3520i$$

$$-1.2224 + 0.8169i$$

$$-1.2224 - 0.8169i$$

$$0.0447 + 1.1526i$$

$$0.0447 - 1.1526i$$

f) $\text{roots}([1 \ 2 \ 10 \ 20 \ 5])$

$$\text{ans} =$$

$$0.0390 + 3.1052i$$

$$0.0390 - 3.1052i$$

$$-1.7881$$

$$-0.2900$$

g) $\text{roots}([1 \ 2 \ 8 \ 12 \ 20 \ 16 \ 16])$

$$\text{ans} =$$

$0.0000 + 2.0000i$

$0.0000 - 2.0000i$

$-1.0000 + 1.0000i$

$-1.0000 - 1.0000i$

$0.0000 + 1.4142i$

$0.0000 - 1.4142i$

(a) Alternatively use the approach in this Chapter's Section 5-4:

1. Activate MATLAB
2. Go to the directory containing the ACSYS software.
3. Type in

Acsys



4. Then press the "transfer function Symbolic button

Transfer Function Symbolic

Enter Transfer Function:

Enter the Numerator and Denominator of the transfer function using a vector of polynomial coefficients, or the numerator or denominator of the transfer function in symbolic form with complex variable 's'. Enter any symbolic variables in the box labeled 'Enter Symbolic Variables.'

ex: For numerator ($s^2 + 3*kp*s + ki^2$):
 enter '[1 , 3*kp, ki^2]' in the Numerator box
 and 'kp ki' in the symbolic variables text box.

ex: The following are all equivalent:
 '(s^2 + 7*s + 12)'
 '[1 7 12]'
 and '(s+4)*(s+3)'.

Enter Symbolic Variables:

Numerator

Denominator

Control Panel

5. Enter the characteristic equation in the denominator and press the “Routh-Hurwitz” push-button.

RH =

[1, 10]

[25, 450]

[-8, 0]

[450, 0]

Two sign changes in the first column. Two roots in RHP=> UNSTABLE

5-5. Use MATLAB Toolbox 5-3-1 to find the roots of the following characteristic equations of linear continuous-data systems and determine the stability condition of the systems.

(a) $s^3 + 10s^2 + 10s + 130 = 0$

- (b) $s^4 + 12s^3 + s^2 + 2s + 10 = 0$
- (c) $s^4 + 12s^3 + 10s^2 + 10s + 10 = 0$
- (d) $s^4 + 12s^3 + s^2 + 10s + 1 = 0$
- (e) $s^6 + 6s^5 + 125s^4 + 100s^3 + 100s^2 + 20s + 10 = 0$
- (f) $s^5 + 125s^4 + 100s^3 + 100s^2 + 20s + 10 = 0$

Solution:

```
(a) >> p=[1 10 10 130]
p =    1    10    10   130
>> roots(p)
ans =
-10.2603 + 0.0000i
0.1301 + 3.5572i
0.1301 - 3.5572i
```

For the rest, use the MATLAB “roots” command same as in the previous problem.

5-6. For each of the characteristic equations of feedback control systems given, use MATLAB to determine the range of K so that the system is asymptotically stable. Determine the value of K so that the system is marginally stable and determine the frequency of sustained oscillation, if applicable.

- (a) $s^4 + 25s^3 + 15s^2 + 20s + K = 0$
- (b) $s^4 + Ks^3 + 2s^2 + (K+1)s + 10 = 0$
- (c) $s^3 + (K+2)s^2 + 2Ks + 10 = 0$
- (d) $s^3 + 20s^2 + 5s + 10K = 0$
- (e) $s^4 + Ks^3 + 5s^2 + 10s + 10K = 0$
- (f) $s^4 + 12.5s^3 + s^2 + 5s + K = 0$

(a) To solve using MATLAB, set the value of K in an iterative process and find the roots such that at least one root changes sign from negative to positive. Then increase resolution if desired.

Example: in this case $0 < K < 12$ (increase resolution by changing the loop to: for $K=11:.1:12$)

```
for K=0:12
K
roots([1 25 15 20 K])
end
```

K =

0

ans =

0

-24.4193

-0.2904 + 0.8572i

-0.2904 - 0.8572i

K =

1

ans =

-24.4192

-0.2645 + 0.8485i

-0.2645 - 0.8485i

-0.0518

K =

2

ans =

-24.4191

-0.2369 + 0.8419i

-0.2369 - 0.8419i

-0.1071

K =

3

ans =

-24.4191

-0.2081 + 0.8379i

-0.2081 - 0.8379i

-0.1648

K =

4

ans =

-24.4190

-0.1787 + 0.8369i

-0.1787 - 0.8369i

-0.2237

K =

5

ans =

-24.4189

-0.1496 + 0.8390i

-0.1496 - 0.8390i

-0.2819

K =

6

ans =

-24.4188

-0.1215 + 0.8438i

-0.1215 - 0.8438i

-0.3381

K =

7

ans =

-24.4188

-0.0951 + 0.8508i

-0.0951 - 0.8508i

-0.3911

K =

8

ans =

-24.4187

-0.0704 + 0.8595i

$$-0.0704 - 0.8595i$$

$$-0.4406$$

$$K =$$

$$9$$

$$\text{ans} =$$

$$-24.4186$$

$$-0.0475 + 0.8692i$$

$$-0.0475 - 0.8692i$$

$$-0.4864$$

$$K =$$

$$10$$

$$\text{ans} =$$

$$-24.4186$$

$$-0.0263 + 0.8796i$$

$$-0.0263 - 0.8796i$$

$$-0.5288$$

$$K =$$

$$11$$

$$\text{ans} =$$

$$-24.4185$$

$$-0.0067 + 0.8905i$$

$$-0.0067 - 0.8905i$$

$$-0.5681$$

$$K =$$

$$12$$

$$\text{ans} =$$

$$-24.4184$$

$$0.0115 + 0.9015i$$

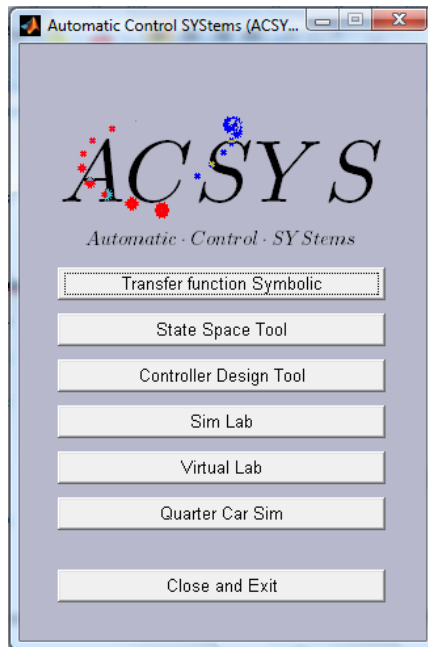
$$0.0115 - 0.9015i$$

$$-0.6046$$

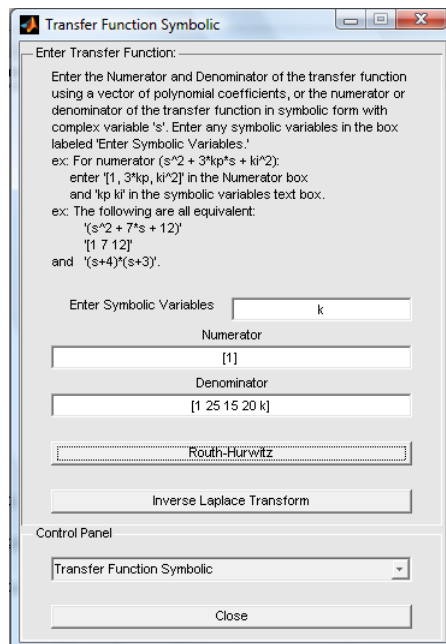
Alternatively use the approach in this Chapter's Section 5-4:

7. Activate MATLAB
8. Go to the directory containing the ACSYS software.
9. Type in

Acsys



10. Then press the “transfer function Symbolic button



11. Enter the characteristic equation in the denominator and press the “Routh-Hurwitz” push-button.

RH =

$$\begin{bmatrix} 1, & 15, & k \\ 25, & 20, & 0 \\ 71/5, & k, & 0 \\ -125/71 * k + 20, & 0, & 0 \\ k, & 0, & 0 \end{bmatrix}$$

12. Find the values of K to make the system unstable following the next steps.

(a) $s^4 + 25s^3 + 15s^2 + 20s + K = 0$

Routh Tabulation:

$$\begin{array}{l} s^4 \quad 1 \qquad \qquad 15 \qquad \qquad K \\ s^3 \quad 25 \qquad \qquad 20 \\ s^2 \quad \frac{375-20}{25} = 14.2 \qquad K \\ s^1 \quad \frac{284-25K}{14.2} = 20-1.76K \qquad 20-1.76K > 0 \text{ or } K < 11.36 \\ s^0 \quad K \qquad \qquad K > 0 \end{array}$$

Thus, the system is stable for $0 < K < 11.36$. When $K = 11.36$, the system is marginally stable.

The

auxiliary equation is $A(s) = 14.2s^2 + 11.36 = 0$. The solution of $A(s) = 0$ is $s^2 = -0.8$. The frequency of oscillation is 0.894 rad/sec.

(b) $s^4 + Ks^3 + 2s^2 + (K+1)s + 10 = 0$

Routh Tabulation:

$$\begin{array}{l} s^4 \quad 1 \qquad \qquad 2 \qquad \qquad 10 \\ s^3 \quad K \qquad \qquad K+1 \qquad \qquad K > 0 \\ s^2 \quad \frac{2K-K-1}{K} = \frac{K-1}{K} \qquad 10 \qquad K > 1 \end{array}$$

$$\begin{array}{rcl}
 s^1 & \frac{-9K^2-1}{K-1} & -9K^2-1 > 0 \\
 s^0 & 10 &
 \end{array}$$

The conditions for stability are: $K > 0$, $K > 1$, and $-9K^2 - 1 > 0$. Since K^2 is always positive, the last condition cannot be met by any real value of K . Thus, the system is unstable for all values of K .

(c) $s^3 + (K+2)s^2 + 2Ks + 10 = 0$

Routh Tabulation:

$$\begin{array}{rcl}
 s^3 & 1 & 2K \\
 s^2 & K+2 & 10 & K > -2 \\
 s^1 & \frac{2K^2+4K-10}{K+2} & K^2+2K-5 > 0 \\
 s^0 & 10 &
 \end{array}$$

The conditions for stability are: $K > -2$ and $K^2 + 2K - 5 > 0$ or $(K+3.4495)(K-1.4495) > 0$, or $K > 1.4495$. Thus, the condition for stability is $K > 1.4495$. When $K = 1.4495$ the system is marginally stable. The auxiliary equation is $A(s) = 3.4495s^2 + 10 = 0$. The solution is $s^2 = -2.899$.

The frequency of oscillation is 1.7026 rad/sec.

(d) $s^3 + 20s^2 + 5s + 10K = 0$

Routh Tabulation:

$$\begin{array}{rcl}
 s^3 & 1 & 5 \\
 s^2 & 20 & 10K \\
 s^1 & \frac{100-10K}{20} = 5-0.5K & 5-0.5K > 0 \text{ or } K < 10 \\
 s^0 & 10K & K > 0
 \end{array}$$

The conditions for stability are: $K > 0$ and $K < 10$. Thus, $0 < K < 10$. When $K = 10$, the system is

marginally stable. The auxiliary equation is $A(s) = 20s^2 + 100 = 0$. The solution of the auxiliary equation is $s^2 = -5$. The frequency of oscillation is 2.236 rad/sec.

(e) $s^4 + Ks^3 + 5s^2 + 10s + 10K = 0$

Routh Tabulation:

s^4	1	5	$10K$	
s^3	K	10		$K > 0$
s^2	$\frac{5K-10}{K}$	$10K$		$5K-10 > 0$ or $K > 2$
s^1	$\frac{\frac{50K-100}{K} - 10K^2}{\frac{5K-10}{K}}$	$\frac{50K-100-10K^3}{5K-10}$		$5K-10-K^3 > 0$
s^0	$10K$			$K > 0$

The conditions for stability are: $K > 0$, $K > 2$, and $5K - 10 - K^3 > 0$.

Use Matlab to solve for k from last condition

```
>> syms k
>> kval=solve(5*k-10+k^3,k);
>> eval(kval)
kval =
    1.4233
   -0.7117 + 2.5533i
   -0.7117 - 2.5533i
```

So $K > 1.4233$.

Thus, the conditions for stability is: $K > 2$

(f) $s^4 + 12.5s^3 + s^2 + 5s + K = 0$

Routh Tabulation:

s^4	1	1	K
s^3	12.5	5	
s^2	$\frac{12.5-5}{12.5} = 0.6$	K	

s^1	$\frac{3-12.5K}{0.6} = 5-20.83K$	$5-20.83K > 0$ or $K < 0.24$
-------	----------------------------------	------------------------------

s^0	K	$K > 0$
-------	-----	---------

The condition for stability is $0 < K < 0.24$. When $K = 0.24$ the system is marginally stable. The auxiliary

equation is $A(s) = 0.6s^2 + 0.24 = 0$. The solution of the auxiliary equation is $s^2 = -0.4$. The frequency of

oscillation is 0.632 rad/sec.

5-7. The loop transfer function of a single-loop feedback control system is given as

$$G(s)H(s) = \frac{K(s+5)}{s(s+2)(1+Ts)}$$

The parameters K and T may be represented in a plane with K as the horizontal axis and T as the vertical axis. Determine the regions in the T -versus- K parameter plane where the closed-loop system is asymptotically stable and where it is unstable. Indicate the boundary on which the system is marginally stable.

The characteristic equation is $Ts^3 + (2T+1)s^2 + (2+K)s + 5K = 0$

Routh Tabulation:

s^3	T	$K+2$	$T > 0$
s^2	$2T+1$	$5K$	$T > -1/2$

$$s^1 \quad \frac{(2T+1)(K+2)-5KT}{2T+1}$$

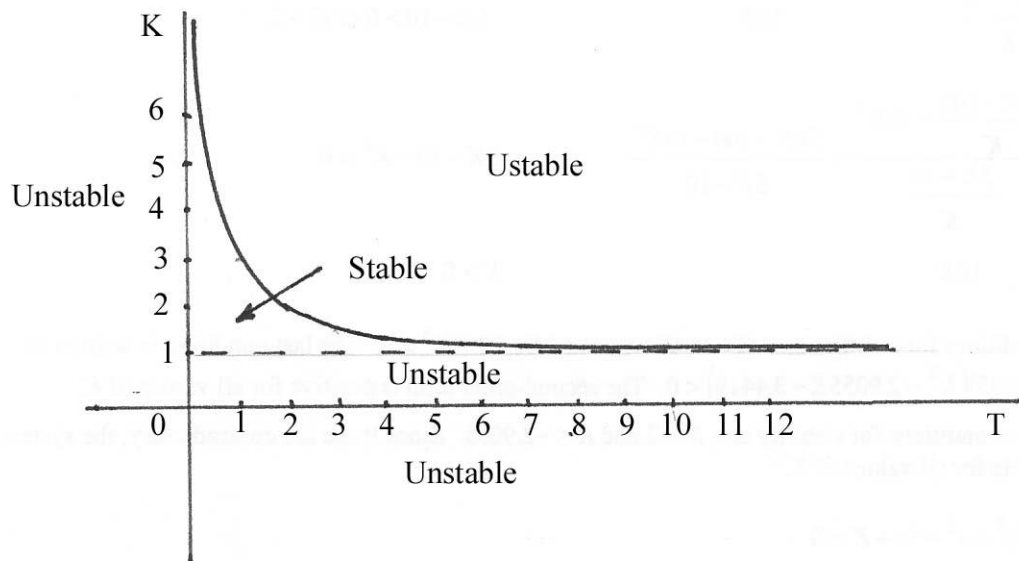
$$K(1-3T)+4T+2 > 0$$

$$s^0 \quad 5K$$

$$K > 0$$

The conditions for stability are: $T > 0$, $K > 0$, and $K < \frac{4T+2}{3T-1}$. The regions of stability in the

T -versus- K parameter plane is shown below.



5-8. Given the forward-path transfer function of unity-feedback control systems, apply the Routh-Hurwitz criterion to determine the stability of the closed-loop system as a function of K . Determine the value of K that will cause sustained constant-amplitude oscillations in the system. Determine the frequency of oscillation.

(a) $G(s) = \frac{K(s+4)(s+20)}{s^3(s+100)(s+500)}$

(b) $G(s) = \frac{K(s+10)(s+20)}{s^2(s+2)}$

$$(c) \quad G(s) = \frac{K}{s(s+10)(s+20)}$$

$$(d) \quad G(s) = \frac{K(s+1)}{s^3 + 2s^2 + 3s + 1}$$

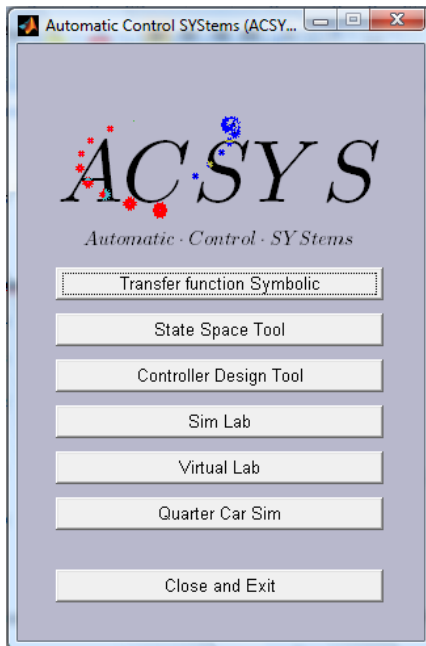
5-9. Use MATLAB to solve Problem 5-8.

Solution 5-8 and 5-9:

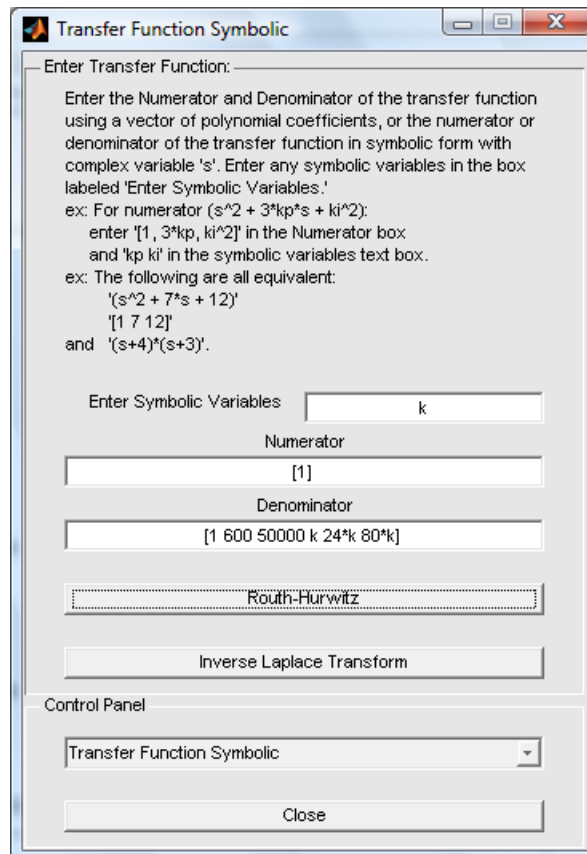
Use the approach in this Chapter's Section 5-4:

1. Activate MATLAB
2. Go to the directory containing the ACSYS software.
3. Type in

Acsys



4. Then press the “transfer function Symbolic button.”



5. Enter the characteristic equation in the denominator and press the “Routh-Hurwitz” push-button.

RH =

```
[1,                                50000,          24*k]
[600,                                k,          80*k]
[-1/600*k+50000,          358/15*k,          0]
[ (35680*k-1/600*k^2)/(-1/600*k+50000),          80*k,          0]
[ 24*k*(k^2-21622400*k+5000000000000000)/(k-30000000)/(35680*k-1/600*k^2)*(-
1/600*k+50000),          0,
0]
[80*k,          0,          0]
```

6. Find the values of K to make the system unstable following the next steps.

(a) Characteristic equation: $s^5 + 600s^4 + 50000s^3 + Ks^2 + 24Ks + 80K = 0$

Routh Tabulation:

s^5	1	50000	$24K$	
s^4	600	K	$80K$	
s^3	$\frac{3 \times 10^7 - K}{600}$	$\frac{14320K}{600}$		$K < 3 \times 10^7$
s^2	$\frac{21408000K - K^2}{3 \times 10^7 - K}$	$80K$		$K < 21408000$
s^1	$\frac{-7.2 \times 10^{16} + 3.113256 \times 10^{11} K - 14400K^2}{600(21408000 - K)}$			$K^2 - 2.162 \times 10^7 K + 5 \times 10^{12} < 0$
s^0	$80K$			$K > 0$

Conditions for stability:

From the s^3 row: $K < 3 \times 10^7$

From the s^2 row: $K < 2.1408 \times 10^7$

From the s^1 row:

$$K^2 - 2.162 \times 10^7 K + 5 \times 10^{12} < 0 \quad \text{or} \quad (K - 2.34 \times 10^5)(K - 2.1386 \times 10^7) < 0$$

Thus, $2.34 \times 10^5 < K < 2.1386 \times 10^7$

From the s^0 row: $K > 0$

Thus, the final condition for stability is: $2.34 \times 10^5 < K < 2.1386 \times 10^7$

When $K = 2.34 \times 10^5$ $\omega = 10.6$ rad/sec.

When $K = 2.1386 \times 10^7$ $\omega = 188.59$ rad/sec.

(b) Characteristic equation: $s^3 + (K + 2)s^2 + 30Ks + 200K = 0$

Routh tabulation:

s^3	1	$30K$	
s^2	$K + 2$	$200K$	$K > -2$
s^1	$\frac{30K^2 - 140K}{K + 2}$		$K > 4.6667$
s^0	$200K$		$K > 0$

Stability Condition: $K > 4.6667$

When $K = 4.6667$, the auxiliary equation is $A(s) = 6.6667s^2 + 933.333 = 0$. The solution is $s^2 = -140$.

The frequency of oscillation is 11.832 rad/sec.

(c) Characteristic equation: $s^3 + 30s^2 + 200s + K = 0$

Routh tabulation:

s^3	1	200	
s^2	30	K	
s^1	$\frac{6000 - K}{30}$		$K < 6000$
s^0	K		$K > 0$

Stability Condition: $0 < K < 6000$

When $K = 6000$, the auxiliary equation is $A(s) = 30s^2 + 6000 = 0$. The solution is $s^2 = -200$.

The frequency of oscillation is 14.142 rad/sec.

(d) Characteristic equation: $s^3 + 2s^2 + (K+3)s + K+1 = 0$

Routh tabulation:

s^3	1	$K+3$	
s^2	2	$K+1$	
s^1	$\frac{K+5}{30}$		$K > -5$
s^0	$K+1$		$K > -1$

Stability condition: $K > -1$. When $K = -1$ the zero element occurs in the first element of the

s^0 row. Thus, there is no auxiliary equation. When $K = -1$, the system is marginally stable, and one

of the three characteristic equation roots is at $s = 0$. There is no oscillation. The system response would increase monotonically.

5-10. A controlled process is modeled by the following state equations.

$$\frac{dx_1(t)}{dt} = x_1(t) - 2x_2(t) \quad \frac{dx_2(t)}{dt} = 10x_1(t) + u(t)$$

The control $u(t)$ is obtained from state feedback such that

$$u(t) = -k_1x_1(t) - k_2x_2(t)$$

where k_1 and k_2 are real constants. Determine the region in the k_1 -versus- k_2 parameter plane in which the closed-loop system is asymptotically stable.

42 State equation: **Open-loop system:** $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 10 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Closed-loop system: $\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t)$

$$\mathbf{A} - \mathbf{B}\mathbf{K} = \begin{bmatrix} 1 & -2 \\ 10 - k_1 & -k_2 \end{bmatrix}$$

Characteristic equation of the closed-loop system:

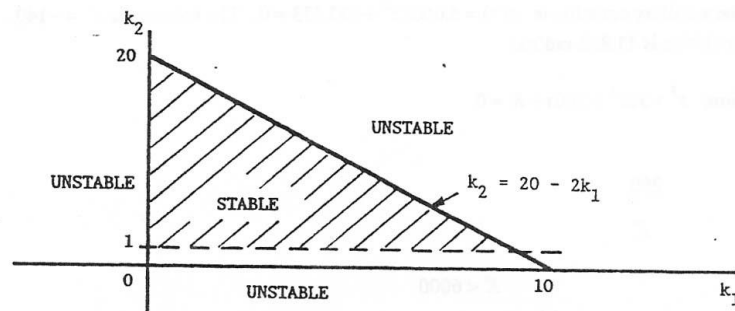
$$|s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}| = \begin{vmatrix} s-1 & 2 \\ -10+k_1 & s+k_2 \end{vmatrix} = s^2 + (k_2-1)s + 20-2k_1-k_2 = 0$$

Stability requirements:

$$k_2 - 1 > 0 \quad \text{or} \quad k_2 > 1$$

$$20 - 2k_1 - k_2 > 0 \quad \text{or} \quad k_2 < 20 - 2k_1$$

Parameter plane:



5-11. A linear time-invariant system is described by the following state equations.

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & -3 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The closed-loop system is implemented by state feedback, so that $u(t) = -\mathbf{K}\mathbf{x}(t)$, where $\mathbf{K} = [k_1 \ k_2 \ k_3]$ and k_1 , k_2 , and k_3 are real constants. Determine the constraints on the elements of \mathbf{K} so that the closed-loop system is asymptotically stable.

) Characteristic equation of closed-loop system:

$$|s\mathbf{I} - \mathbf{A} + \mathbf{BK}| = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ k_1 & k_2 + 4 & s + k_3 + 3 \end{vmatrix} = s^3 + (k_3 + 3)s^2 + (k_2 + 4)s + k_1 = 0$$

Routh Tabulation:

s^3	1	0	$k_2 + 4$
s^2	$k_3 + 3$	k_1	$k_3 + 3 > 0$ or $k_3 > -3$
s^1	$\frac{(k_3 + 3)(k_2 + 4) - k_1}{k_3 + 3}$	$(k_3 + 3)(k_2 + 4) - k_1 > 0$	
s^0	k_1	$k_1 > 0$	

Stability Requirements:

$$k_3 > -3, \quad k_1 > 0, \quad (k_3 + 3)(k_2 + 4) - k_1 > 0$$

5-12. Given the system in state equation form,

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

$$(a) \quad \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$(b) \quad \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Can the system be stabilized by state feedback $u(t) = -\mathbf{K}\mathbf{x}(t)$, where $\mathbf{K} = [k_1 \ k_2 \ k_3]$?

(a) Since \mathbf{A} is a diagonal matrix with distinct eigenvalues, the states are decoupled from each other. The second row of \mathbf{B} is zero; thus, the second state variable, x_2 is uncontrollable. Since the uncontrollable state has the eigenvalue at -3 which is stable, and the unstable state x_3 with the eigenvalue at -2 is controllable, the system is stabilizable.

(b) Since the uncontrollable state x_1 has an unstable eigenvalue at 1 , the system is not stabilizable.

5-13. Consider the open-loop system in Fig. 5P-13(a).

where $\frac{d^2 y}{dt^2} - \frac{g}{l} y = z$ and $f(t) = \tau \frac{dz}{dt} + z$.

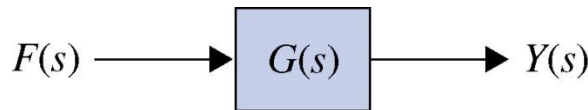


Figure 5P-13(a)

Our goal is to stabilize this system so the closed-loop feedback control will be defined as shown in the block diagram in Fig. 5P-13(b).

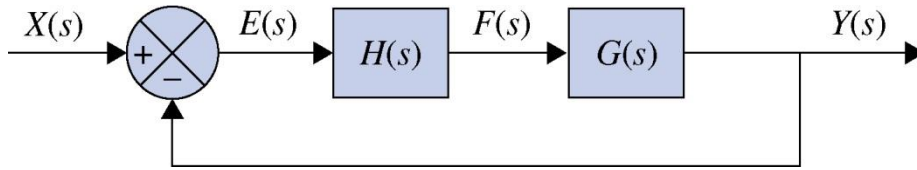


Figure 5P-13(b)

Assuming $f(t) = k_p e + k_d \frac{de}{dt}$.

- Find the open-loop transfer function.
- Find the closed-loop transfer function.
- Find the range of k_p and k_d in which the system is stable.
- Suppose $\frac{g}{l} = 10$ and $\tau = 0.1$. If $y(0) = 10$ and $\frac{dy}{dt} = 0$, then plot the step response of the system with three different values for k_p and k_d . Then show that some values are better than others; however, all values must satisfy the Routh-Hurwitz criterion.

5-45) a)

$$G(s) = \frac{Y(s)}{F(s)}$$

If $\frac{d^2 y}{dt^2} - \frac{g}{l} y = z$, then $s^2 Y(s) - \frac{g}{l} Y(s) = Z(s)$ or $Y(s) = \frac{Z(s)}{s^2 - \frac{g}{l}}$

If $f(t) = \frac{\tau dt}{dt} + z$, then $F(s) = (\tau s + 1)Z(s)$. As a result:

$$G(s) = \frac{\frac{Z(s)}{s^2 - \frac{g}{l}}}{(\tau s + 1)Z(s)} = \frac{1}{\left(s^2 - \frac{g}{l}\right)(\tau s + 1)}$$

$$\text{b) } \begin{cases} F(s) = (\tau s + 1)Z(s) \\ F(s) = (K_p + K_d s)E(s) \end{cases} \Rightarrow Z(s) = \frac{K_p + K_d s}{\tau s + 1} E(s)$$

As a result:

$$\frac{Y(s)}{E(s)} = G(s)H(s) = \frac{K_p + K_d s}{(\tau s + 1)\left(s^2 - \frac{g}{l}\right)}$$

$$\frac{Y(s)}{X(s)} = \frac{G(s)H(s)}{1 + G(s)H(s)} = \frac{K_p + K_d s}{(\tau s + 1) \left(s^2 - \frac{g}{l} \right) + K_p + K_d s}$$

$$\frac{Y(s)}{X(s)} = \frac{G(s)H(s)}{(1 + G(s)H(s))} = \frac{(K_p + K_d s)}{((\tau s + 1)(s^2 - g / l) + K_p + K_d s)}$$

$$= \frac{(K_p + K_d s)}{(\tau s^3 + (\tau(-g / l) + 1)s^2 + K_d s - g / l + K_p)}$$

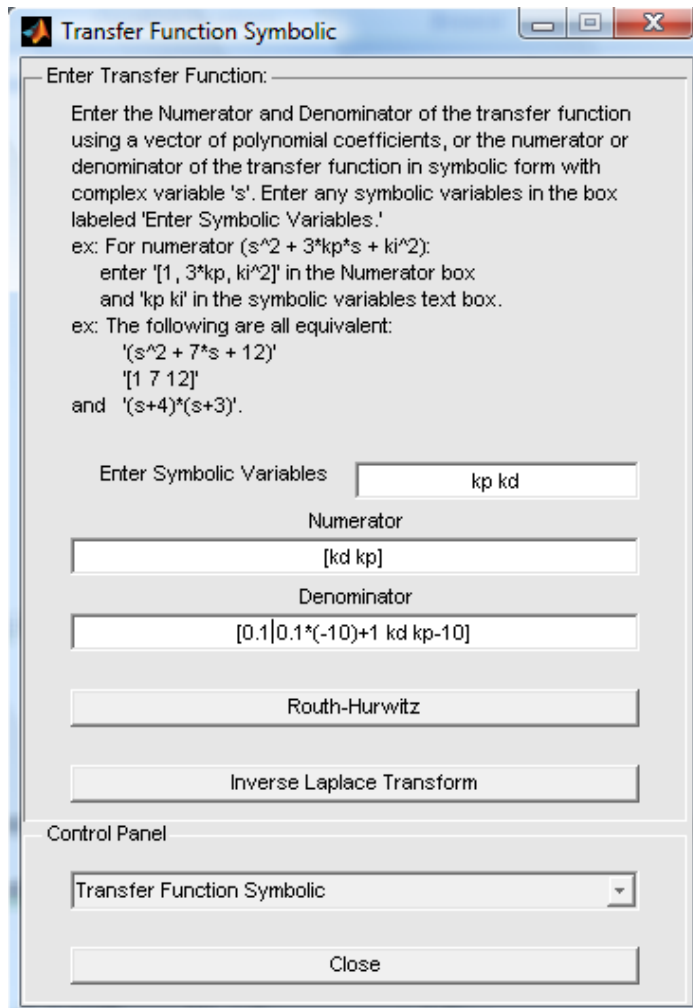
Use the approach in this Chapter's Section 5-4:

1. Activate MATLAB
2. Go to the directory containing the ACSYS software.
3. Type in

Acsys



4. Then press the “transfer function Symbolic button.”



5. Enter the characteristic equation in the denominator and press the “Routh-Hurwitz” push-button.

RH =

$$\begin{bmatrix} 1/10, & kd \\ eps, & kp-10 \\ (-1/10*kp+1+kd*eps)/eps, & 0 \\ kp-10, & 0 \end{bmatrix}$$

For the choice of g/l or τ the system will be unstable. The quantity $\tau g/l$ must be >1 .

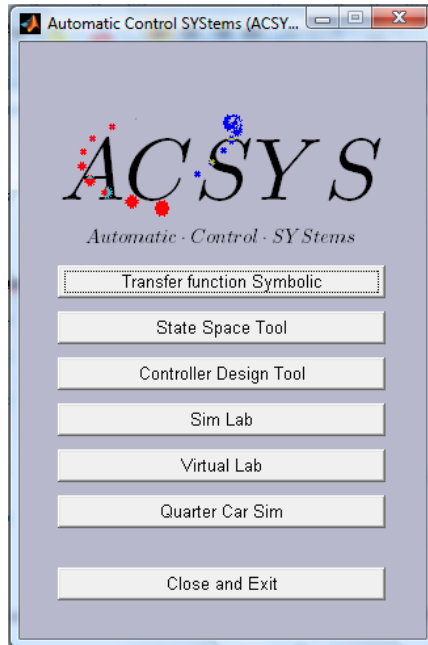
Increase $\tau g/l$ to 1.1 and repeat the process.

- a) Use the ACSYS toolbox as in section 5-4 to find the inverse Laplace transform. Then plot the time response by selecting the parameter values. **Or use toolbox 5-4-1.**

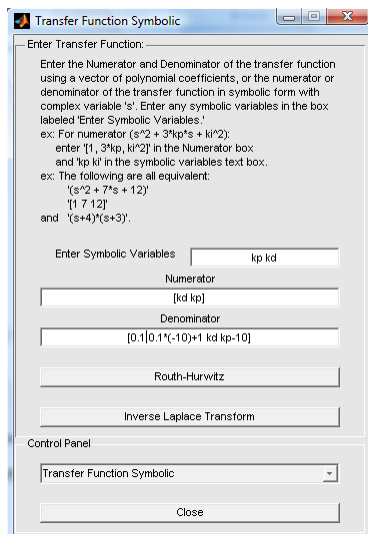
Use the approach in this Chapter’s Section 5-4:

1. Activate MATLAB
2. Go to the directory containing the ACSYS software.
3. Type in

Acsys



4. Then press the “transfer function Symbolic button.”



5. Enter the characteristic equation in the denominator and press the “Inverse Laplace Transform” push-button.

Inverse Laplace Transform

$$G(s) =$$

$$\begin{bmatrix} kd & kp \\ \hline 3 & 3 \end{bmatrix} \begin{bmatrix} 1/10s + s kd + kp - 10 & 1/10s + s kd + kp - 10 \end{bmatrix}$$

G(s) factored:

$$\begin{bmatrix} kd & kp \\ 10 \hline 3 & 3 \end{bmatrix} \begin{bmatrix} s + 10s kd + 10kp - 100 & s + 10s kd + 10kp - 100 \end{bmatrix}$$

Inverse Laplace Transform:

$$g(t) = \text{matrix}([10*kd*\sum(1/(3*_alpha^2+10*kd)*exp(_alpha*t),_alpha=\text{RootOf}(_Z^3+10*_Z*kd+10*kp-100)), 10*kp*\sum(1/(3*_alpha^2+10*kd)*exp(_alpha*t),_alpha=\text{RootOf}(_Z^3+10*_Z*kd+10*kp-100))])$$

While MATLAB is having a hard time with this problem, **it is easy to see the solution will be unstable for all values of Kp and Kd.** Stability of a **linear** system is independent of its initial conditions. For different values of g/l and τ , you may solve the problem similarly – assign all values (including Kp and Kd) and then find the inverse Laplace transform of the system. Find the time response and apply the initial conditions.

Lets chose $g/l=1$ and keep $\tau=0.1$, take $Kd=1$ and $Kp=10$.

$$\begin{aligned} \frac{Y(s)}{X(s)} &= \frac{G(s)H(s)}{(1+G(s)H(s))} = \frac{(K_p + K_d s)}{((\tau s + 1)(s^2 - g/l) + K_p + K_d s)} \\ &= \frac{(10 + s)}{(0.1s^3 + (0.1(-1) + 1)s^2 + s - 1 + 10)} = \frac{(10 + s)}{(0.1s^3 + 0.9s^2 + s + 9)} \end{aligned}$$

Using ACSYS:

RH =

$$[1/10, 1]$$

$$[9/10, \quad 9]$$

$$[\quad 9/5, \quad 0]$$

$$[\quad 9, \quad 0]$$

Hence the system is **stable**

Inverse Laplace Transform

G(s) =

$$\frac{s + 10}{\frac{1}{10}s^3 + \frac{9}{10}s^2 + s + 9}$$

G factored:

Zero/pole/gain:

$$\frac{10(s+10)}{(s+9)(s^2 + 10)}$$

Inverse Laplace Transform:

$$g(t) = -10989/100000 \cdot \exp(-2251801791980457/40564819207303340847894502572032 \cdot t) \cdot \cos(79057/25000 \cdot t) + 868757373/250000000 \cdot \exp(-2251801791980457/40564819207303340847894502572032 \cdot t) \cdot \sin(79057/25000 \cdot t) + 10989/100000 \cdot \exp(-9 \cdot t)$$

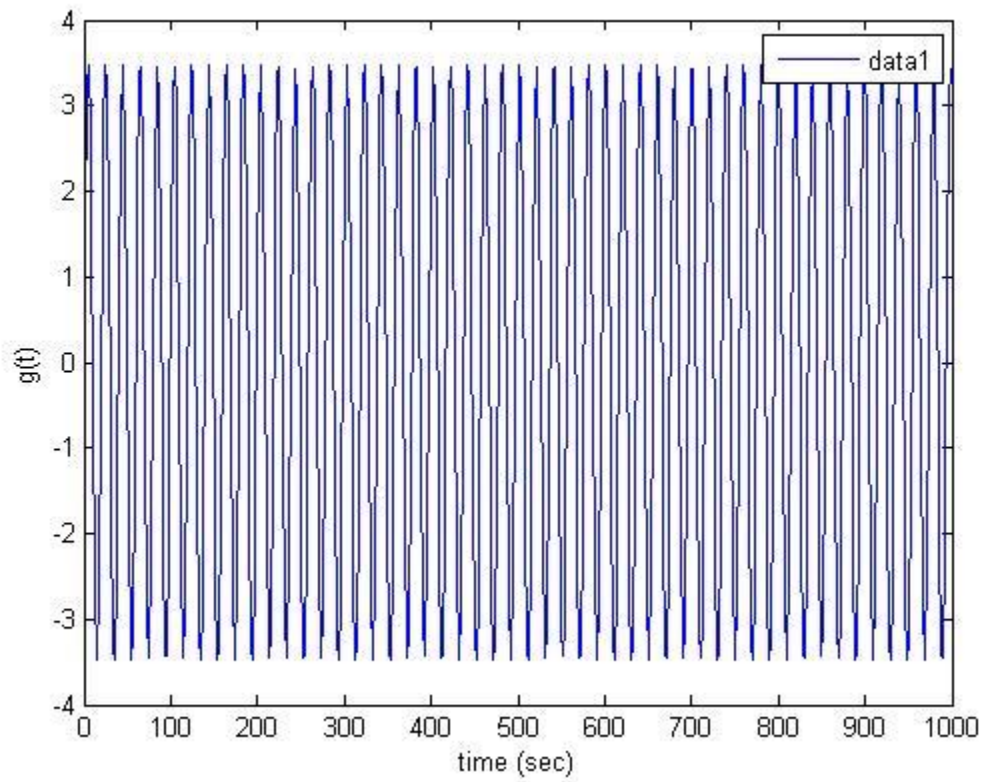
Use this MATLAB code to plot the time response:

```
for i=1:1000
    t=0.1*i;
    tf(i)=-10989/100000*exp(-
2251801791980457/40564819207303340847894502572032*t)*cos(79057/25000*t)+8
68757373/250000000*exp(-
2251801791980457/40564819207303340847894502572032*t)*sin(79057/25000*t)+10
989/100000*exp(-9*t);
```

end

figure(3)

plot(1:1000,tf)



lets choose $\frac{g}{l} = 10$ and $\tau = 0.1$.

- 5-14.** The block diagram of a motor-control system with tachometer feedback is shown in Fig. 5P-14. Find the range of the tachometer constant K_t so that the system is asymptotically stable.

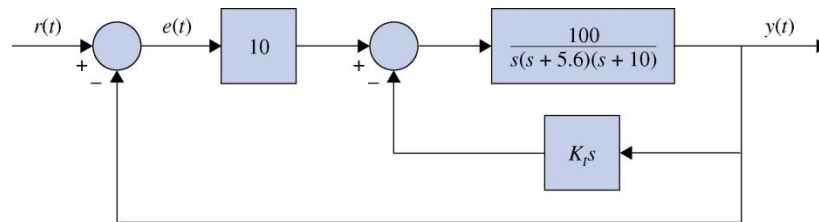


Figure 5P-14

Solution: Using block diagram reduction, the transfer function of the system is:

$$\frac{Y(s)}{R(s)} = \frac{1000}{s^3 + 15.6s^2 + (56 + 100K_t)s + 1000}$$

The characteristic equation is: $s^3 + 15.6s^2 + (56 + 100K_t)s + 1000 = 0$

Routh Tabulation:

s^3	1	$56 + 100K_t$	
s^2	15.6	1000	
s^1	$\frac{873.6 + 1560K_t - 1000}{15.6}$		$1560K_t - 126.4 > 0$
s^0	1000		

Stability Requirements: $K_t > 0.081$

- 5-15.** The block diagram of a control system is shown in Fig. 5P-15. Find the region in the K -versus- α plane for the system to be asymptotically stable. (Use K as the vertical and α as the horizontal axis.)

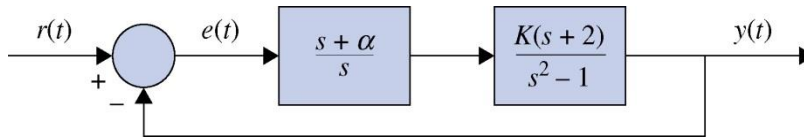


Figure 5P-15

Solution: Using block diagram reduction, the closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{K(s+2)(s+\alpha)}{s^3 + Ks^2 + (2K + \alpha K - 1)s + 2\alpha K}$$

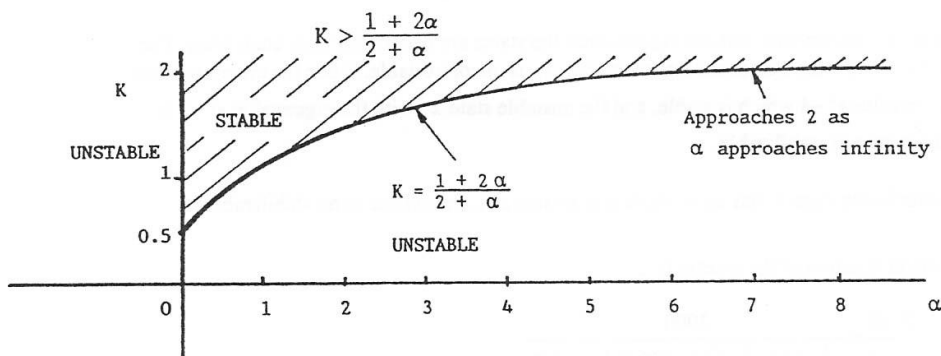
The characteristic equation: $s^3 + Ks^2 + (2K + \alpha K - 1)s + 2\alpha K = 0$

Routh Tabulation:

s^3	1	$2K + \alpha K - 1$	
s^2	K	$2\alpha K$	$K > 0$
s^1	$\frac{(2 + \alpha)K^2 - K - 2\alpha K}{K}$		$(2 + \alpha)K - 1 - 2\alpha > 0$
s^0	$2\alpha K$		$\alpha > 0$

Stability Requirements: $\alpha > 0, \quad K > 0, \quad K > \frac{1 + 2\alpha}{2 + \alpha}.$

K-versus- α Parameter Plane:



5-16. The conventional Routh-Hurwitz criterion gives information only on the location of the zeros of a polynomial $F(s)$ with respect to the left half and right half of the s -plane. Devise a linear transformation $s = f(p, \alpha)$, where p is a complex

variable, so that the Routh-Hurwitz criterion can be applied to determine whether $F(s)$ has zeros to the right of the line $s = -\alpha$, where α is a positive real number. Apply the transformation to the following characteristic equations to determine how many roots are to the right of the line $s = -1$ in the s -plane.

(a) $F(s) = s^2 + 5s + 3 = 0$

(b) $s^3 + 3s^2 + 3s + 1 = 0$

(c) $F(s) = s^3 + 4s^2 + 3s + 10 = 0$

(d) $s^3 + 4s^2 + 4s + 4 = 0$

Let $s_1 = s + \alpha$, then when $s = -\alpha$, $s_1 = 0$. This transforms the $s = -\alpha$ axis in the s -plane onto the imaginary axis of the s_1 -plane.

(a) $F(s) = s^2 + 5s + 3 = 0$ Let $s = s_1 - 1$ We get $(s_1 - 1)^2 + 5(s_1 - 1) + 3 = 0$
Or $s_1^2 + 3s_1 - 1 = 0$

Routh Tabulation:	s_1^2	1	-1
	s_1^1	3	
	s_1^0	-1	

Since there is one sign change in the first column of the Routh tabulation, there is one root in the region to the right of $s = -1$ in the s -plane. The roots are at -3.3028 and 0.3028 .

(b) $F(s) = s^3 + 3s^2 + 3s + 1 = 0$ Let $s = s_1 - 1$ We get
 $(s_1 - 1)^3 + 3(s_1 - 1)^2 + 3(s_1 - 1) + 1 = 0$
Or $s_1^3 = 0$. The three roots in the s_1 -plane are all at $s_1 = 0$. Thus, $F(s)$ has three roots at $s = -1$.

(c) $F(s) = s^3 + 4s^2 + 3s + 10 = 0$ Let $s = s_1 - 1$ We get
 $(s_1 - 1)^3 + 4(s_1 - 1)^2 + 3(s_1 - 1) + 10 = 0$
Or $s_1^3 + s_1^2 - 2s_1 + 10 = 0$

Routh Tabulation:	s_1^3	1	-2
	s_1^2	1	10
	s_1^1	-12	
	s_1^0	10	

in the Since there are two sign changes in the first column of the Routh tabulation, $F(s)$ has two roots region to the right of $s = -1$ in the s -plane. The roots are at -3.8897 , $-0.0552 + j1.605$, and $-0.0552 - j1.6025$.

(d) $F(s) = s^3 + 4s^2 + 4s + 4 = 0$ Let $s = s_1 - 1$ We get
 $(s_1 - 1)^3 + 4(s_1 - 1)^2 + 4(s_1 - 1) + 4 = 0$
 Or $s_1^3 + s_1^2 - s_1 + 3 = 0$

Routh Tabulation:

s_1^3	1	-1
s_1^2	1	3
s_1^1	-4	
s_1^0	3	

in the Since there are two sign changes in the first column of the Routh tabulation, $F(s)$ has two roots region to the right of $s = -1$ in the s -plane. The roots are at -3.1304 , $-0.4348 + j1.0434$, and $-0.4348 - j1.04348$.

5-17. The payload of a space-shuttle-pointing control system is modeled as a pure mass M . The payload is suspended by magnetic bearings so that no friction is encountered in the control. The attitude of the payload in the y direction is controlled by magnetic actuators located at the base. The total force produced by the magnetic actuators is $f(t)$. The controls of the other degrees of motion are independent and are not considered here. Because there are experiments located on the payload, electric power must be brought to the payload through cables. The linear spring with spring constant K_s is used to model the cable attachment. The dynamic system model for the control of the y -axis motion is shown in Figure 5P-17. The force equation of motion in the y -direction is

$$f(t) = K_s y(t) + M \frac{d^2 y(t)}{dt^2}$$

where $K_s = 0.5 \text{ N-m/m}$ and $M = 500 \text{ kg}$. The magnetic actuators are controlled through state feedback, so that

$$f(t) = -K_p y(t) - K_D \frac{dy(t)}{dt}$$

(a) Draw a functional block diagram for the system.

- (b) Find the characteristic equation of the closed-loop system.
- (c) Find the region in the K_D -versus- K_P plane in which the system is asymptotically stable.

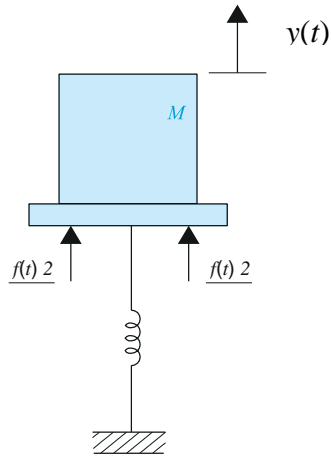
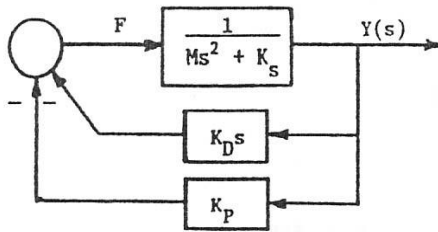


Figure 5P-17

(a) Block diagram:



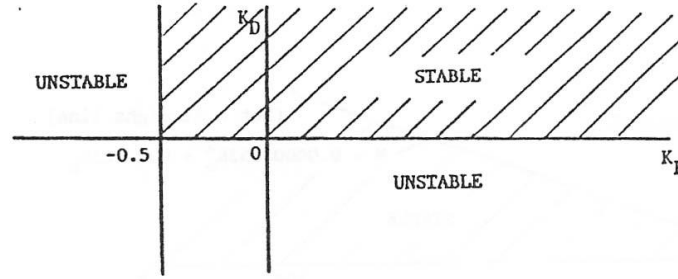
(b) Characteristic equation:

$$Ms^2 + K_D s + K_s + K_P = 0$$

$$500s^2 + K_D s + 500 + K_P = 0$$

(c) For stability, $K_D > 0$, $0.5 + K_P > 0$. Thus, $K_P > -0.5$

Stability Region:



5-18. An inventory-control system is modeled by the following differential equations:

$$\frac{dx_1(t)}{dt} = -x_2(t) + u(t)$$

$$\frac{dx_2(t)}{dt} = -Ku(t)$$

where $x_1(t)$ is the level of inventory; $x_2(t)$, the rate of sales of product; $u(t)$, the production rate; and K , a real constant. Let the output of the system be $y(t) = x_1(t)$ and $r(t)$ be the reference set point for the desired inventory level. Let $u(t) = r(t) - y(t)$. Determine the constraint on K so that the closed-loop system is asymptotically stable.

5-19. Use MATLAB to solve Problem 5-18.

5-20. Use MATLAB to

(a) Generate symbolically the time function of $f(t)$

$$f(t) = 5 + 2e^{-2t} \sin\left(2t + \frac{\pi}{4}\right) - 4e^{-2t} \cos\left(2t - \frac{\pi}{2}\right) + 3e^{-4t}$$

(b) Generate symbolically $G(s) = \frac{(s+1)}{s(s+2)(s^2+2s+2)}$

(c) Find the Laplace transform of $f(t)$ and name it $F(s)$.

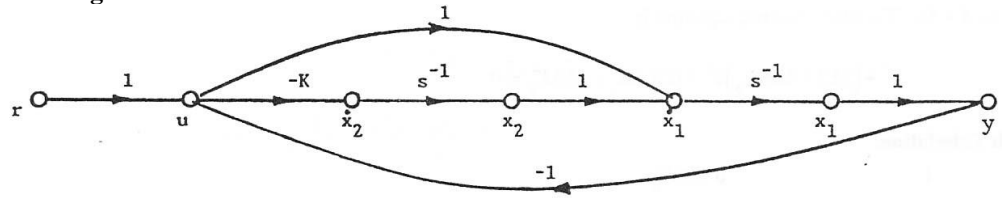
(d) Find the inverse Laplace transform of $G(s)$ and name it $g(t)$.

(e) If $G(s)$ is the forward-path transfer function of unity-feedback control systems, find the transfer function of the closed-loop system and apply the Routh-Hurwitz criterion to determine its stability.

(f) If $F(s)$ is the forward-path transfer function of unity-feedback

control systems, find the transfer function of the closed-loop system and apply the Routh-Hurwitz criterion to determine its stability.

(g) State diagram:



(h)

(i)

(j)

(k)

(l)

Characteristic equation:

$$\Delta = 1 + s + Ks^2$$

(m)

$$s^2 + s + K = 0$$

(n)

(o)

Stability requirement:

$$K > 0$$

CHAPTER 6

IMPORTANT COMPONENTS OF FEEDBACK CONTROL SYSTEMS

PROBLEMS

6-1. Write the force equations of the linear translational systems shown in Fig. 6P-1.

(a) Draw state diagrams using a minimum number of integrators. Write the state equations from the state diagrams.

(b) Define the state variables as follows:

(i) $x_1 = y_2$, $x_2 = dy_2/dt$, $x_3 = y_1$, and $x_4 = dy_1/dt$

(ii) $x_1 = y_2$, $x_2 = y_1$, and $x_3 = dy_1/dt$

(iii) $x_1 = y_1$, $x_2 = y_2$, and $x_3 = dy_2/dt$

Write the state equations and draw the state diagram with these state variables. Find the transfer functions $Y_1(s)/F(s)$ and $Y_2(s)/F(s)$.

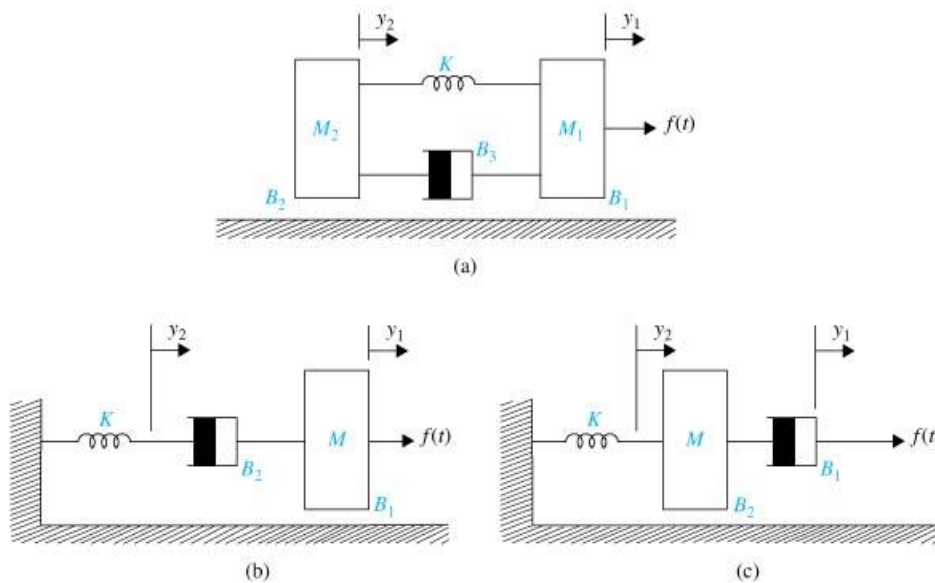


Figure 6P-1

(a) Force equations:

$$f(t) = M_1 \frac{d^2 y_1}{dt^2} + B_1 \frac{dy_1}{dt} + B_3 \left(\frac{dy_1}{dt} - \frac{dy_2}{dt} \right) + K(y_1 - y_2)$$

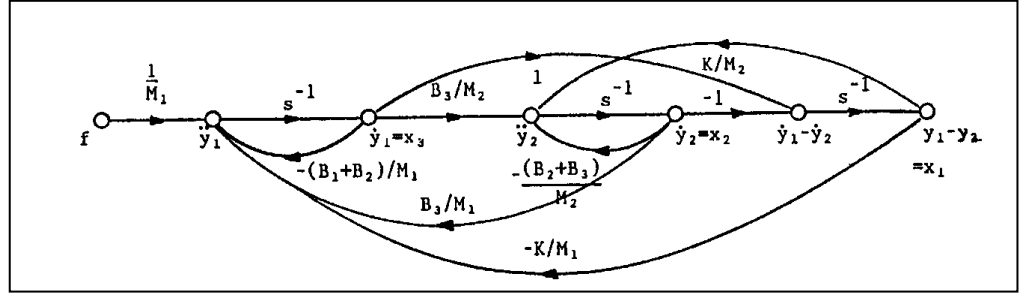
$$B_3 \left(\frac{dy_1}{dt} - \frac{dy_2}{dt} \right) + K(y_1 - y_2) + M_2 \frac{d^2 y_2}{dt^2} + B_2 \frac{dy_2}{dt}$$

Rearrange the equations as follows:

$$\frac{d^2 y_1}{dt^2} = -\frac{(B_1 + B_3)}{M_1} \frac{dy_1}{dt} + \frac{B_3}{M_1} \frac{dy_2}{dt} - \frac{K}{M_1} (y_1 - y_2) + \frac{f}{M_1}$$

$$\frac{d^2 y_2}{dt^2} = \frac{B_3}{M_2} \frac{dy_1}{dt} - \frac{(B_2 + B_3)}{M_2} \frac{dy_2}{dt} + \frac{K}{M_2} (y_1 - y_2)$$

(i) State diagram: Since $y_1 - y_2$ appears as one unit, the minimum number of integrators is three.



State equations: Define the state variables as $x_1 = y_1 - y_2$, $x_2 = \frac{dy_2}{dt}$, $x_3 = \frac{dy_1}{dt}$.

$$\frac{dx_1}{dt} = -x_2 + x_3 \quad \frac{dx_2}{dt} = \frac{K}{M_2} x_1 - \frac{(B_2 + B_3)}{M_2} x_2 + \frac{B_3}{M_2} x_3 \quad \frac{dx_3}{dt} = -\frac{K}{M_1} x_1 + \frac{B_3}{M_1} x_2 - \frac{(B_1 + B_3)}{M_1} x_3 + \frac{1}{M} f$$

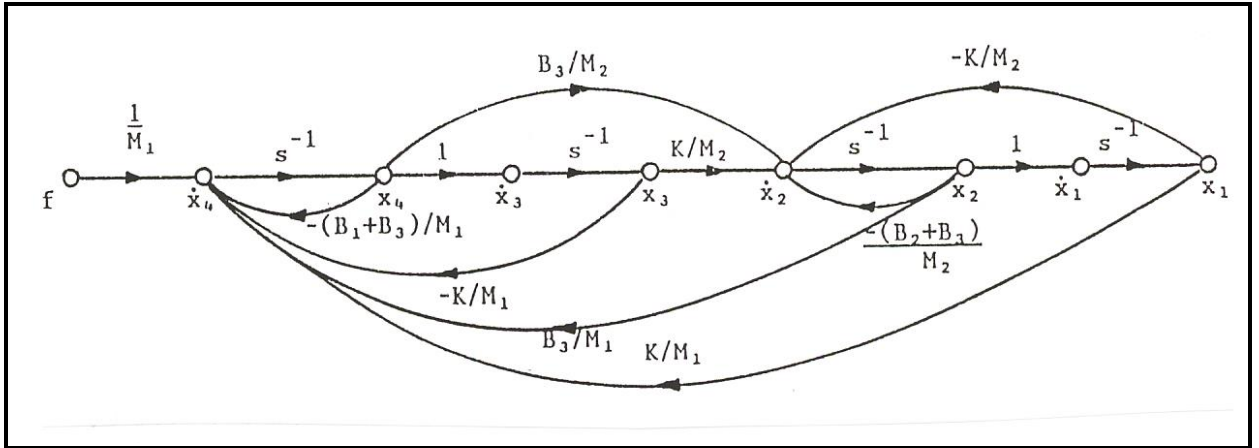
(ii) State variables: $x_1 = y_2$, $x_2 = \frac{dy_2}{dt}$, $x_3 = y_1$, $x_4 = \frac{dy_1}{dt}$.

State equations:

$$\frac{dx_1}{dt} = x_2 \quad \frac{dx_2}{dt} = -\frac{K}{M_2} x_1 - \frac{B_2 + B_3}{M_2} x_2 + \frac{K}{M_2} x_3 + \frac{B_3}{M_2} x_4$$

$$\frac{dx_3}{dt} = x_4 \quad \frac{dx_4}{dt} = \frac{K}{M_1} x_1 + \frac{B_3}{M_1} x_2 - \frac{K}{M_1} x_3 - \frac{B_1 + B_3}{M_1} x_4 + \frac{1}{M_1} f$$

State diagram:



Transfer functions:

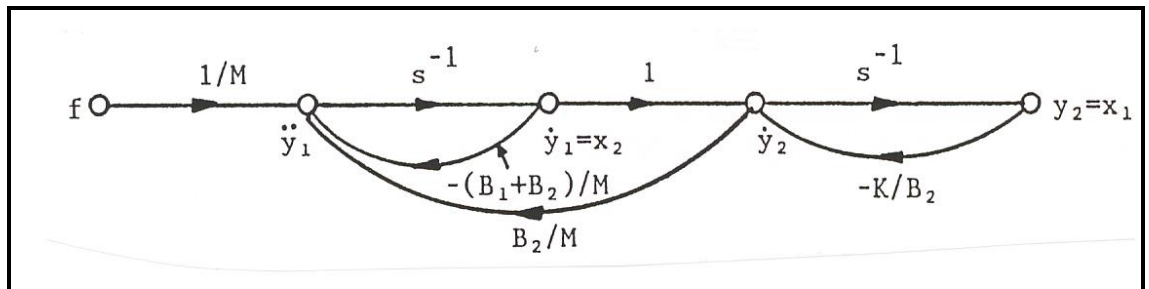
$$\frac{Y_1(s)}{F(s)} = \frac{M_2 s^2 + (B_2 + B_3)s + K}{s \{ M_1 M_2 s^3 + [(B_1 + B_3)M_2 + (B_2 + B_3)M_1] s^2 + [K(M_1 + M_2) + B_1 B_2 + B_2 B_3 + B_1 B_3] s + (B_1 + B_2)K \}}$$

$$\frac{Y_2(s)}{F(s)} = \frac{B_3 s + K}{s \{ M_1 M_2 s^3 + [(B_1 + B_3)M_2 + (B_2 + B_3)M_1] s^2 + [K(M_1 + M_2) + B_1 B_2 + B_2 B_3 + B_1 B_3] s + (B_1 + B_2)K \}}$$

(b) Force equations:

$$\frac{d^2 y_1}{dt^2} = -\frac{(B_1 + B_2)}{M} \frac{dy_1}{dt} + \frac{B_2}{M} \frac{dy_2}{dt} + \frac{1}{M} f \quad \frac{dy_2}{dt} = \frac{dy_1}{dt} - \frac{K}{B_2} y_2$$

(i) State diagram:



Define the outputs of the integrators as state variables, $x_1 = y_2$, $x_2 = \frac{dy_1}{dt}$.

State equations:

$$\frac{dx_1}{dt} = -\frac{K}{B_2} x_1 + x_2 \quad \frac{dx_2}{dt} = -\frac{K}{M} x_1 - \frac{B_1}{M} x_2 + \frac{1}{M} f$$

(ii) **State equations: State variables:** $x_1 = y_2, \quad x_2 = \dot{y}_1, \quad x_3 = \frac{dy_1}{dt}$.

$$\frac{dx_1}{dt} = -\frac{K}{B_2}x_1 + x_3 \quad \frac{dx_2}{dt} = x_3 \quad \frac{dx_3}{dt} = -\frac{K}{M}x_1 - \frac{B_1}{M}x_3 + \frac{1}{M}f$$

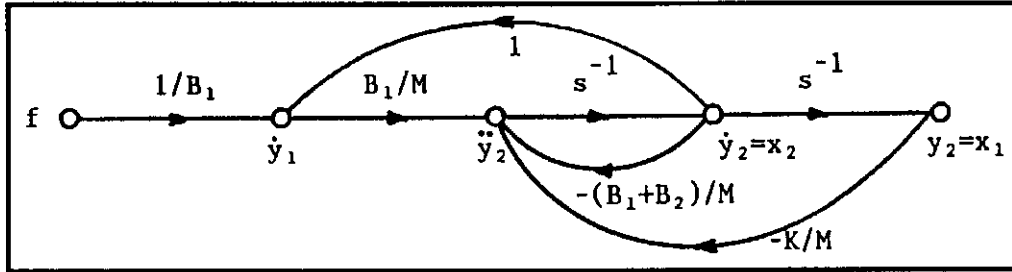
Transfer functions:

$$\frac{Y_1(s)}{F(s)} = \frac{B_2 s + K}{s[MB_2 s^2 + (B_1 B_2 + KM)s + (B_1 + B_2)K]} \quad \frac{Y_2(s)}{F(s)} = \frac{B_2}{MB_2 s^2 + (B_1 B_2 + KM)s + (B_1 + B_2)K}$$

(c) **Force equations:**

$$\frac{dy_1}{dt} = \frac{dy_2}{dt} + \frac{1}{B_1}f \quad \frac{d^2 y_2}{dt^2} = -\frac{(B_1 + B_2)}{M} \frac{dy_2}{dt} + \frac{B_1}{M} \frac{dy_2}{dt} + \frac{B_1}{M} \frac{dy_1}{dt} - \frac{K}{M} y_2$$

(i) **State diagram:**



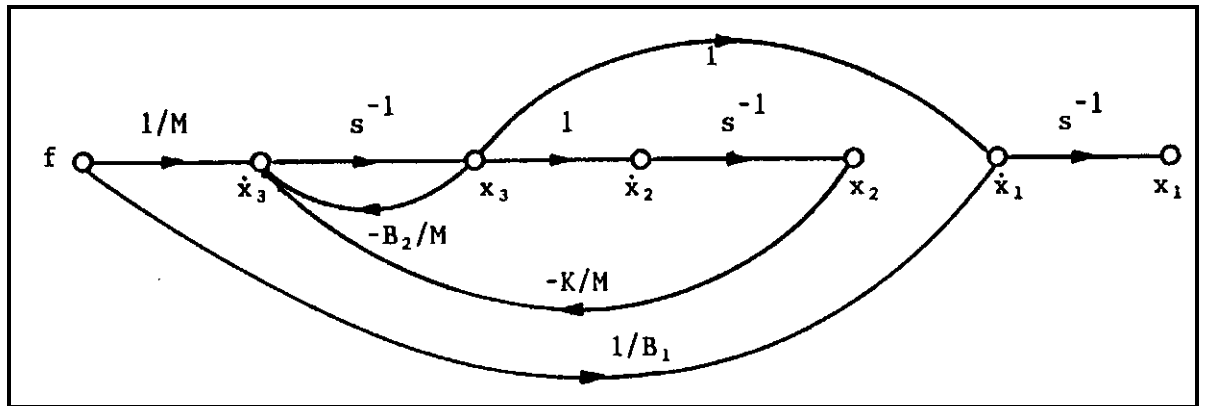
State equations: Define the outputs of integrators as state variables.

$$\frac{dx_1}{dt} = x_2 \quad \frac{dx_2}{dt} = -\frac{K}{M}x_1 - \frac{B_2}{M}x_2 + \frac{1}{M}f$$

(ii) **State equations: state variables:** $x_1 = y_1, \quad x_2 = y_2, \quad x_3 = \frac{dy_2}{dt}$.

$$\frac{dx_1}{dt} = x_3 + \frac{1}{B_1}f \quad \frac{dx_2}{dt} = x_3 \quad \frac{dx_3}{dt} = -\frac{K}{M}x_2 - \frac{B_2}{M}x_3 + \frac{1}{M}f$$

State diagram:



Transfer functions:

$$\frac{Y_1(s)}{F(s)} = \frac{Ms^2 + (B_1 + B_2)s + K}{B_1s(Ms^2 + B_2s + K)} \quad \frac{Y_2(s)}{F(s)} = \frac{1}{Ms^2 + B_2s + K}$$

6-2. Write the force equations of the linear translational system shown in Fig. 6P-2. Draw the state diagram using a minimum number of integrators. Write the state equations from the state diagram. Find the transfer functions $Y_1(s)/F(s)$ and $Y_2(s)/F(s)$. Set $Mg = 0$ for the transfer functions.

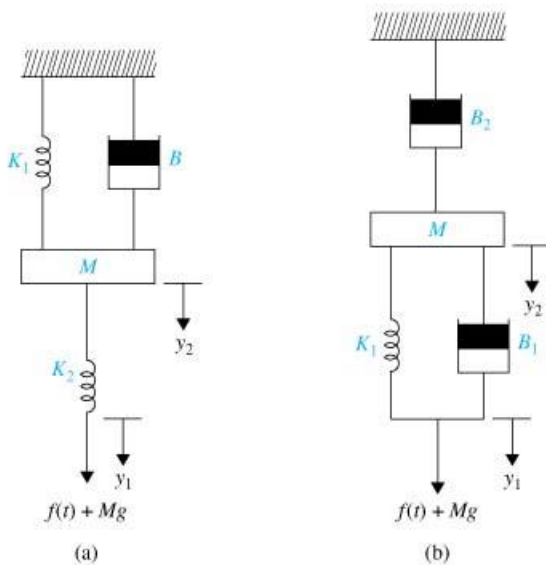
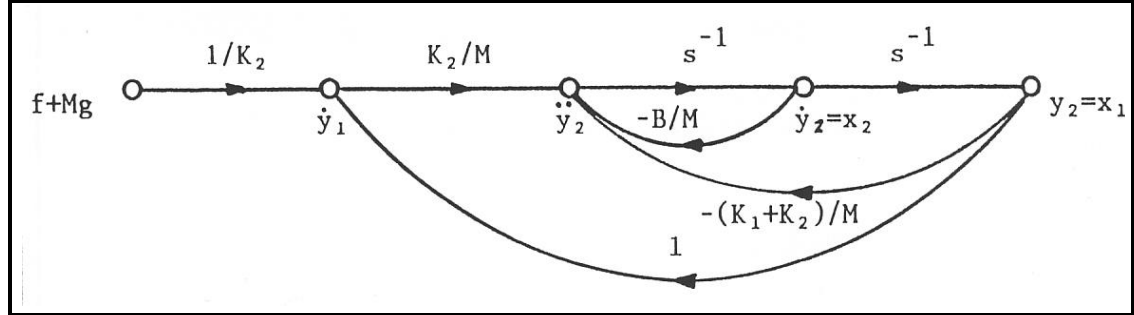


Figure 6P-2

(a) Force equations:

$$y_1 = \frac{1}{K_2}(f + Mg) + y_2 \quad \frac{d^2 y_2}{dt^2} = -\frac{B}{M} \frac{dy_2}{dt} - \frac{K_1 + K_2}{M} y_2 + \frac{K_2}{M} y_1$$

State diagram:



State equations: Define the state variables as: $x_1 = y_2$, $x_2 = \frac{dy_2}{dt}$.

$$\frac{dx_1}{dt} = x_2 \quad \frac{dx_2}{dt} = -\frac{K_1}{M} x_1 - \frac{B}{M} x_2 + \frac{1}{M}(f + Mg)$$

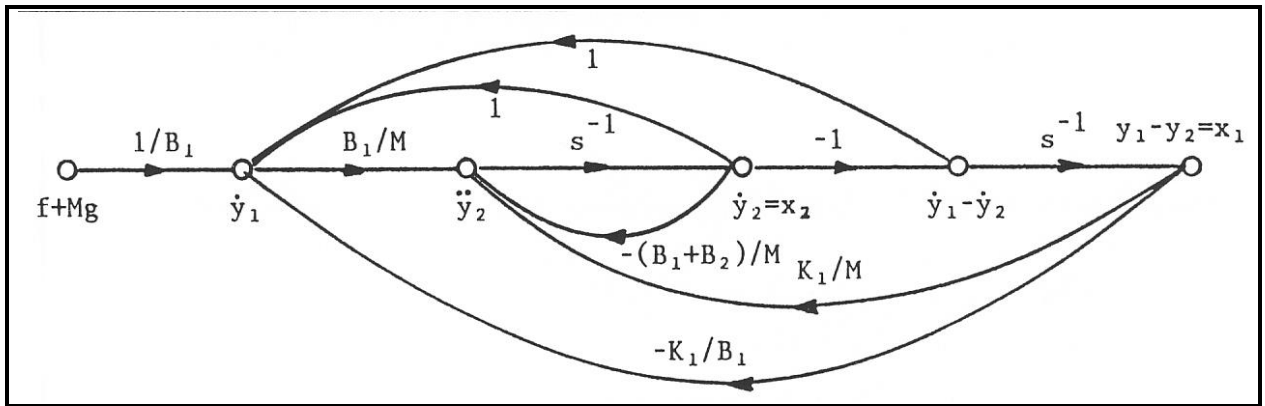
Transfer functions:

$$\frac{Y_1(s)}{F(s)} = \frac{s^2 + Bs + K_1 + K_2}{K_2(Ms^2 + Bs + K_1)} \quad \frac{Y_2(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K_1}$$

(b) Force equations:

$$\frac{dy_1}{dt} = \frac{1}{B_1} [f(t) + Mg] + \frac{dy_2}{dt} - \frac{K_1}{B_1} (y_1 - y_2) \quad \frac{d^2 y_2}{dt^2} = \frac{B_1}{M} \left(\frac{dy_1}{dt} - \frac{dy_2}{dt} \right) + \frac{K_1}{M} (y_1 - y_2) - \frac{B_2}{M} (y_1 - y_2) - \frac{B_2}{M} \frac{dy_2}{dt}$$

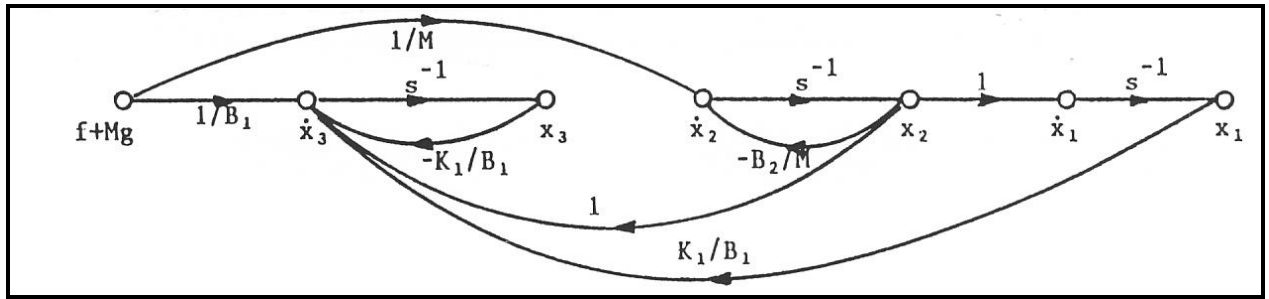
State diagram: (With minimum number of integrators)



To obtain the transfer functions $Y_1(s)/F(s)$ and $Y_2(s)/F(s)$, we need to redefine the state variables as:

$$x_1 = y_2, \quad x_2 = dy_2/dt, \text{ and } x_3 = y_1.$$

State diagram:



Transfer functions:

$$\frac{Y_1(s)}{F(s)} = \frac{Ms^2 + (B_1 + B_2)s + K_1}{s^2 [MB_1s + (B_1B_2 + MK_1)]} \quad \frac{Y_2(s)}{F(s)} = \frac{Bs + K_1}{s^2 [MB_1s + (B_1B_2 + MK_1)]}$$

6-3. Write the torque equations of the rotational systems shown in Fig. 6P-3. Draw state diagrams using a minimum number of integrators. Write the state equations from the state diagrams. Find the transfer function $\Theta(s)/T(s)$ for the system in (a). Find the transfer functions $\Theta_1(s)/T(s)$ and $\Theta_2(s)/T(s)$ for the systems in parts (b), (c), (d), and (e).

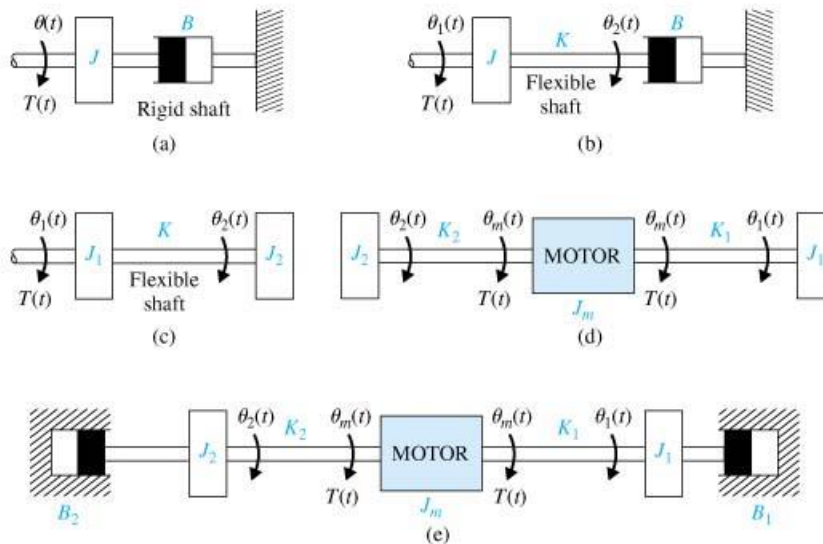


Figure 6P-3

(a) Torque equation:

State diagram:

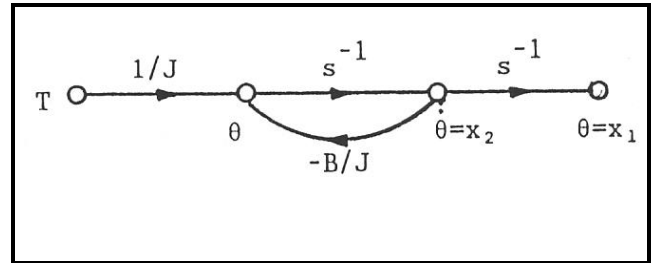
$$\frac{d^2\theta}{dt^2} = -\frac{B}{J} \frac{d\theta}{dt} + \frac{1}{J} T(t)$$

State equations:

$$\frac{dx_1}{dt} = x_2 \quad \frac{dx_2}{dt} = -\frac{B}{J} x_2 + \frac{1}{J} T$$

Transfer function:

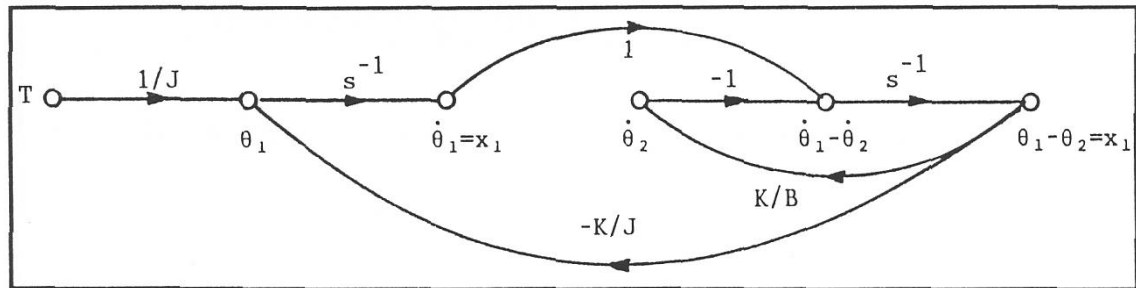
$$\frac{\Theta(s)}{T(s)} = \frac{1}{s(Js + B)}$$



(b) Torque equations:

$$\frac{d^2\theta_1}{dt^2} = -\frac{K}{J}(\theta_1 - \theta_2) + \frac{1}{J} T \quad K(\theta_1 - \theta_2) = B \frac{d\theta_2}{dt}$$

State diagram: (minimum number of integrators)



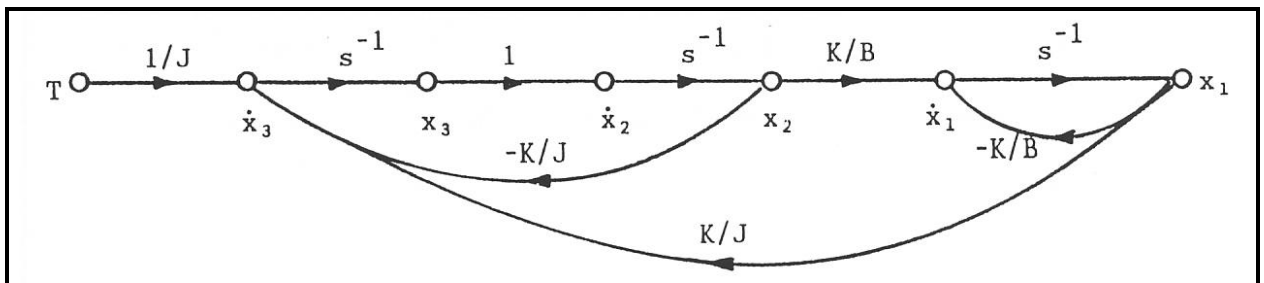
State equations:

$$\frac{dx_1}{dt} = -\frac{K}{B} x_1 + x_2 \quad \frac{dx_2}{dt} = -\frac{K}{J} x_1 + \frac{1}{J} T$$

State equations: Let $x_1 = \theta_2$, $x_2 = \theta_1$, and $x_3 = \frac{d\theta_1}{dt}$.

$$\frac{dx_1}{dt} = -\frac{K}{B} x_1 + \frac{K}{B} x_2 \quad \frac{dx_2}{dt} = x_3 \quad \frac{dx_3}{dt} = \frac{K}{J} x_1 - \frac{K}{J} x_2 + \frac{1}{J} T$$

State diagram:



Transfer functions:

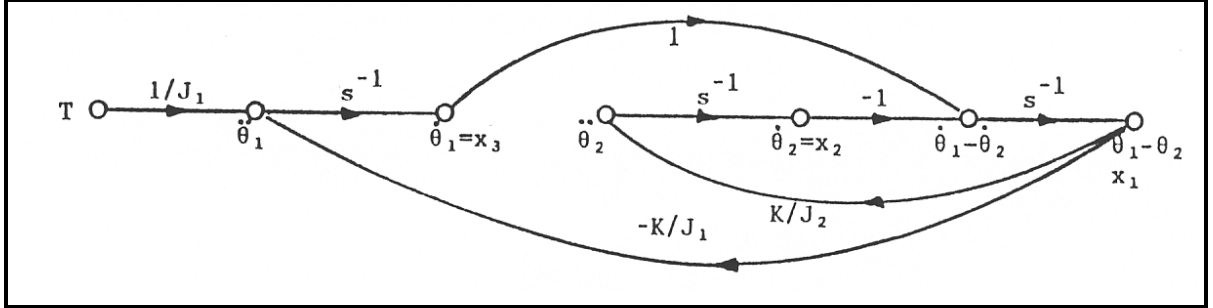
$$\frac{\Theta_1(s)}{T(s)} = \frac{Bs + K}{s(BJs^2 + JKs + BK)}$$

$$\frac{\Theta_2(s)}{T(s)} = \frac{K}{s(BJs^2 + JKs + BK)}$$

(c) Torque equations:

$$T(t) = J_1 \frac{d^2 \theta_1}{dt^2} + K(\theta_1 - \theta_2) \quad K(\theta_1 - \theta_2) = J_2 \frac{d^2 \theta_2}{dt^2}$$

State diagram:



State equations: state variables: $x_1 = \theta_2$, $x_2 = \frac{d\theta_2}{dt}$, $x_3 = \theta_1$, $x_4 = \frac{d\theta_1}{dt}$.

$$\frac{dx_1}{dt} = x_2 \quad \frac{dx_2}{dt} = -\frac{K}{J_2}x_1 + \frac{K}{J_2}x_3 \quad \frac{dx_3}{dt} = x_4 \quad \frac{dx_4}{dt} = \frac{K}{J_1}x_1 - \frac{K}{J_1}x_3 + \frac{1}{J_1}T$$

Transfer functions:

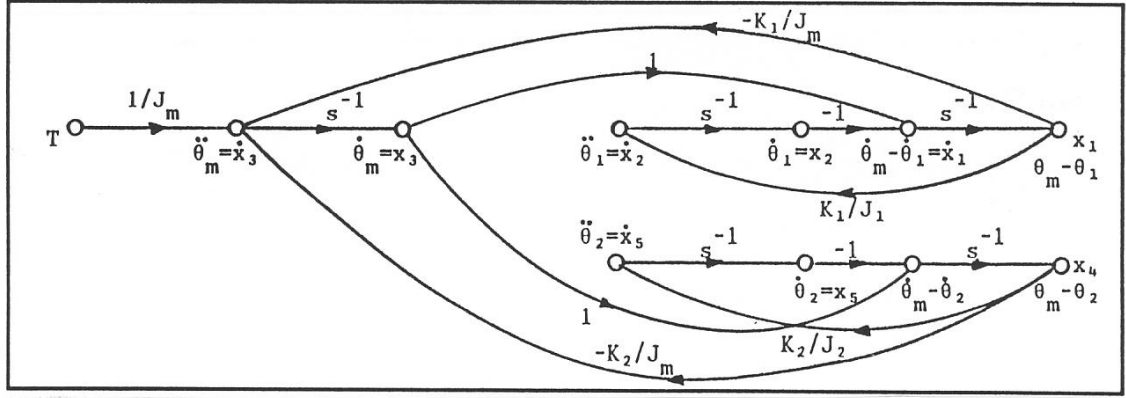
$$\frac{\Theta_1(s)}{T(s)} = \frac{J_2 s^2 + K}{s^2 [J_1 J_2 s^2 + K(J_1 + J_2)]}$$

$$\frac{\Theta_2(s)}{T(s)} = \frac{K}{s^2 [J_1 J_2 s^2 + K(J_1 + J_2)]}$$

(d) Torque equations:

$$T(t) = J_m \frac{d^2 \theta_m}{dt^2} + K_1(\theta_m - \theta_1) + K_2(\theta_m - \theta_2) \quad K_1(\theta_m - \theta_1) = J_1 \frac{d^2 \theta_1}{dt^2} \quad K_2(\theta_m - \theta_2) = J_2 \frac{d^2 \theta_2}{dt^2}$$

State diagram:



State equations: $x_1 = \theta_m - \theta_1$, $x_2 = \frac{d\theta_1}{dt}$, $x_3 = \frac{d\theta_m}{dt}$, $x_4 = \theta_m - \theta_2$, $x_5 = \frac{d\theta_2}{dt}$.

$$\frac{dx_1}{dt} = -x_2 + x_3 \quad \frac{dx_2}{dt} = \frac{K_1}{J_1} x_1 \quad \frac{dx_3}{dt} = -\frac{K_1}{J_m} x_1 - \frac{K_2}{J_m} x_4 + \frac{1}{J_m} T \quad \frac{dx_4}{dt} = x_3 - x_5 \quad \frac{dx_5}{dt} = \frac{K_2}{J_2} x_4$$

Transfer functions:

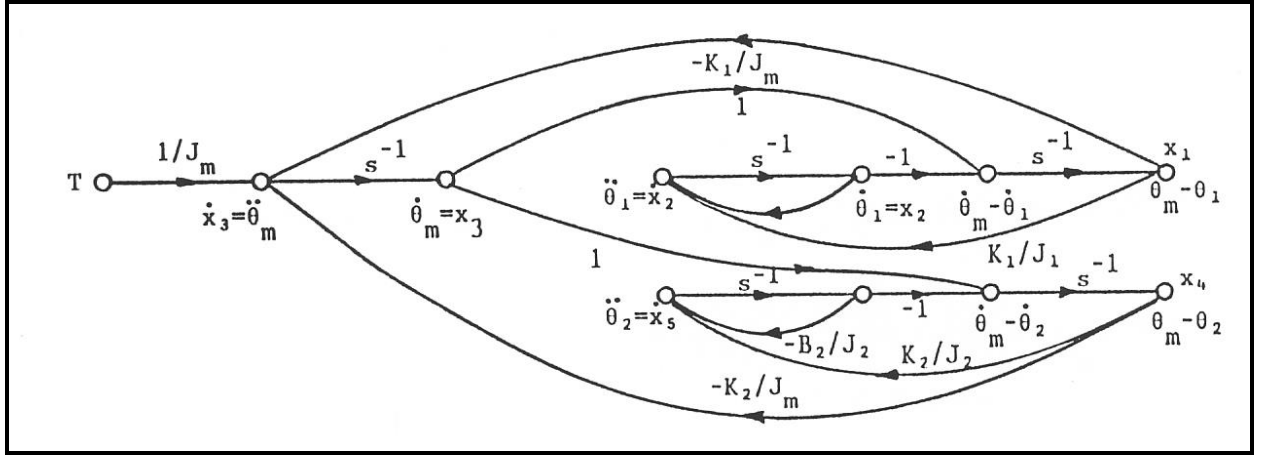
$$\frac{\Theta_1(s)}{T(s)} = \frac{K_1(J_2 s^2 + K_2)}{s^2 \left[s^4 + (K_1 J_2 J_m + K_2 J_1 J_m + K_1 J_1 J_2 + K_2 J_1 J_2) s^2 + K_1 K_2 (J_m + J_1 + J_2) \right]}$$

$$\frac{\Theta_2(s)}{T(s)} = \frac{K_2(J_1 s^2 + K_1)}{s^2 \left[s^4 + (K_1 J_2 J_m + K_2 J_1 J_m + K_1 J_1 J_2 + K_2 J_1 J_2) s^2 + K_1 K_2 (J_m + J_1 + J_2) \right]}$$

(e) Torque equations:

$$\frac{d^2 \theta_m}{dt^2} = -\frac{K_1}{J_m} (\theta_m - \theta_1) - \frac{K_2}{J_m} (\theta_m - \theta_2) + \frac{1}{J_m} T \quad \frac{d^2 \theta_1}{dt^2} = \frac{K_1}{J_1} (\theta_m - \theta_1) - \frac{B_1}{J_1} \frac{d\theta_1}{dt} \quad \frac{d^2 \theta_2}{dt^2} = \frac{K_2}{J_2} (\theta_m - \theta_1) - \frac{B_2}{J_2} \frac{d\theta_2}{dt}$$

State diagram:



State variables: $x_1 = \theta_m - \theta_1$, $x_2 = \frac{d\theta_1}{dt}$, $x_3 = \frac{d\theta_m}{dt}$, $x_4 = \theta_m - \theta_2$, $x_5 = \frac{d\theta_2}{dt}$.

State equations:

$$\frac{dx_1}{dt} = -x_2 + x_3 \quad \frac{dx_2}{dt} = \frac{K_1}{J_1} x_1 - \frac{B_1}{J_1} x_2 \quad \frac{dx_3}{dt} = -\frac{K_1}{J_m} x_1 - \frac{K_2}{J_m} x_4 + \frac{1}{J_m} T \quad \frac{dx_4}{dt} = x_3 - x_5 \quad \frac{dx_5}{dt} = \frac{K_2}{J_2} x_4 - \frac{B_2}{J_2} x_5$$

Transfer functions:

$$\frac{\Theta_1(s)}{T(s)} = \frac{K_1 (J_2 s^2 + B_2 s + K_2)}{\Delta(s)} \quad \frac{\Theta_2(s)}{T(s)} = \frac{K_2 (J_1 s^2 + B_1 s + K_1)}{\Delta(s)}$$

$$\Delta(s) = s^2 \{ J_1 J_2 J_m s^4 + J_m (B_1 + B_2) s^3 + [(K_1 J_2 + K_2 J_1) J_m + (K_1 + K_2) J_1 J_2 + B_1 B_2 J_m] s^2 + [(B_1 K_2 + B_2 K_1) J_m + B_1 K_2 J_2 + B_2 K_1 J_1] s + K_1 K_2 (J_m + J_1 + J_2) \}$$

6-4. An open-loop motor control system is shown in Fig. 6P-4. The potentiometer has a maximum range of 10 turns ($20\pi\text{rad}$). Find the transfer functions $E_o(s)/T_m(s)$. The following parameters and variables are defined: $\theta_m(t)$ is the motor displacement; $\theta_L(t)$, the load displacement; $T_m(t)$, the motor torque; J_m , the motor inertia; B_m , the motor viscous-friction coefficient; B_p , the potentiometer viscous-friction coefficient; $e_o(t)$, the output voltage, and K , the torsional spring constant.

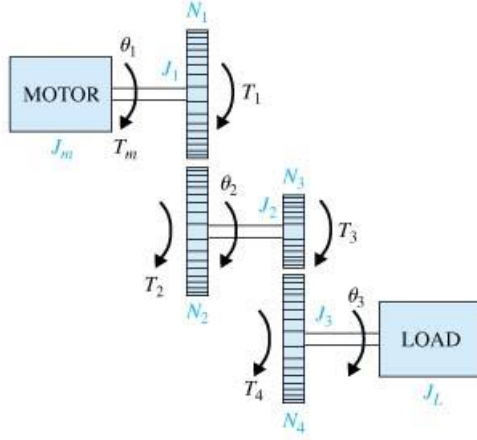


Figure 6P-5

(a)

$$T_m(t) = J_m \frac{d^2\theta_1}{dt^2} + T_1 \quad T_1 = \frac{N_1}{N_2} T_2 \quad T_3 = \frac{N_3}{N_4} T_4 \quad T_4 = J_L \frac{d^2\theta_3}{dt^2} \quad T_2 = T_3 \quad \theta_2 = \frac{N_1}{N_2} \theta_1$$

$$\theta_3 = \frac{N_1 N_3}{N_2 N_4} \theta_1 \quad T_2 = \frac{N_3}{N_4} T_4 = \frac{N_3}{N_4} J_L \frac{d^2\theta_3}{dt^2} \quad T_m = J_m \frac{d^2\theta_1}{dt^2} + \frac{N_1 N_3}{N_2 N_4} T_4 = \left[J_m + \left[\frac{N_1 N_3}{N_2 N_4} \right]^2 J_L \right] \frac{d^2\theta_1}{dt^2}$$

(b)

$$T_m = J_m \frac{d^2\theta_1}{dt^2} + T_1 \quad T_2 = J_2 \frac{d^2\theta_2}{dt^2} + T_3 \quad T_4 = (J_3 + J_L) \frac{d^2\theta_3}{dt^2} \quad T_1 = \frac{N_1}{N_2} T_2 \quad T_3 = \frac{N_3}{N_4} T_4$$

$$\theta_2 = \frac{N_1}{N_2} \theta_1 \quad \theta_3 = \frac{N_1 N_3}{N_2 N_4} \theta_1 \quad T_2 = J_2 \frac{d^2\theta_2}{dt^2} + \frac{N_3}{N_4} T_4 = J_2 \frac{d^2\theta_2}{dt^2} + \frac{N_3}{N_4} (J_3 + J_L) \frac{d^2\theta_3}{dt^2}$$

$$T_m(t) = J_m \frac{d^2\theta_1}{dt^2} + \frac{N_1}{N_2} \left(J_2 \frac{d^2\theta_2}{dt^2} + \frac{N_3}{N_4} (J_3 + J_L) \frac{d^2\theta_3}{dt^2} \right) = \left[J_m + \left(\frac{N_1}{N_2} \right)^2 J_2 + \left(\frac{N_1 N_3}{N_2 N_4} \right)^2 (J_3 + J_L) \right] \frac{d^2\theta_1}{dt^2}$$

6-6. A vehicle towing a trailer through a spring-damper coupling hitch is shown in Fig. 6P-6. The following parameters and variables are defined: M is the mass of the trailer; K_h , the spring constant of the hitch; B_h , the viscous damping coefficient of the hitch; B_t , the viscous-friction coefficient of the trailer; $y_1(t)$, the displacement of the towing vehicle; $y_2(t)$, the displacement of the trailer; and $f(t)$, the force of the towing vehicle.

(a) Write the differential equation of the system.

- (b) Write the state equations by defining the following state variables:
 $x_1(t) = y_1(t) - y_2(t)$ and $x_2(t) = \frac{dy_2(t)}{dt}$.

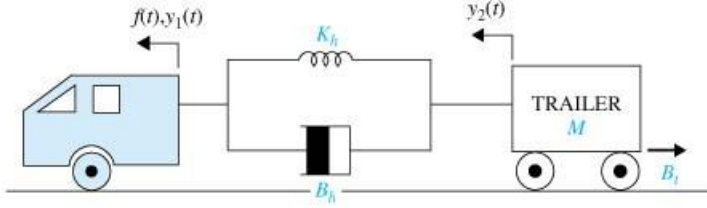


Figure 6P-6

- (a) Force equations:

$$f(t) = K_h (y_1 - y_2) + B_h \left(\frac{dy_1}{dt} - \frac{dy_2}{dt} \right) \quad K_h (y_1 - y_2) + B_h \left(\frac{dy_1}{dt} - \frac{dy_2}{dt} \right) = M \frac{d^2 y_2}{dt^2} + B_t \frac{dy_2}{dt}$$

- (b) State variables: $x_1 = y_1 - y_2$, $x_2 = \frac{dy_2}{dt}$

State equations:

$$\frac{dx_1}{dt} = -\frac{K_h}{B_h} x_1 + \frac{1}{B_h} f(t) \quad \frac{dx_2}{dt} = -\frac{B_t}{M} x_2 + \frac{1}{M} f(t)$$

6-7. Figure 6P-7 shows a motor-load system coupled through a gear train with gear ratio $n = N_1/N_2$. The motor torque is $T_m(t)$, and $T_L(t)$ represents a load torque.

- (a) Find the optimum gear ratio n^* such that the load acceleration $\alpha_L = d^2\theta_L/dt^2$ is maximized.

- (b) Repeat part (a) when the load torque is zero.

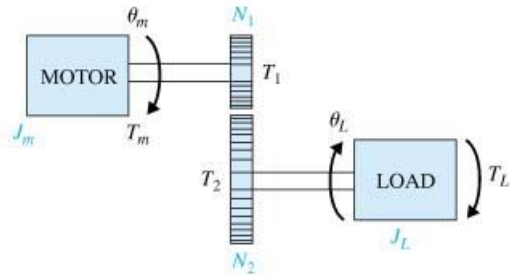


Figure 6P-7

- (a)

$$T_m = J_m \frac{d^2 \theta_m}{dt^2} + T_1 \quad T_2 = J_L \frac{d^2 \theta_L}{dt^2} + T_L \quad T_1 = \frac{N_1}{N_2} T_2 = n T_2 \quad \theta_m N_1 = \theta_L N_2$$

$$T_m = J_m \frac{d^2 \theta_m}{dt^2} + n J_L \frac{d^2 \theta_L}{dt^2} + n T_L = \left(\frac{J_m}{n} + n J_L \right) \alpha_L + n T_L \quad \text{Thus, } \alpha_L = \frac{n T_m - n^2 T_L}{J_m + n^2 J_L}$$

$$\text{Set } \frac{\partial \alpha_L}{\partial n} = 0. \quad (T_m - 2n T_L)(J_m + n^2 J_L) - 2n J_L (n T_m - n^2 T_L) = 0 \quad \text{Or,}$$

$$n^2 + \frac{J_m T_L}{J_L T_m} n - \frac{J_m}{J_L} = 0$$

$$\text{Optimal gear ratio: } n^* = -\frac{J_m T_L}{2 J_L T_m} + \frac{\sqrt{J_m^2 T_L^2 + 4 J_m J_L T_m^2}}{2 J_L T_m} \quad \text{where the + sign has been}$$

chosen.

(b) When $T_L = 0$, the optimal gear ratio is

$$n^* = \sqrt{J_m / J_L}$$

6-8. Figure 6P-8 shows the simplified diagram of the printwheel control system of a word processor. The printwheel is controlled by a dc motor through belts and pulleys. Assume that the belts are rigid. The following parameters and variables are defined: $T_m(t)$

is the motor torque; $\theta_m(t)$, the motor displacement; $y(t)$, the linear displacement of the printwheel; J_m , the motor inertia; B_m , the motor viscous-friction coefficient; r , the pulley radius; M , the mass of the printwheel.

(a) Write the differential equation of the system.

(b) Find the transfer function $Y(s)/T_m(s)$.

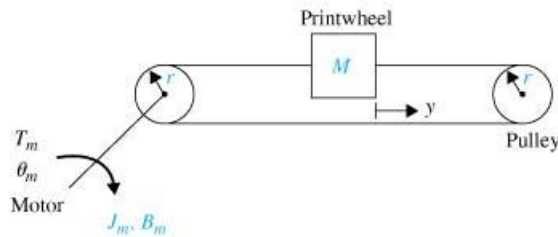


Figure 6P-8

(a) Torque equation about the motor shaft: Relation between linear and rotational displacements:

$$T_m = J_m \frac{d^2 \theta_m}{dt^2} + M r^2 \frac{d^2 \theta_m}{dt^2} + B_m \frac{d \theta_m}{dt} \quad y = r \theta_m$$

(b) Taking the Laplace transform of the equations in part (a), with zero initial conditions, we have

$$T_m(s) = (J_m + Mr^2)s^2\Theta_m(s) + B_ms\Theta_m(s) \quad Y(s) = r\Theta_m(s)$$

Transfer function:

$$\frac{Y(s)}{T_m(s)} = \frac{r}{s[(J_m + Mr^2)s + B_m]}$$

6-9. Figure 6P-9 shows the diagram of a printwheel system with belts and pulleys. The belts are modeled as linear springs with spring constants K_1 and K_2 .

- (a) Write the differential equations of the system using θ_m and y as the dependent variables.
- (b) Write the state equations using $x_1 = r\theta_m - y$, $x_2 = dy/dt$, and $x_3 = \omega_m = d\theta_m/dt$ as the state variables.
- (c) Draw a state diagram for the system.
- (d) Find the transfer function $Y(s)/T_m(s)$.
- (e) Find the characteristic equation of the system.

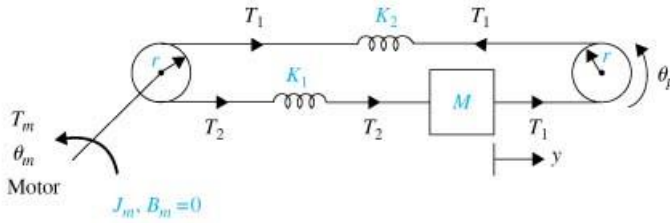


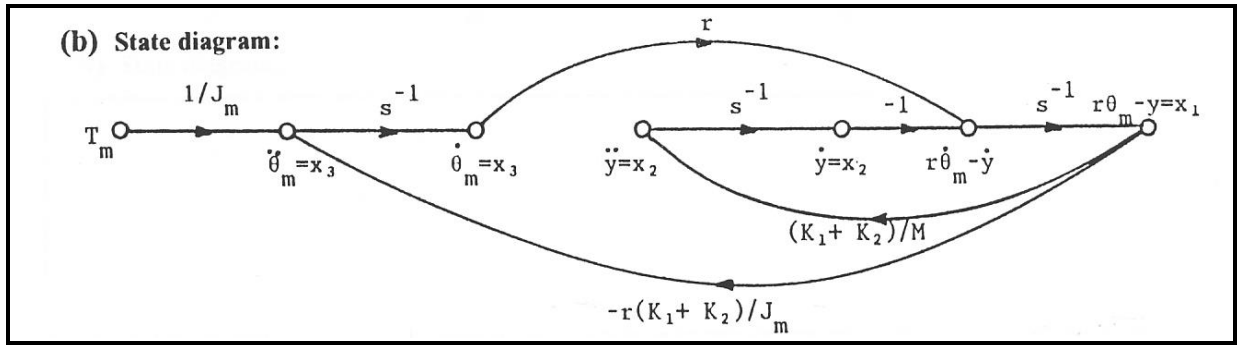
Figure 6P-9

(a)

$$T_m = J_m \frac{d^2\theta_m}{dt^2} + r(T_1 - T_2) \quad T_1 = K_2(r\theta_m - r\theta_p) = K_2(r\theta_m - y) \quad T_2 = K_1(y - r\theta_m)$$

$$T_1 - T_2 = M \frac{d^2y}{dt^2} \quad \text{Thus, } T_m = J_m \frac{d^2\theta_m}{dt^2} + r(K_1 + K_2)(r\theta_m - y)$$

$$M \frac{d^2y}{dt^2} = (K_1 + K_2)(r\theta_m - y)$$



(c) State equations:

$$\frac{dx_1}{dt} = rx_3 - x_2 \quad \frac{dx_2}{dt} = \frac{K_1 + K_2}{M} x_1 \quad \frac{dx_3}{dt} = \frac{-r(K_1 + K_2)}{J_m} x_1 + \frac{1}{J_m} T_m$$

(d) Transfer function:

$$\frac{Y(s)}{T_m(s)} = \frac{r(K_1 + K_2)}{s^2 [J_m Ms^2 + (K_1 + K_2)(J_m + rM)]}$$

(e) Characteristic equation:

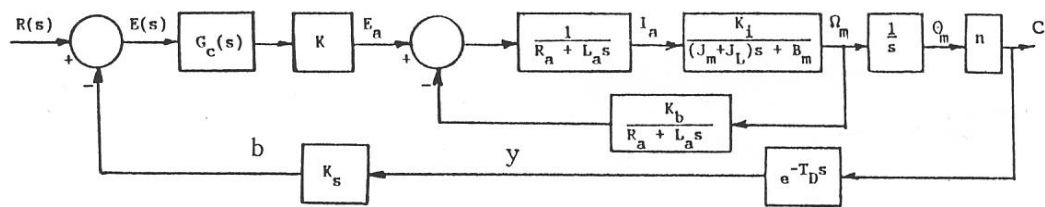
$$s^2 [J_m Ms^2 + (K_1 + K_2)(J_m + rM)] = 0$$

6-18) (a) System equations:

$$T_m = K_i i_a = (J_m + J_L) \frac{d\omega_m}{dt} + B_m \omega_m \quad e_a = R_a i_a + L_a \frac{di_a}{dt} + K_b \omega_m \quad y = n\theta_m \quad y = y(t - T_D)$$

$$T_D = \frac{d}{V} \text{ (sec)} \quad e = r - b \quad b = K_s y \quad E_a(s) = KG_c(s)E(s)$$

Block diagram:



(b) Forward-path transfer function:

$$\frac{Y(s)}{E(s)} = \frac{KK_i n G_c(s) e^{-T_b s}}{s \{ (R_a + L_a s) [(J_m + J_L) s + B_m] + K_b K_i \}}$$

Closed-loop transfer function:

$$\frac{Y(s)}{R(s)} = \frac{KK_i n G_c(s) e^{-T_b s}}{s (R_a + L_a s) [(J_m + J_L) s + B_m] + K_b K_i s + K G_c(s) K_i n e^{-T_b s}}$$

6-10. The schematic diagram of a motor-load system is shown in Fig. 6P-10. The following parameters and variables are defined: $T_m(t)$ is the motor torque; $\omega_m(t)$, the motor velocity; $\theta_m(t)$, the motor displacement; $\omega_L(t)$, the load velocity; $\theta_L(t)$, the load displacement; K , the torsional spring constant; J_m , the motor inertia; B_m , the motor viscous-friction coefficient; and B_L , the load viscous-friction coefficient.

(a) Write the torque equations of the system.

(b) Find the transfer functions $\Theta_L(s)/T_m(s)$ and $\Theta_m(s)/T_m(s)$.

(c) Find the characteristic equation of the system.

(d) Let $T_m(t) = T_m$ be a constant applied torque; show that $\omega_m = \omega_L = \text{constant}$ in the steady state. Find the steady-state speeds ω_m and ω_L .

(e) Repeat part (d) when the value of J_L is doubled, but J_m stays the same.

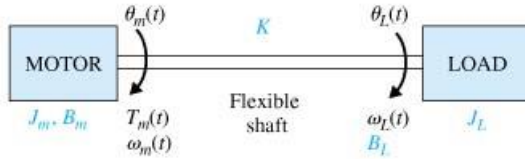
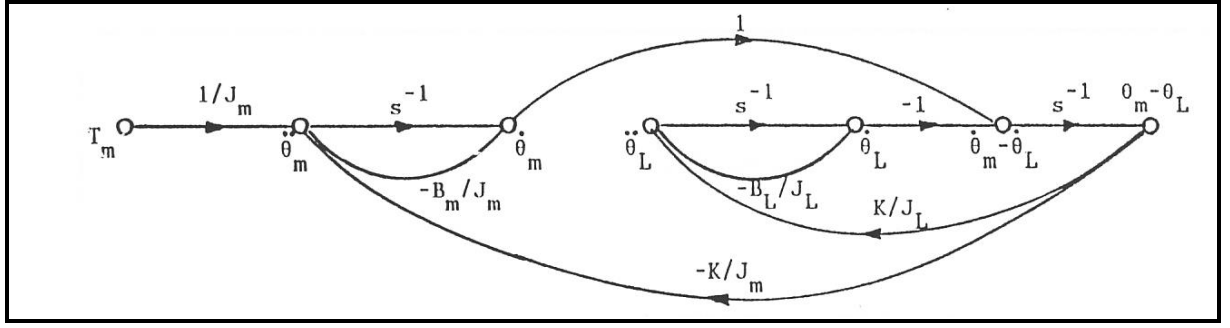


Figure 6P-10

(a) **Torque equations:**

$$T_m(t) = J_m \frac{d^2 \theta_m}{dt^2} + B_m \frac{d \theta_m}{dt} + K (\theta_m - \theta_L) \quad K (\theta_m - \theta_L) = J_L \frac{d^2 \theta_L}{dt^2} + B_L \frac{d \theta_L}{dt}$$

State diagram:



(b) Transfer functions:

$$\frac{\Theta_L(s)}{T_m(s)} = \frac{K}{\Delta(s)} \quad \frac{\Theta_m(s)}{T_m(s)} = \frac{J_L s^2 + B_L s + K}{\Delta(s)} \quad \Delta(s) = s \left[J_m J_L s^3 + (B_m J_L + B_L J_m) s^2 + (K J_m + K J_L + B_m B_L) s + B_m K \right]$$

(c) Characteristic equation: $\Delta(s) = 0$

(d) Steady-state performance: $T_m(t) = T_m = \text{constant}$. $T_m(s) = \frac{T_m}{s}$.

$$\lim_{t \rightarrow \infty} \omega_m(t) = \lim_{s \rightarrow 0} s \Omega_m(s) = \lim_{s \rightarrow 0} \frac{J_L s^2 + B_L s + K}{J_m J_L s^3 + (B_m J_L + B_L J_m) s^2 + (K J_m + K J_L + B_m B_L) s + B_m K} = \frac{1}{B_m}$$

Thus, in the steady state, $\omega_m = \omega_L$.

(e) The steady-state values of ω_m and ω_L do not depend on J_m and J_L .

6-11. The schematic diagram of a control system containing a motor coupled to a tachometer and an inertial load is shown in Fig. 6P-11. The following parameters and variables are defined: T_m is the motor torque; J_m , the motor inertia; J_t , the tachometer inertia; J_L , the load inertia; K_1 and K_2 , the spring constants of the shafts; θ_t , the tachometer displacement; θ_m , the motor velocity; θ_L , the load displacement; ω_t , the tachometer velocity; ω_L , the load velocity; and B_m , the motor viscous-friction coefficient.

(a) Write the state equations of the system using $\theta_L, \omega_L, \theta_t, \omega_t, \theta_m$, and ω_m as the state variables (in the listed order). The motor torque T_m is the input.

(b) Draw a signal flow diagram with T_m at the left and ending with θ_L on the far right. The state diagram should have a total of 10 nodes. Leave out the initial states.

- (c) Find the following transfer functions: $\frac{\Theta_L(s)}{T_m(s)}$ $\frac{\Theta_t(s)}{T_m(s)}$ $\frac{\Theta_m(s)}{T_m(s)}$
- (d) Find the characteristic equation of the system.

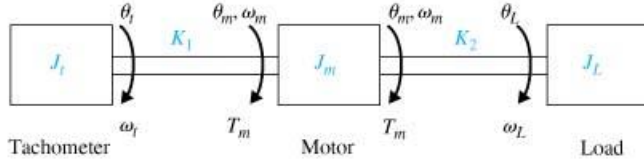
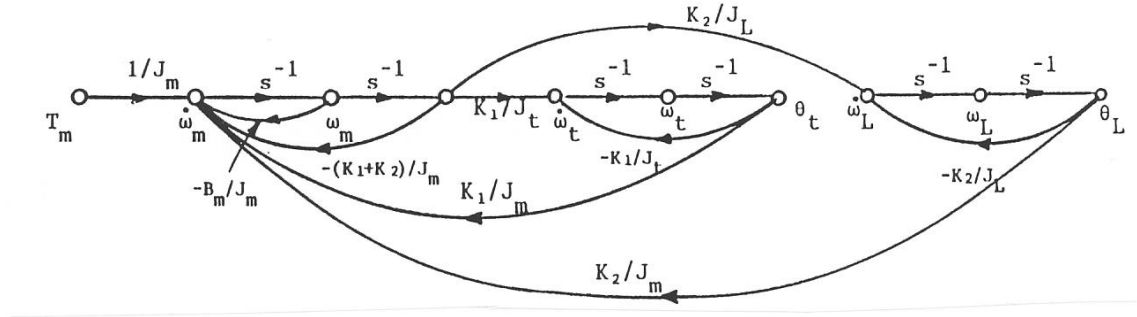


Figure 6P-11

- (a) State equations:

$$\begin{aligned} \frac{d\theta_L}{dt} &= \omega_L & \frac{d\omega_L}{dt} &= \frac{K_2}{J_L} \theta_m - \frac{K_2}{J_L} \theta_L & \frac{d\theta_t}{dt} &= \omega_t & \frac{d\omega_t}{dt} &= \frac{K_1}{J_t} \theta_m - \frac{K_1}{J_t} \theta_t \\ \frac{d\theta_m}{dt} &= \omega_m & \frac{d\omega_m}{dt} &= -\frac{B_m}{J_m} \omega_m - \frac{(K_1 + K_2)}{J_m} \theta_m + \frac{K_1}{J_m} \theta_t + \frac{K_2}{J_m} \theta_L + \frac{1}{J_m} T_m \end{aligned}$$

- (b) State diagram:



- (c) Transfer functions:

$$\frac{\Theta_L(s)}{T_m(s)} = \frac{K_2(J_t s^2 + K_1)}{\Delta(s)} \quad \frac{\Theta_t(s)}{T_m(s)} = \frac{K_1(J_L s^2 + K_2)}{\Delta(s)} \quad \frac{\Theta_m(s)}{T_m(s)} = \frac{J_t J_L s^4 + (K_1 J_L + K_2 J_t) s^2 + K_1 K_2}{\Delta(s)}$$

$$\begin{aligned} \Delta(s) &= s[J_m J_L s^5 + B_m J_L J_t s^4 + (K_1 J_L J_t + K_2 J_L J_t + K_1 J_m J_L + K_2 J_m J_t) s^3 \\ &\quad + B_m J_L (K_1 + K_2) s^2 + K_1 K_2 (J_L + J_t + J_m) s + B_m K_1 K_2] = 0 \end{aligned}$$

(d) Characteristic equation: $\Delta(s) = 0$.

6-12. The voltage equation of a dc motor is written as

$$e_a(t) = R_a i_a(t) + L_a \frac{di_a(t)}{dt} + K_b \omega_m(t)$$

where $e_a(t)$ is the applied voltage; $i_a(t)$, the armature current; R_a , the armature resistance; L_a , the armature inductance; K_b , the back-emf constant; $\omega_m(t)$, the motor velocity; and $\omega_n(t)$, the reference input voltage. Taking the Laplace transform on both sides of the voltage equation, with zero initial conditions and solving for $\Omega_m(s)$, we get

$$\Omega_m(s) = \frac{E_a(s) - (R_a + L_a s) I_a(s)}{K_b}$$

which shows that the velocity information can be generated by feeding back the armature voltage and current. The block diagram in Fig. 6P-12 shows a dc-motor system, with voltage and current feedbacks, for speed control.

(a) Let K_1 be a very high gain amplifier. Show that when $H_i(s)/H_e(s) = -(R_a + L_a s)$, the motor velocity $\omega_m(t)$ is totally independent of the load-disturbance torque T_L .

(b) Find the transfer function between $\Omega_m(s)$ and $\Omega_r(s)$ ($T_L = 0$) when $H_i(s)$ and $H_e(s)$ are selected as in part (a).

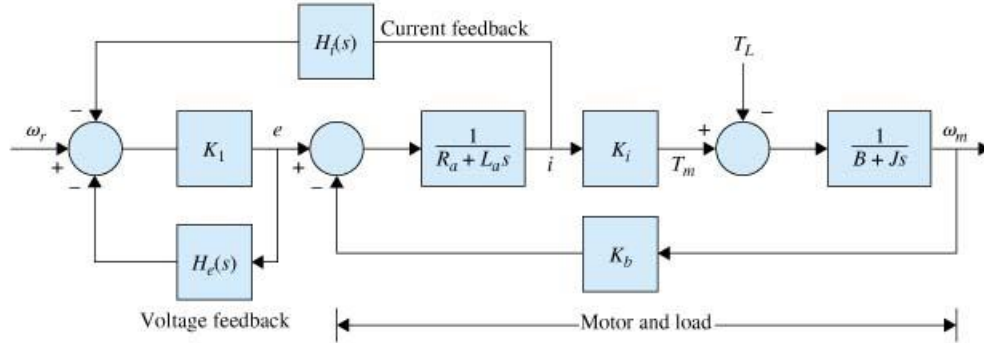


Figure 6P-12

(a)

$$\left. \frac{\Omega_m(s)}{T_L(s)} \right|_{\omega_r=0} = \frac{-1}{B + Js} \left(1 + K_1 H_e(s) + \frac{K_1 H_i(s)}{R_a + L_a s} \right) \frac{-K_i}{B + Js} \left(H_e(s) + \frac{H_i(s)}{R_a + L_a s} \right) \frac{1}{\Delta(s)} = 0$$

Thus,

$$\begin{aligned}
H_e(s) &= -\frac{H_i(s)}{R_a + L_a s} & \frac{H_i(s)}{H_e(s)} &= -(R_a + L_a s) \\
\text{(b)} \quad \frac{\Omega_m(s)}{\Omega_r(s)} \Big|_{T_e=0} &= \frac{\frac{K_i K_i}{(R_a + L_a s)(B + Js)}}{\Delta(s)} \\
\Delta(s) &= 1 + K_i H_e(s) + \frac{K_i K_b}{(R_a + L_a s)(B + Js)} + \frac{K_i H_i(s)}{R_a + L_a s} + \frac{K_i K_i K_b H_e(s)}{(R_a + L_a s)(B + Js)} \\
&= 1 + \frac{K_i K_b}{(R_a + L_a s)(B + Js)} + \frac{K_i K_i}{(R_a + L_a s)(B + Js)} \\
\frac{\Omega_m(s)}{\Omega_r(s)} \Big|_{T_e=0} &= \frac{K_i K_i}{(R_a + L_a s)(B + Js) + K_i K_b + K_i K_i K_b H_e(s)} \cong \frac{1}{K_b H_e(s)}
\end{aligned}$$

6-13. This problem deals with the attitude control of a guided missile. When traveling through the atmosphere, a missile encounters aerodynamic forces that tend to cause instability in the attitude of the missile. The basic concern from the flight-control standpoint is the lateral force of the air, which tends to rotate the missile about its center of gravity. If the missile centerline is not aligned with the direction in which the center of gravity C is traveling, as shown in Fig. 6P-13, with angle θ , which is also called the angle of attack, a side force is produced by the drag of the air through which the missile travels. The total force F_a may be considered to be applied at the center of pressure P . As shown in Fig. 6P-20, this side force has a tendency to cause the missile to tumble end over end, especially if the point P is in front of the center of gravity C . Let the angular acceleration of the missile about the point C , due to the side force, be denoted by α_F . Normally, α_F is directly proportional to the angle of attack θ and is given by

$$\alpha_F = \frac{K_F d_1}{J} \theta$$

where K_F is a constant that depends on such parameters as dynamic pressure, velocity of the missile, air density, and so on, and

J = missile moment of inertia about C

d_1 = distance between C and P

The main objective of the flight-control system is to provide the stabilizing action to counter the effect of the side force. One of the standard control means is to use gas injection at the tail of the missile to deflect the direction of the rocket engine thrust T_s , as shown in the figure.

- (a) Write a torque differential equation to relate among T_s , δ , θ , and the system parameters given. Assume that δ is very small, so that $\sin \delta(t)$ is approximated by $\delta(t)$.
- (b) Assume that T_s is a constant torque. Find the transfer function $\Theta(s)/\Delta(s)$, where $\Theta(s)$ and $\Delta(s)$ are the Laplace transforms of $\theta(t)$ and $\delta(t)$, respectively. Assume that $\delta(t)$ is very small.
- (c) Repeat parts (a) and (b) with points C and P interchanged. The d_1 in the expression of α_F should be changed to d_2 .

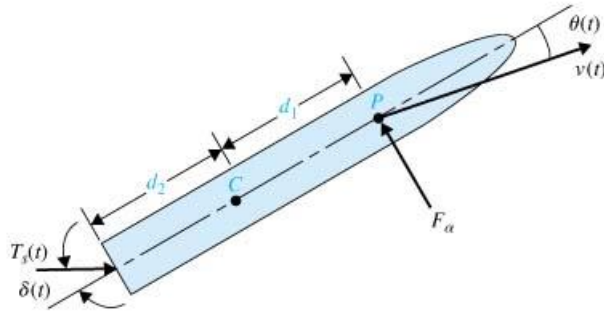


Figure 6P-13

- (a) **Torque equation:** (About the center of gravity C)

$$J \frac{d^2 \theta}{dt^2} = T_s d_2 \sin \delta + F_a d_1 \quad F_a d_1 = J \alpha_1 = K_F d_1 \theta \quad \sin \delta \cong \delta$$

$$\text{Thus,} \quad J \frac{d^2 \theta}{dt^2} = T_s d_2 \delta + K_F d_1 \theta \quad J \frac{d^2 \theta}{dt^2} - K_F d_1 \theta = T_s d_2 \delta$$

$$(b) \quad Js^2 \Theta(s) - K_F d_1 \Theta(s) = T_s d_2 \Delta(s)$$

- (c) With C and P interchanged, the torque equation about C is:

$$T_s (d_1 + d_2) \delta + F_a d_2 = J \frac{d^2 \theta}{dt^2} \quad T_s (d_1 + d_2) \delta + K_F d_2 \theta = J \frac{d^2 \theta}{dt^2}$$

$$Js^2 \Theta(s) - K_F d_2 \Theta(s) = T_s (d_1 + d_2) \Delta(s) \quad \frac{\Theta(s)}{\Delta(s)} = \frac{T_s (d_1 + d_2)}{Js^2 - K_F d_2}$$

6-14. Figure 6P-14(a) shows the schematic diagram of a dc-motor control system for the control of a printwheel of a word processor. The load in this case is the printwheel, which is directly coupled to the motor shaft. The following parameters and variables are defined: K_s is the error-detector gain (V/rad); K_i , the torque constant (oz-in./A); K , the amplifier gain (V/V); K_b , the back-emf constant (V/rad/sec); n , the gear train ratio $= \theta_2 / \theta_m = T_m / T_2$; B_m , the motor viscous-friction coefficient (oz-in.-sec); J_m , the motor

inertia (oz-in.-sec²); K_L the torsional spring constant of the motor shaft (oz-in./rad); and J_L , the load inertia (oz-in.-sec²).

(a) Write the cause-and-effect equations of the system. Rearrange these equations into the form of state equations with $x_1 = \theta_o$, $x_2 = \dot{\theta}_o$, $x_3 = \theta_m$, $x_4 = \dot{\theta}_m$, and $x_5 = i_a$.

(b) Draw a state diagram using the nodes shown in Fig. 3P-38(b).

(c) Derive the forward-path transfer function (with the outer feedback path open):

$G(s) = \Theta_o(s)/\Theta_e(s)$. Find the closed-loop transfer function $M(s) = \Theta_o(s)/\Theta_r(s)$.

(e) Repeat part (c) when the motor shaft is rigid, that is, $K_L = \infty$. Show that you can obtain the solutions by taking the limit as K_L approaches infinity in the results in part (c).

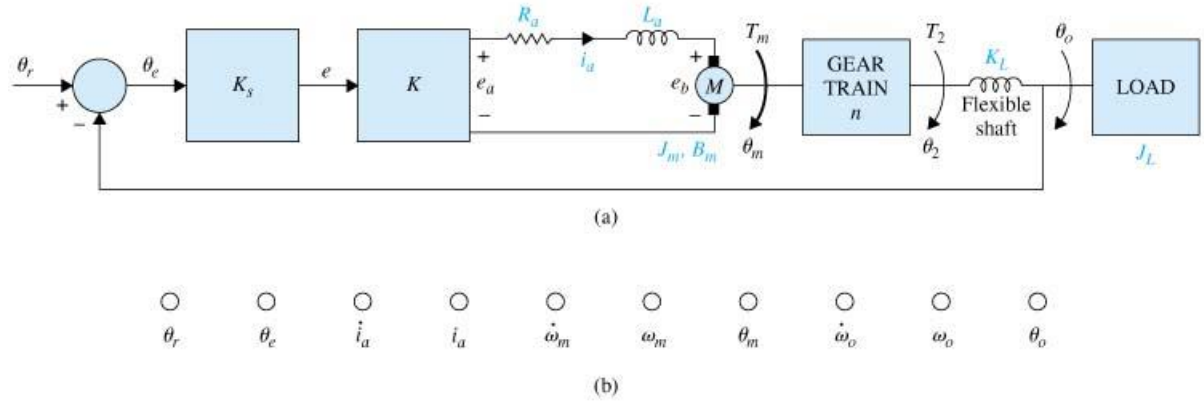


Figure 6P-14

(a) Cause-and-effect equations: $\theta_e = \theta_r - \theta_o$ $e = K_s \theta_e$ $e_a = Ke$

$$\frac{di_a}{dt} = -\frac{R_a}{L_a} i_a + \frac{1}{L_a} (e_a - e_b) \quad T_m = K_i i_a$$

$$\frac{d^2 \theta_m}{dt^2} = -\frac{B_m}{J_m} \frac{d\theta_m}{dt} + \frac{1}{J} T_m - \frac{nK_L}{J_m} (n\theta_m - \theta_o) \quad T_2 = \frac{T_m}{n} \quad \theta_2 = n\theta_m$$

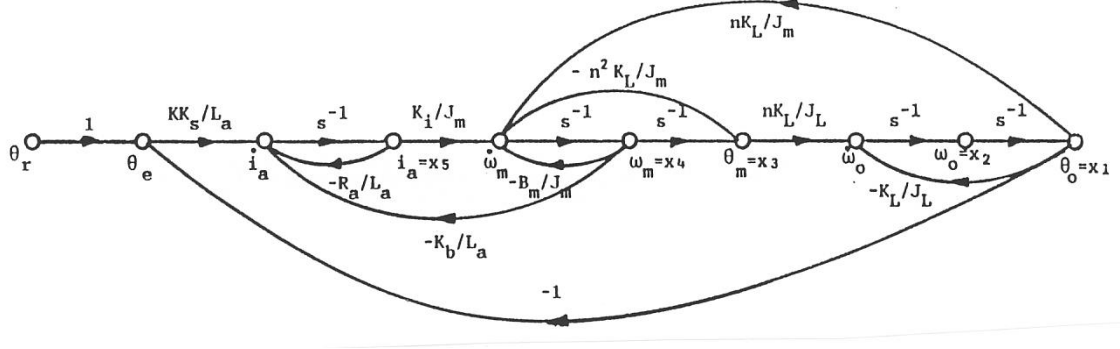
$$\frac{d^2 \theta_o}{dt^2} = \frac{K_L}{J_L} (\theta_2 - \theta_o)$$

State variables: $x_1 = \theta_o$, $x_2 = \dot{\theta}_o$, $x_3 = \theta_m$, $x_4 = \dot{\theta}_m$, $x_5 = i_a$

State equations:

$$\begin{aligned}\frac{dx_1}{dt} &= x_2 & \frac{dx_2}{dt} &= -\frac{K_L}{J_L}x_1 + \frac{nK_L}{J_L}x_3 & \frac{dx_3}{dt} &= x_4 \\ \frac{dx_4}{dt} &= -\frac{nK_L}{J_m}x_1 - \frac{n^2K_L}{J_m}x_3 - \frac{B_m}{J_m}x_4 + \frac{K_i}{J_m}x_5 & \frac{dx_5}{dt} &= -\frac{KK_s}{L_a}x_1 - \frac{K_b}{L_a}x_4 - \frac{R_a}{L_a}x_5 + \frac{KK_s}{L_a}\theta_r\end{aligned}$$

(b) State diagram:



(c) Forward-path transfer function:

$$\frac{\Theta_o(s)}{\Theta_e(s)} = \frac{KK_s K_i nK_L}{s \left[J_m J_L L_a s^4 + J_L (R_a J_m + B_m J_m + B_m L_a) s^3 + (n^2 K_L L_a J_L + K_L J_m L_a + B_m R_a J_L) s^2 + \overline{(n^2 R_a K_L J_L + R_a K_L J_m + B_m K_L L_a)} s + K_i K_b K_L + R_a B_m K_L \right]}$$

Closed-loop transfer function:

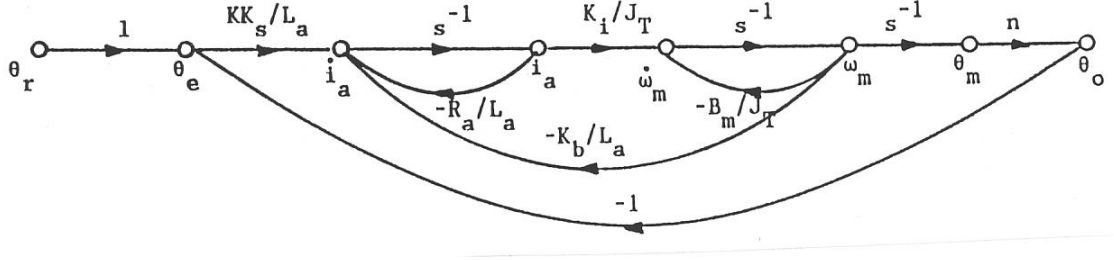
$$\frac{\Theta_o(s)}{\Theta_r(s)} = \frac{KK_s K_i nK_L}{J_m J_L L_a s^5 + J_L (R_a J_m + B_m J_m + B_m L_a) s^4 + (n^2 K_L L_a J_L + K_L J_m L_a + B_m R_a J_L) s^3 + \overline{(n^2 R_a K_L J_L + R_a K_L J_m + B_m K_L L_a)} s^2 + (K_i K_b K_L + R_a B_m K_L) s + nKK_s K_i K_L}$$

(d) $K_L = \infty$, $\theta_o = \theta_2 = n\theta_m$. J_L is reflected to motor side so $J_T = J_m + n^2 J_L$.

State equations:

$$\begin{aligned}\frac{d\omega_m}{dt} &= -\frac{B_m}{J_T}\omega_m + \frac{K_i}{J_T}i_a & \frac{d\theta_m}{dt} &= \omega_m & \frac{di_a}{dt} &= -\frac{R_a}{L_a}i_a + \frac{KK_s}{L_a}\theta_r - \frac{KK_s}{L_a}n\theta_m - \frac{K_b}{L_a}\omega_m\end{aligned}$$

State diagram:



Forward-path transfer function:

$$\frac{\Theta_o(s)}{\Theta_e(s)} = \frac{KK_s K_i n}{s \left[J_T L_a s^2 + (R_a J_T + B_m L_a) s + R_a B_m + K_i K_b \right]}$$

Closed-loop transfer function:

$$\frac{\Theta_o(s)}{\Theta_r(s)} = \frac{KK_s K_i n}{J_T L_a s^3 + (R_a J_T + B_m L_a) s^2 + (R_a B_m + K_i K_b) s + KK_s K_i n}$$

From part (c), when

$K_L = \infty$, all the terms without K_L in $\Theta_o(s)/\Theta_e(s)$ and $\Theta_o(s)/\Theta_r(s)$ can be neglected.

The same results as above are obtained.

6-15. The schematic diagram of a voice-coil motor (VCM), used as a linear actuator in a disk memory-storage system, is shown in Fig. 6P-15(a). The VCM consists of a cylindrical permanent magnet (PM) and a voice coil. When current is sent through the coil, the magnetic field of the PM interacts with the current-carrying conductor, causing the coil to move linearly. The voice coil of the VCM in Fig. 6P-15(a) consists of a primary coil and a shorted-turn coil. The latter is installed for the purpose of effectively reducing the electric constant of the device. Fig. 6P-15(b) shows the equivalent circuit of the coils. The following parameters and variables are defined: $e_a(t)$ is the applied coil voltage; $i_a(t)$, the primary-coil current; $i_s(t)$, the shorted-turn coil current; R_a , the primary-coil resistance; L_a , the primary-coil inductance; L_{as} , the mutual inductance between the primary and shorted-turn coils; $v(t)$, the velocity of the voice coil; $y(t)$, the displacement of the voice coil; $f(t) = K_i v(t)$, the force of the voice coil; K_i , the force constant; K_b , the back-emf constant; $e_b(t) = K_b v(t)$, the back emf; M_T , the total mass of the voice coil and load; and B_T , the total viscous-friction coefficient of the voice coil and load.

- Write the differential equations of the system.
- Draw a block diagram of the system with $E_a(s)$, $I_a(s)$, $I_s(s)$, $V(s)$, and $Y(s)$ as variables.
- Derive the transfer function $Y(s)/E_a(s)$.

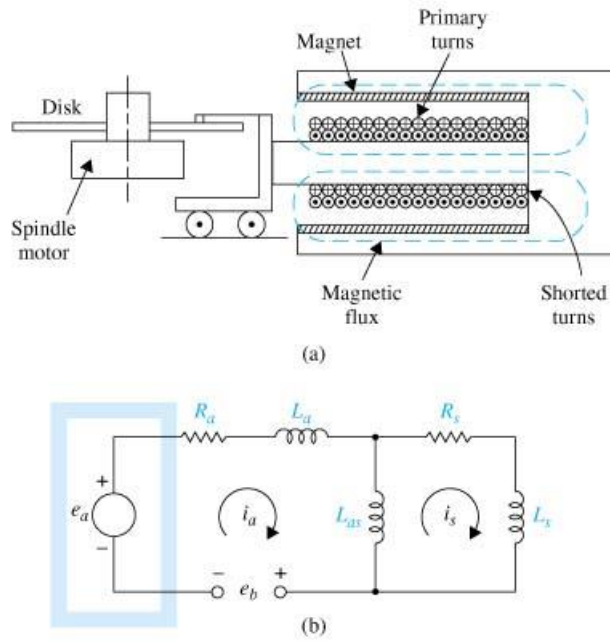


Figure 6P-15

(a) System equations:

$$f = K_i i_a = M_T \frac{dv}{dt} + B_T v \quad e_a = R_a i_a + (L_a + L_{as}) \frac{di_a}{dt} - L_{as} \frac{di_s}{dt} + e_b \quad 0 = R_s i_s + (L_s + L_{as}) \frac{di_s}{dt} - L_{as} \frac{di_a}{dt}$$

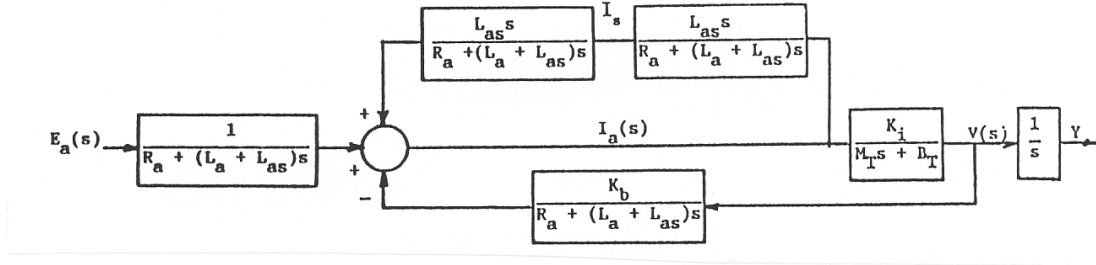
(b) Take the Laplace transform on both sides of the last three equations, with zero initial conditions, we have

$$K_i I_a(s) = (M_T s + B_T) V(s) \quad E_a(s) = [R_a + (L_a + L_{as}) s] I_a(s) - L_{as} s I_s(s) + K_b V(s) \\ 0 = -L_{as} s I_a(s) + [R_s + s(L_s + L_{as})] I_s(s)$$

Rearranging these equations, we get

$$V(s) = \frac{K_i}{M_T s + B_T} I_a(s) \quad Y(s) = \frac{V(s)}{s} = \frac{K_i}{s(M_T s + B_T)} I_a(s) \\ I_a(s) = \frac{1}{R_a + (L_a + L_{as}) s} [E_a(s) + L_{as} s I_s(s) - K_b V(s)] \quad I_s(s) = \frac{L_{as} s}{R_s + (L_s + L_{as}) s} I_a(s)$$

Block diagram:



(c) Transfer function:

$$\frac{Y(s)}{E_a(s)} = \frac{K_i [R_s + (L_s + L_{as})s]}{s [R_a + (L_a + L_{as})s] [R_s + (L_s + L_{as})s] (M_I s + B_T) + K_i K_b [R_s + (L_a + L_{as})s] - L_{as}^2 s^2 (M_I s + B_T)}$$

6-16. A dc-motor position-control system is shown in Fig. 6P-16 a). The following parameters and variables are defined: e is the error voltage; e_r , the reference input; θ_L , the load position; K_A , the amplifier gain; e_a , the motor input voltage; e_b , the back emf; i_a , the motor current; T_m , the motor torque; J_m , the motor inertia = 0.03 oz-in.-s²; B_m , the motor viscous-friction coefficient = 10 oz-in.-s²; K_L , the torsional spring constant = 50,000 oz-in./rad; J_L , the load inertia = 0.05 oz-in.-s²; K_i , the motor torque constant = 21 oz-in./A; K_b , the back-emf constant = 15.5 V/1000 rpm; K_s , the error-detector gain = $E/2\pi$; E , the error-detector applied voltage = 2π V; R_a , the motor resistance = 1.15Ω ; and $\theta_e = \theta_r - \theta_L$.

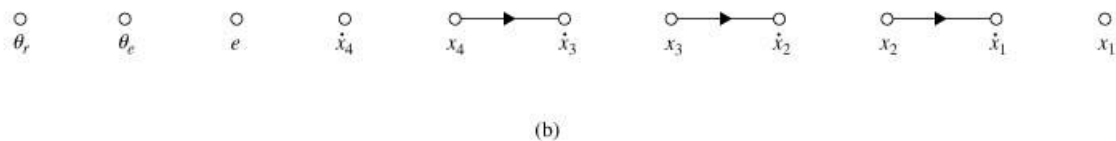
(a) Write the state equations of the system using the following state variables:

$$x_1 = \theta_L, x_2 = d\theta_L / dt = \omega_L, x_3 = \theta_s, \text{ and } x_4 = d\theta_m / dt = \omega_m.$$

(b) Draw a signal flow diagram using the nodes shown in Fig. 6P-16(b).

(c) Derive the forward-path transfer function $G(s) = \Theta_L(s) / \Theta_e(s)$ when the outer feedback path from θ_L is opened. Find the poles of $G(s)$.

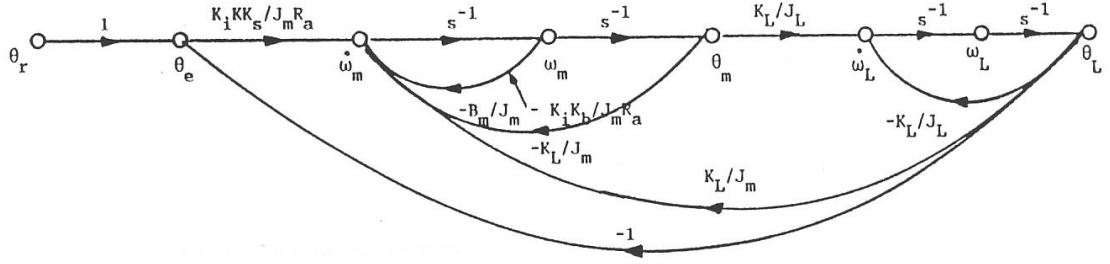
(d) Derive the closed-loop transfer function $M(s) = \Theta_L(s) / \Theta_e(s)$. Find the poles of $M(s)$ when $K_A = 1, 2738$, and 5476. Locate these poles in the s -plane, and comment on the significance of these values of K_A .



(a) Cause-and-effect equations:

State equations:

(b) State diagram:



(c) Forward-path transfer function:

$$G(s) = \frac{K_i K K_s K_L}{s \left[J_m J_L R_a s^3 + (B_m R_a + K_i K_b) J_L s^2 + R_a K_L (J_L + J_m) s + K_L (B_m R_a + K_i K_b) \right]}$$

$$J_m R_a J_L = 0.03 \times 1.15 \times 0.05 = 0.001725 \quad B_m R_a J_L = 10 \times 1.15 \times 0.05 = 0.575 \quad K_i K_b J_L = 21 \times 0.148 \times 0.05 = 0.1554$$

$$R_a K_L J_L = 1.15 \times 50000 \times 0.05 = 2875 \quad R_a K_L J_m = 1.15 \times 50000 \times 0.03 = 1725 \quad K_i K K_s K_L = 21 \times 1 \times 50000 K = 1050000 K$$

$$K_L (B_m R_a + K_i K_b) = 50000(10 \times 1.15 + 21 \times 0.148) = 730400$$

$$G(s) = \frac{608.7 \times 10^6 K}{s(s^3 + 423.42s^2 + 2.6667 \times 10^6 s + 4.2342 \times 10^8)}$$

(d) Closed-loop transfer function:

$$M(s) = \frac{\Theta_L(s)}{\Theta_r(s)} = \frac{G(s)}{1 + G(s)} = \frac{K_i K K_s K_L}{J_m J_L R_a s^4 + (B_m R_a + K_i K_b) J_L s^3 + R_a K_L (J_L + J_m) s^2 + K_L (B_m R_a + K_i K_b) s + K_i K K_s K_L}$$

$$M(s) = \frac{6.087 \times 10^8 K}{s^4 + 423.42s^3 + 2.6667 \times 10^6 s^2 + 4.2342 \times 10^8 s + 6.087 \times 10^8 K}$$

Characteristic equation roots:

$K = 1$	$K = 2738$	$K = 5476$
$s = -1.45$	$s = \pm j1000$	$s = 405 \pm j1223.4$
$s = -159.88$	$s = -211.7 \pm j1273.5$	$s = -617.22 \pm j1275$
$s = -131.05 \pm j1614.6$		

6-17. Figure 6P-17(a) shows the setup of the temperature control of an air-flow system. The hot-water reservoir supplies the water that flows into the heat exchanger for heating the air. The temperature sensor senses the air temperature T_{AO} and sends it to be compared with the reference temperature T_r . The temperature error T_e is sent to the

controller, which has the transfer function $G_c(s)$. The output of the controller, $u(t)$, which is an electric signal, is converted to a pneumatic signal by a transducer. The output of the actuator controls the water-flow rate through the three-way valve. Figure 6P-17(b) shows the block diagram of the system.

The following parameters and variables are defined: dM_w is the flow rate of the heating fluid = $k_M u$, $K_M = 0.054$ kg/sec/V; T_w , the water temperature = $K_R dM_w$; $K_R = 65^\circ\text{C/kg/sec}$; and T_{AO} the output air temperature.

Heat-transfer equation between water and air:

$$\tau_c \frac{dT_{AO}}{dt} = T_w - T_{AO} \quad \tau_c = 10 \text{ seconds}$$

Temperature sensor equation:

$$\tau_s \frac{dT_s}{dt} = T_{AO} - T_s \quad \tau_s = 2 \text{ seconds}$$

- (a) Draw a functional block diagram that includes all the transfer functions of the system.
- (b) Derive the transfer function $T_{AO}(s)/T_r(s)$ when $G_c(s) = 1$.

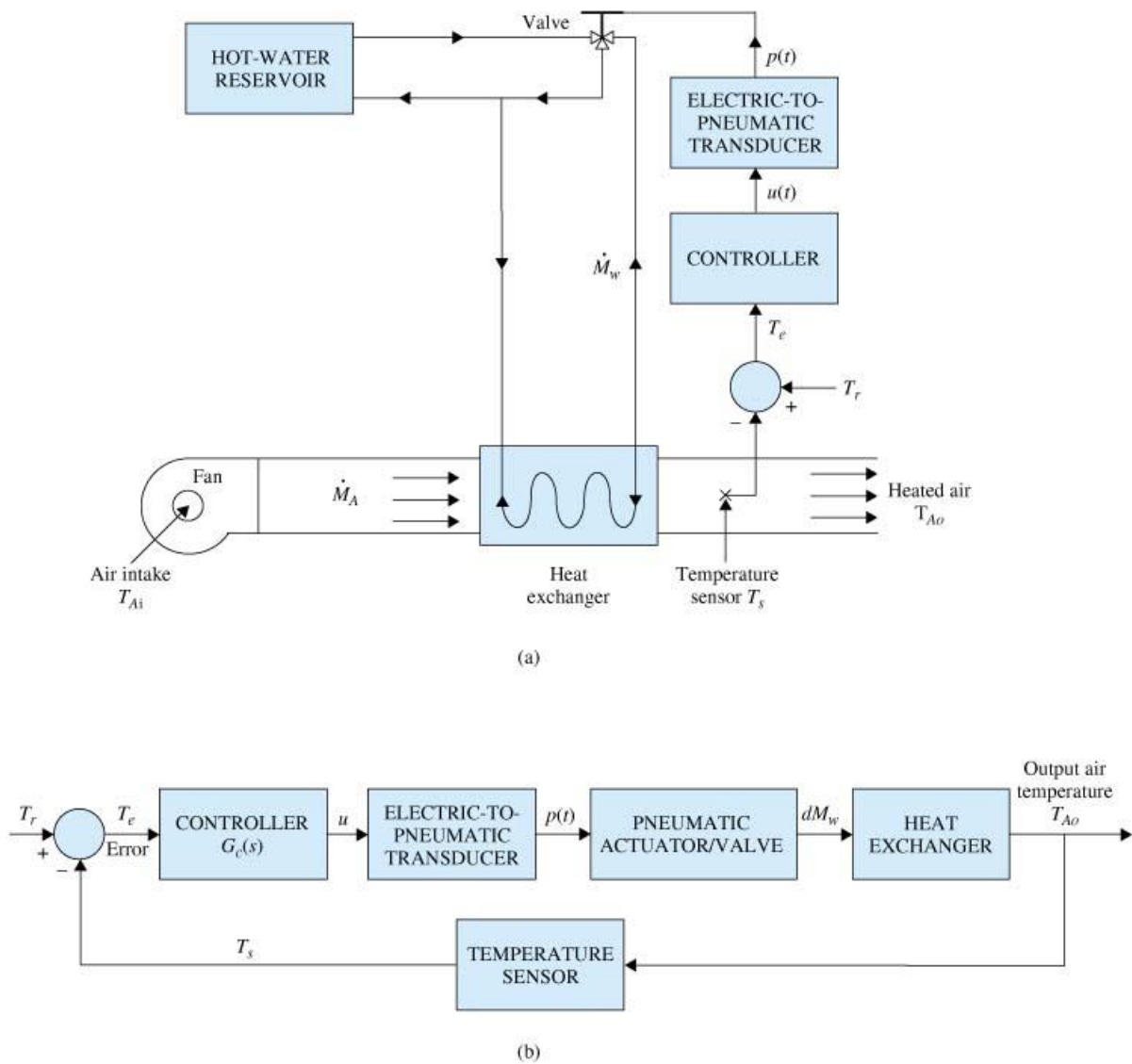
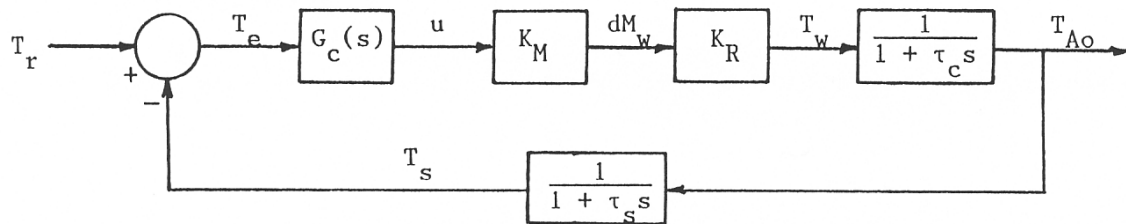


Figure 6P-17

(a) Block diagram:



(b) Transfer function:

$$\frac{T_{AO}(s)}{T_r(s)} = \frac{K_M K_R}{(1 + \tau_c s)(1 + \tau_s s) + K_m K_R} = \frac{3.51}{20s^2 + 12s + 4.51}$$

6-18. The objective of this problem is to develop a linear analytical model of the automobile engine for idle-speed control system shown in Fig. 1-2. The input of the system is the throttle position that controls the rate of air flow into the manifold (see Fig. 6P-18). Engine torque is developed from the buildup of manifold pressure due to air intake and the intake of the air/gas mixture into the cylinder. The engine variations are as follows:

$q_i(t)$ = amount of air flow across throttle into manifold

$dq_i(t)/dt$ = rate of air flow across throttle into manifold

$q_m(t)$ = average air mass in manifold

$q_o(t)$ = amount of air leaving intake manifold through intake valves

$dq_o(t)/dt$ = rate of air leaving intake manifold through intake valves

$T(t)$ = engine torque

T_d = disturbance torque due to application of auto accessories = constant

$\omega(t)$ = engine speed

$\alpha(t)$ = throttle position

τ_D = time delay in engine

J_e = inertia of engine

The following assumptions and mathematical relations between the engine variables are given:

1. The rate of air flow into the manifold is linearly dependent on the throttle position:

$$\frac{dq_i(t)}{dt} = K_1 \alpha(t) \quad K_1 = \text{proportional constant}$$

2. The rate of air flow leaving the manifold depends linearly on the air mass in the manifold and the engine speed:

$$\frac{dq_o(t)}{dt} = K_2 q_m(t) + K_3 \omega(t) \quad K_2, K_3 = \text{constants}$$

3. A pure time delay of τ_D seconds exists between the change in the manifold air mass and the engine torque:

$$T(t) = K_4 q_m(t - \tau_D) \quad K_4 = \text{constant}$$

4. The engine drag is modeled by a viscous-friction torque $B\omega(t)$, where B is the viscous-friction coefficient.

5. The average air mass $q_m(t)$ is determined from

$$q_m(t) = \int \left(\frac{dq_i(t)}{dt} - \frac{dq_o(t)}{dt} \right) dt$$

6. The equation describing the mechanical components is

$$T(t) = J \frac{d\omega(t)}{dt} + B\omega(t) + T_d$$

(a) Draw a functional block diagram of the system with $\alpha(t)$ as the input, $\omega(t)$ as the output, and T_d as the disturbance input. Show the transfer function of each block.

(b) Find the transfer function $\Omega(s)/\alpha(s)$ of the system.

(c) Find the characteristic equation and show that it is not rational with constant coefficients.

(d) Approximate the engine time delay by

$$e^{-\tau_D s} \cong \frac{1 - \tau_D s / 2}{1 + \tau_D s / 2}$$

and repeat parts (b) and (c).

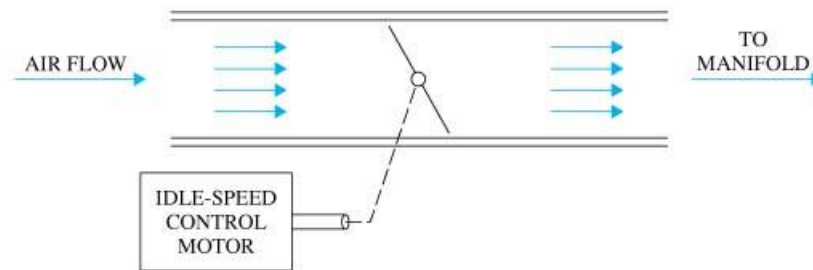
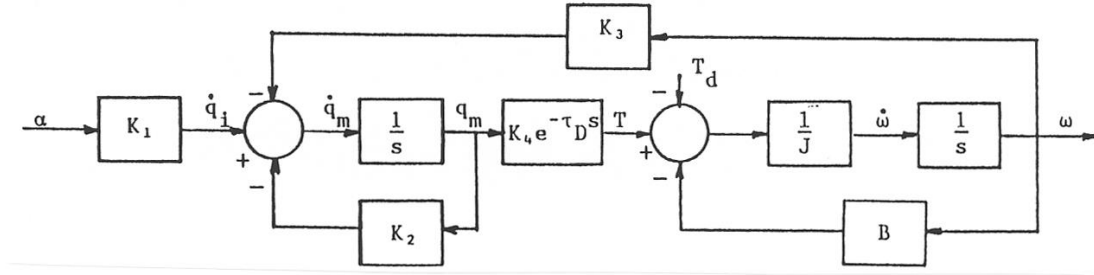


Figure 6P-18

(a) Block diagram:



(b) Transfer function:

$$\frac{\Omega(s)}{\alpha(s)} = \frac{K_1 K_4 e^{-\tau_D s}}{Js^2 + (JK_L + B)s + K_2 B + K_3 K_4 e^{-\tau_D s}}$$

(c) Characteristic equation:

$$Js^2 + (JK_L + B)s + K_2 B + K_3 K_4 e^{-\tau_D s} = 0$$

(d) Transfer function:

$$\frac{\Omega(s)}{\alpha(s)} \cong \frac{K_1 K_4 (2 - \tau_D s)}{\Delta(s)}$$

Characteristic equation:

$$\Delta(s) \cong J\tau_D s^3 + (2J + JK_2\tau_D + B\tau_D)s^2 + (2JK_2 + 2B - \tau_D K_2 B - \tau_D K_3 K_4)s + 2(K_2 B + K_3 K_4) = 0$$

6-19. Phase-locked loops are control systems used for precision motor-speed control. The basic elements of a phase-locked loop system incorporating a dc motor is shown in Fig. 6P-19(a). An input pulse train represents the reference frequency or desired output speed. The digital encoder produces digital pulses that represent motor speed. The phase detector compares the motor speed and the reference frequency and sends an error voltage to the filter (controller) that governs the dynamic response of the system. Phase detector gain = K_p , encoder gain = K_e , counter gain = $1/N$, and dc-motor torque constant = K_t . Assume zero inductance and zero friction for the motor.

(a) Derive the transfer function $E_c(s)/E(s)$ of the filter shown in Fig. 6P-19(b). Assume that the filter sees infinite impedance at the output and zero impedance at the input.

(b) Draw a functional block diagram of the system with gains or transfer functions in the blocks.

(c) Derive the forward-path transfer function $\Omega_m(s)/E(s)$ when the feedback path is open.

(d) Find the closed-loop transfer function $\Omega_m(s)/F_r(s)$.

(e) Repeat parts (a), (c), and (d) for the filter shown in Fig. 6P-19(c).

(f) The digital encoder has an output of 36 pulses per revolution. The reference frequency f_r is fixed at 120 pulse/sec. Find K_e in pulse/rad. The idea of using the counter N is that with f_r fixed, various desired output speeds can be attained by changing the value of N . Find N if the desired output speed is 200 rpm. Find N if the desired output speed is 1800 rpm.

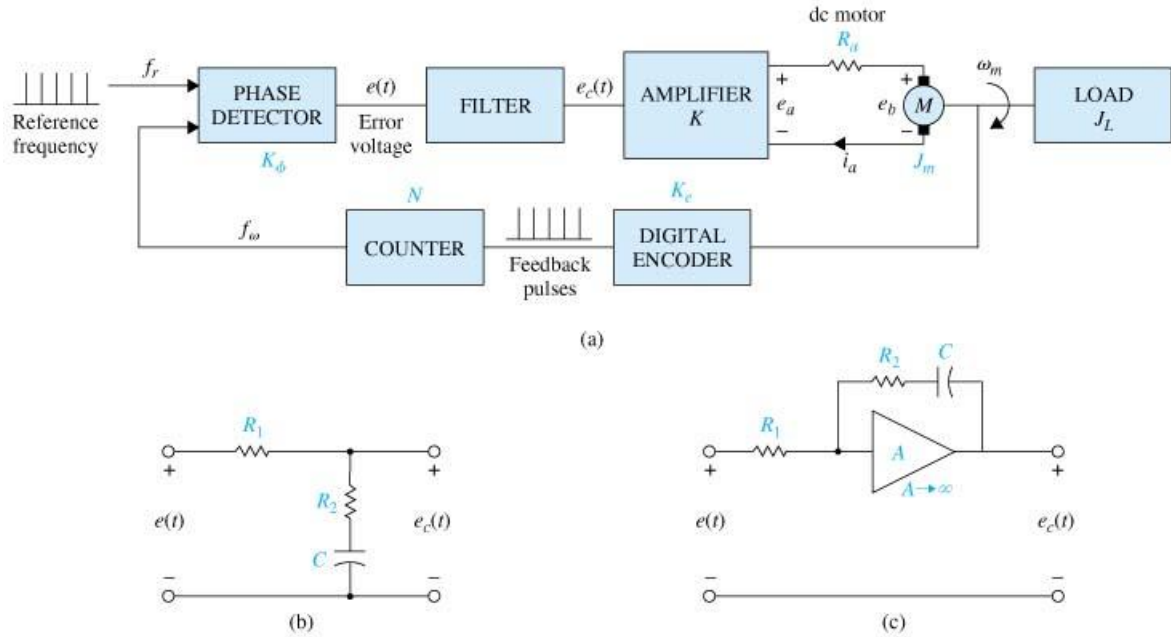
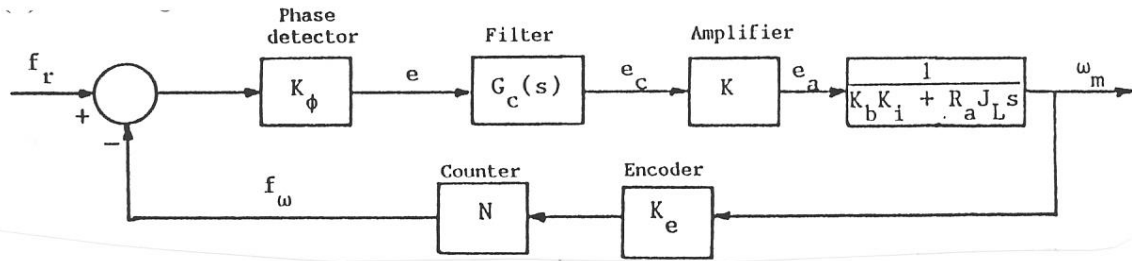


Figure 6P-19

(a) Transfer function:

$$G(s) = \frac{E_c(s)}{E(s)} = \frac{1 + R_2Cs}{1 + (R_1 + R_2)Cs}$$

(b) Block diagram:



(c) Forward-path transfer function:

$$\frac{\Omega_m(s)}{E(s)} = \frac{K(1 + R_2Cs)}{[1 + (R_1 + R_2)Cs](K_b K_i + R_a J_L s)}$$

(d) Closed-loop transfer function:

$$\frac{\Omega_m(s)}{F_r(s)} = \frac{K_\phi K (1 + R_2 Cs)}{[1 + (R_1 + R_2) Cs] (K_b K_i + R_a J_L s) + K_\phi K K_e N (1 + R_2 Cs)}$$

(e) $G_c(s) = \frac{E_c(s)}{E(s)} = \frac{(1 + R_2 Cs)}{R_1 Cs}$

Forward-path transfer function:

$$\frac{\Omega_m(s)}{E(s)} = \frac{K (1 + R_2 Cs)}{R_1 Cs (K_b K_i + R_a J_L s)}$$

Closed-loop transfer function:

$$\frac{\Omega_m(s)}{F_r(s)} = \frac{K_\phi K (1 + R_2 Cs)}{R_1 Cs (K_b K_i + R_a J_L s) + K_\phi K K_e N (1 + R_2 Cs)}$$

$$K_e = 36 \text{ pulses / rev} = 36 / 2\pi \text{ pulses / rad} = 5.73 \text{ pulses / rad.}$$

(f) $f_r = 120 \text{ pulses / sec}$ $\omega_m = 200 \text{ RPM} = 200(2\pi / 60) \text{ rad / sec}$

$$f_\omega = NK_e \omega_m = 120 \text{ pulses / sec} = N(36 / 2\pi)200(2\pi / 60) = 120N \text{ pulses / sec}$$

Thus, $N = 1$. For

$$\omega_m = 1800 \text{ RPM, } 120 = N(36 / 2\pi)1800(2\pi / 60) = 1080N. \text{ Thus, } N = 9.$$

6-20. The linearized model of a robot arm system driven by a dc motor is shown in Fig. 6P-20. The system parameters and variables are given as follows:

<i>DC Motor</i>	<i>Robot Arm</i>
$T_m = \text{motor torque} = K_i i_a$	$J_L = \text{inertia of arm}$
$K_i = \text{torque constant}$	$T_L = \text{disturbance torque on arm}$
$i_a = \text{armature current of motor}$	$\theta_L = \text{arm displacement}$
$J_m = \text{motor inertia}$	$K = \text{torsional spring constant}$
$B_m = \text{motor viscous-friction coefficient}$	$\theta_m = \text{motor-shaft displacement}$
$B = \text{viscous-friction coefficient of shaft between the motor and arm}$	
$B_L = \text{viscous-friction coefficient of the robot arm shaft}$	

(a) Write the differential equations of the system with $i_a(t)$ and $T_L(t)$ as input and $\theta_m(t)$ and $\theta_L(t)$ as outputs.

(b) Draw an SFG using $I_a(s)$, $T_L(s)$, $\Theta_m(s)$, and $\Theta_L(s)$ as node variables.

(c) Express the transfer-function relations as

$$\begin{bmatrix} \mathbf{Q}_m(s) \\ \mathbf{Q}_L(s) \end{bmatrix} = \mathbf{G}(s) \begin{bmatrix} I_a(s) \\ -T_L(s) \end{bmatrix}$$

Find $\mathbf{G}(s)$.

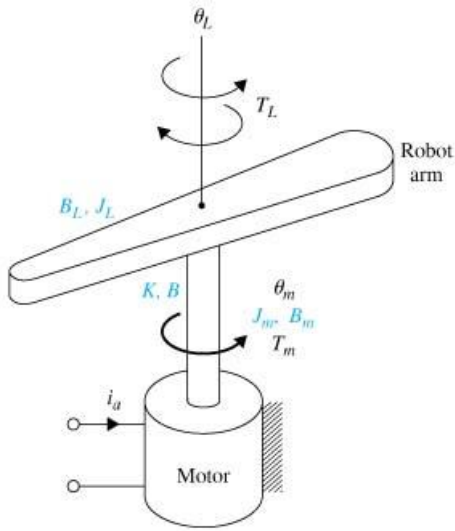


Figure 6P-20

(a) **Differential equations:**

$$K_i i_a = J_m \frac{d^2 \theta_m}{dt^2} + B_m \frac{d\theta_m}{dt} + K(\theta_m - \theta_L) + B \left(\frac{d\theta_m}{dt} - \frac{d\theta_L}{dt} \right)$$

$$K(\theta_m - \theta_L) + B \left(\frac{d\theta_m}{dt} - \frac{d\theta_L}{dt} \right) = \left(J_L \frac{d^2 \theta_L}{dt^2} + B_L \frac{d\theta_L}{dt} \right) + T_L$$

(b) Take the Laplace transform of the differential equations with zero initial conditions, we get

$$K_i I_a(s) = (J_m s^2 + B_m s + Bs + K) \Theta_m(s) + (Bs + K) \Theta_L(s)$$

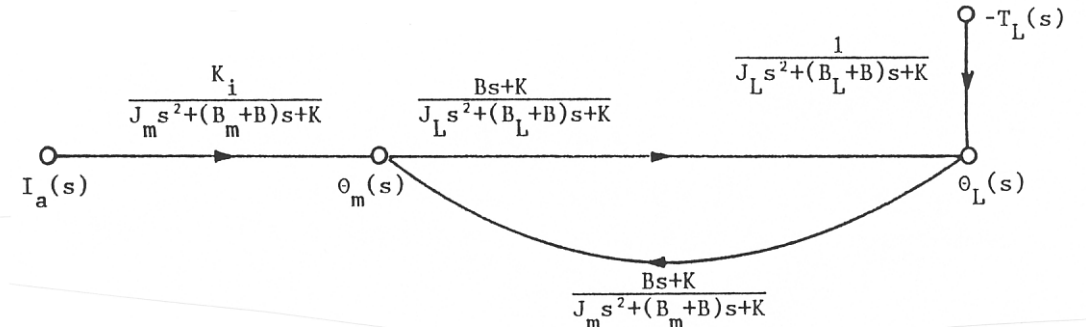
$$(Bs + K) \Theta_m(s) - (Bs + K) \Theta_L(s) = (J_L s^2 + B_L s) s \Theta_L(s) + T_L(s)$$

Solving for $\Theta_m(s)$ and $\Theta_L(s)$ from the last two equations, we have

$$\Theta_m(s) = \frac{K_i}{J_m s^2 + (B_m + B)s + K} I_a(s) + \frac{Bs + K}{J_m s^2 + (B_m + B)s + K} \Theta_L(s)$$

$$\Theta_L(s) = \frac{Bs + K}{J_L s^2 + (B_L + B)s + K} \Theta_m(s) - \frac{T_L(s)}{J_L s^2 + (B_L + B)s + K}$$

Signal flow graph:



(c) Transfer matrix:

$$\begin{bmatrix} \Theta_m(s) \\ \Theta_L(s) \end{bmatrix} = \frac{1}{\Delta_o(s)} \begin{bmatrix} K_i [J_L s^2 + (B_L + B)s + K] & Bs + K \\ K_i (Bs + K) & J_m s^2 + (B_m + B)s + K \end{bmatrix} \begin{bmatrix} I_a(s) \\ -T_L(s) \end{bmatrix}$$

$$\Delta_o(s) = J_L J_m s^3 + [J_L (B_m + B) + J_m (B_L + B)] s^2 + [B_L B_m + (B_L + B_m) B + (J_m + J_L) K] s + K (B_L + B)$$

6-21. The following differential equations describe the motion of an electric train in a traction system:

$$\frac{dx(t)}{dt} = v(t)$$

$$\frac{dv(t)}{dt} = -k(v) - g(x) + f(t)$$

where

$x(t)$ = linear displacement of train

$v(t)$ = linear velocity of train

$k(v)$ = resistance force on train [odd function of v , with the properties: $k(0) = 0$ and $dk(v)/dv = 0$].

$g(x)$ = gravitational force for a non-level track or due to curvature of track

$f(t)$ = tractive force

The electric motor that provides the tractive force is described by the following equations:

$$e(t) = K_b \phi(t) v(t) + R_a i_a(t)$$

$$f(t) = K_i \phi(t) i_a(t)$$

where $e(t)$ is the applied voltage; $i_a(t)$, the armature current; $i_f(t)$, the field current; R_a , the armature resistance; $\phi(t)$, the magnetic flux from a separately excited field = $K_f i_f(t)$; and K_i , the force constant.

(a) Consider that the motor is a dc series motor with the armature and field windings connected in series, so that $i_a(t) = i_f(t)$, $g(x) = 0$, $k(v) = Bv(t)$, and $R_a = 0$. Show that the system is described by the following nonlinear state equations:

$$\frac{dx(t)}{dt} = v(t)$$

$$\frac{dv(t)}{dt} = -Bv(t) + \frac{K_i}{K_b^2 K_f v^2(t)} e^2(t)$$

(b) Consider that for the conditions stated in part (a), $i_a(t)$ is the input of the system [instead of $e(t)$]. Derive the state equations of the system.

(c) Consider the same conditions as in part (a) but with $\phi(t)$ as the input. Derive the state equations.

(a) Nonlinear differential equations:

$$\frac{dx(t)}{dt} = v(t) \quad \frac{dv(t)}{dt} = -k(v) - g(x) + f(t) = -Bv(t) + f(t)$$

With $R_a = 0$,

$$\phi(t) = \frac{e(t)}{K_b v(t)} = K_f i_f(t) = K_f i_a(t) \quad \text{Then, } i_a(t) = \frac{e(t)}{K_b K_f v(t)}$$

$$f(t) = K_i \phi(t) i_a(t) = \frac{K_i e^2(t)}{K_b^2 K_f v^2(t)}. \quad \text{Thus, } \frac{dv(t)}{dt} = -Bv(t) + \frac{K_i}{K_b^2 K_f v^2(t)} e^2(t)$$

(b) State equations: $i_a(t)$ as input.

$$\frac{dx(t)}{dt} = v(t) \quad \frac{dv(t)}{dt} = -Bv(t) + K_i K_f i_a^2(t)$$

(c) State equations: $\phi(t)$ as input.

$$f(t) = K_i K_f i_a^2(t) \quad i_a(t) = i_f(t) = \frac{\phi(t)}{K_f}$$

$$\frac{dx(t)}{dt} = v(t) \quad \frac{dv(t)}{dt} = -Bv(t) + \frac{K_i}{K_f} \phi^2(t)$$

6-22. Figure 6P-22(a) shows a well-known “broom-balancing” system. The objective of the control system is to maintain the broom in the upright position by means of the force $u(t)$ applied to the car as shown. In practical applications, the system is analogous to a one-dimensional control problem of balancing a unicycle or a missile immediately after launching. The free-body diagram of the system is shown in Fig. 6P-22(b), where

f_x = force at broom base in horizontal direction

f_y = force at broom base in vertical direction

M_b = mass of broom

g = gravitational acceleration

M_c = mass of car

J_b = moment of inertia of broom about center of gravity $CG = M_b L^2/3$

(a) Write the force equations in the x and the y directions at the pivot point of the broom. Write the torque equation about the center of gravity CG of the broom. Write the force equation of the car in the horizontal direction.

(b) Express the equations obtained in part (a) as state equations by assigning the state variables as $x_1 = \theta, x_2 = d\theta/dt, x_3 = x$, and $x_4 = dx/dt$. Simplify these equations for small θ by making the approximations $\sin \theta \cong \theta$ and $\cos \theta \cong 1$. **(c)** Obtain a small-signal linearized state-equation model for the system in the form of

$$\frac{d\Delta \mathbf{x}(t)}{dt} = \mathbf{A} * \Delta \mathbf{x}(t) + \mathbf{B} * \Delta \mathbf{r}(t)$$

at the equilibrium point $x_{01}(t) = 1, x_{02}(t) = 0, x_{03}(t) = 0$, and $x_{04}(t) = 0$.

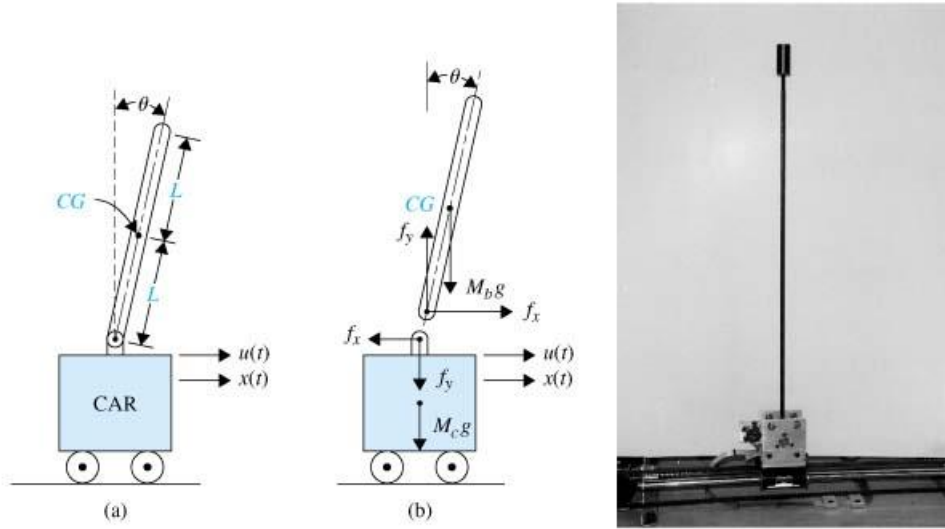


Figure 6P-22

(a) Nonlinear differential equations:

$$\frac{dx(t)}{dt} = v(t) \quad \frac{dv(t)}{dt} = -k(v) - g(x) + f(t) = -Bv(t) + f(t)$$

With $R_a = 0$,

$$\phi(t) = \frac{e(t)}{K_b v(t)} = K_f i_f(t) = K_f i_a(t) \quad \text{Then, } i_a(t) = \frac{e(t)}{K_b K_f v(t)}$$

$$f(t) = K_i \phi(t) i_a(t) = \frac{K_i e^2(t)}{K_b^2 K_f v^2(t)}. \quad \text{Thus, } \frac{dv(t)}{dt} = -Bv(t) + \frac{K_i}{K_b^2 K_f v^2(t)} e^2(t)$$

(b) State equations: $i_a(t)$ as input.

$$\frac{dx(t)}{dt} = v(t) \quad \frac{dv(t)}{dt} = -Bv(t) + K_i K_f i_a^2(t)$$

(c) State equations: $\phi(t)$ as input.

$$f(t) = K_i K_f i_a^2(t) \quad i_a(t) = i_f(t) = \frac{\phi(t)}{K_f}$$

$$\frac{dx(t)}{dt} = v(t) \quad \frac{dv(t)}{dt} = -Bv(t) + \frac{K_i}{K_f} \phi^2(t)$$

6-23. Figure 6P-23 shows the schematic diagram of a ball-suspension control system. The steel ball is suspended in the air by the electromagnetic force generated by the electromagnet. The objective of the control is to keep the metal ball suspended at the nominal equilibrium position by controlling the current in the magnet with the voltage $e(t)$. The practical application of this system is the magnetic levitation of trains or magnetic bearings in high-precision control systems.

The resistance of the coil is R , and the inductance is $L(y) = L/y(t)$, where L is a constant. The applied voltage $e(t)$ is a constant with amplitude E .

(a) Let E_{eq} be a nominal value of E . Find the nominal values of $y(t)$ and $dy(t)/dt$ at equilibrium.

(b) Define the state variables at $x_1(t) = i(t)$, $x_2(t) = y(t)$, and $x_3(t) = dy(t)/dt$. Find the nonlinear state equations in the form of

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(\mathbf{x}, e)$$

(c) Linearize the state equations about the equilibrium point and express the linearized state equations as

$$\frac{d\Delta\mathbf{x}(t)}{dt} = \mathbf{A}^* \Delta\mathbf{x}(t) + \mathbf{B}^* \Delta e(t)$$

The force generated by the electromagnet is $Ki^2(t)/y(t)$, where K is a proportional constant, and the gravitational force on the steel ball is Mg .

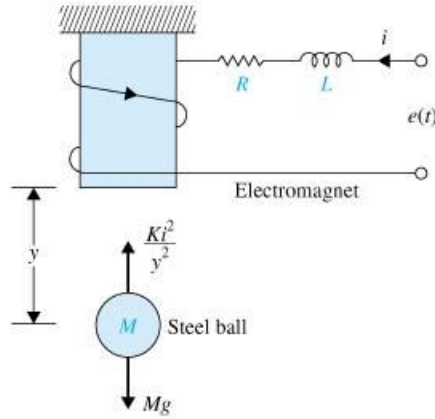


Figure 6P-23

Differential equations: $\left[L(y) = \frac{L}{y} \right]$

$$e(t) = Ri(t) + \frac{d[L(y)i(t)]}{dt} = Ri(t) + i(t) \frac{dL(y)}{dy} \frac{dy(t)}{dt} + \frac{L}{y} \frac{di(t)}{dt} = Ri(t) - \frac{L}{y^2} i(t) \frac{dy(t)}{dt} + \frac{L}{y} \frac{di(t)}{dt}$$

$$My(t) = Mg - \frac{Ki^2(t)}{y^2(t)} \quad \text{At equilibrium, } \frac{di(t)}{dt} = 0, \quad \frac{dy(t)}{dt} = 0, \quad \frac{d^2y(t)}{dt^2} = 0$$

$$\text{Thus, } i_{eq} = \frac{E_{eq}}{R} \quad \frac{dy_{eq}}{dt} = 0 \quad y_{eq} = \frac{E_{eq}}{R} \sqrt{\frac{K}{Mg}}$$

(b) Define the state variables as $x_1 = i$, $x_2 = y$, and $x_3 = \frac{dy}{dt}$.

$$\text{Then, } x_{1eq} = \frac{E_{eq}}{R} \quad x_{2eq} = \frac{E_{eq}}{R} \sqrt{\frac{K}{Mg}} \quad x_{3eq} = 0$$

The differential equations are written in state equation form:

$$\frac{dx_1}{dt} = -\frac{R}{L}x_1x_2 + \frac{x_1x_3}{x_2} + \frac{x_2}{L}e = f_1 \quad \frac{dx_2}{dt} = x_3 = f_2 \quad \frac{dx_3}{dt} = g - \frac{K}{M} \frac{x_1^2}{x_2^2} = f_3$$

(c) **Linearization:**

$$\frac{\partial f_1}{\partial x_1} = -\frac{R}{L}x_{2eq} + \frac{x_{3eq}}{x_{2eq}} = -\frac{E_{eq}}{L} \sqrt{\frac{K}{Mg}} \quad \frac{\partial f_1}{\partial x_2} = -\frac{R}{L}x_{1eq} - \frac{x_1x_3}{x_2^2} + \frac{E_{eq}}{L} = 0 \quad \frac{\partial f_1}{\partial x_3} = \frac{x_{1eq}}{x_{2eq}} = \sqrt{\frac{Mg}{K}}$$

$$\frac{\partial f_2}{\partial x_1} = \frac{x_{2eq}}{L} = \frac{1}{L} \sqrt{\frac{K}{Mg}} \frac{E_{eq}}{R} \quad \frac{\partial f_2}{\partial x_2} = 0 \quad \frac{\partial f_2}{\partial x_3} = 0 \quad \frac{\partial f_2}{\partial e} = 1 \quad \frac{\partial f_2}{\partial x_3} = 0$$

$$\frac{\partial f_3}{\partial x_1} = -\frac{2K}{M} \frac{x_{1eq}}{x_{2eq}^2} = -\frac{2Rg}{E_{eq}} \quad \frac{\partial f_3}{\partial x_2} = \frac{2K}{M} \frac{x_{1eq}^2}{x_{2eq}^3} = \frac{2Rg}{E_{eq}} \sqrt{\frac{Mg}{K}} \quad \frac{\partial f_3}{\partial e} = 0$$

The linearized state equations about the equilibrium point are written as:

$$\Delta \dot{\mathbf{x}} = \mathbf{A}^* \Delta \mathbf{x} + \mathbf{B}^* \Delta e$$

$$\mathbf{A}^* = \begin{bmatrix} -\frac{E_{eq}}{L} \sqrt{\frac{K}{Mg}} & 0 & \sqrt{\frac{Mg}{K}} \\ 0 & 0 & 0 \\ -\frac{2Rg}{E_{eq}} & \frac{2Rg}{E_{eq}} \sqrt{\frac{Mg}{K}} & 0 \end{bmatrix} \quad \mathbf{B}^* = \begin{bmatrix} \frac{E_{eq}}{RL} \sqrt{\frac{K}{Mg}} \\ 0 \\ 0 \end{bmatrix}$$

6-24. Figure 6P-24(a) shows the schematic diagram of a ball-suspension system. The steel ball is suspended in the air by the electromagnetic force generated by the electromagnet. The objective of the control is to keep the metal ball suspended at the nominal position by controlling the current in the electromagnet. When the system is at the stable equilibrium point, any small perturbation of the ball position from its floating

equilibrium position will cause the control to return the ball to the equilibrium position. The free-body diagram of the system is shown in Fig. 6P-24(b), where

M_1 = mass of electromagnet = 2.0

M_2 = mass of steel ball = 1.0

B = viscous-friction coefficient of air = 0.1

K = proportional constant of electromagnet = 1.0

g = gravitational acceleration = 32.2

Assume all units are consistent. Let the stable equilibrium values of the variable, $i(t)$, $y_1(t)$, and $y_2(t)$ be I , Y_1 , and Y_2 , respectively. The state variables are defined as $x_1(t) = y_1(t)$, $x_2(t) = dy_1(t)/dt$, $x_3(t) = y_2(t)$, and $x_4(t) = dy_2(t)/dt$.

(a) Given $Y_1 = 1$, find I and Y_2 .

(b) Write the nonlinear state equations of the system in the form of $dx(t)/dt = f(x, i)$.

(c) Find the state equations of the linearized system about the equilibrium state I , Y_1 , and Y_2 in the form:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A} * \Delta\mathbf{x}(t) + \mathbf{B} * \Delta i(t)$$

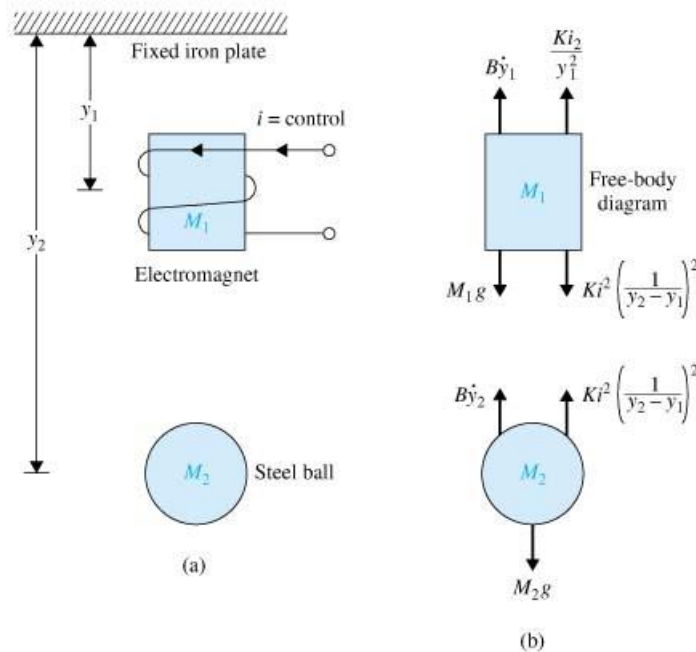


Figure 6P-24

(a) Differential equations:

$$M_1 \frac{d^2 y_1(t)}{dt^2} = M_1 g - B \frac{dy_1(t)}{dt} - \frac{Ki^2(t)}{y_1^2(t)} + Ki^2(t) \frac{1}{[y_2(t) - y_1(t)]^2}$$

$$M_2 \frac{d^2 y_2(t)}{dt^2} = M_2 g - B \frac{dy_2(t)}{dt} - \frac{Ki^2(t)}{[y_2(t) - y_1(t)]^2}$$

Define the state variables as $x_1 = y_1$, $x_2 = \frac{dy_1}{dt}$, $x_3 = y_2$, $x_4 = \frac{dy_2}{dt}$.

The state equations are:

$$\frac{dx_1}{dt} = x_2 \quad M_1 \frac{dx_2}{dt} = M_1 g - Bx_2 - \frac{Ki^2}{x_1^2} + \frac{Ki^2}{(x_3 - x_1)^2} \quad \frac{dx_3}{dt} = x_4 \quad M_2 \frac{dx_4}{dt} = M_2 g - Bx_4 - \frac{Ki^2}{(x_3 - x_1)^2}$$

At equilibrium, $\frac{dx_1}{dt} = 0$, $\frac{dx_2}{dt} = 0$, $\frac{dx_3}{dt} = 0$, $\frac{dx_4}{dt} = 0$. Thus, $x_{2eq} = 0$ and $x_{4eq} = 0$.

$$M_1 g - \frac{KI^2}{X_1^2} + \frac{KI^2}{(X_3 - X_1)^2} = 0 \quad M_2 g - \frac{KI^2}{(X_3 - X_1)^2} = 0$$

Solving for I , with $X_1 = 1$, we have

$$Y_2 = X_3 = 1 + \left(\frac{M_1 + M_2}{M_2} \right)^{1/2} \quad I = \left(\frac{(M_1 + M_2)g}{K} \right)^{1/2}$$

(b) Nonlinear state equations:

$$\frac{dx_1}{dt} = x_2 \quad \frac{dx_2}{dt} = g - \frac{B}{M_1} x_2 - \frac{K}{M_1 x_1^2} i^2 + \frac{Ki^2}{M_1 (x_3 - x_1)^2} \quad \frac{dx_3}{dt} = x_4 \quad \frac{dx_4}{dt} = g - \frac{B}{M_2} x_4 - \frac{Ki^2}{M_2 (x_3 - x_1)^2}$$

(c) Linearization:

$$\begin{array}{ccccc} \frac{\partial f_1}{\partial x_1} = 0 & \frac{\partial f_1}{\partial x_2} = 0 & \frac{\partial f_1}{\partial x_3} = 0 & \frac{\partial f_1}{\partial x_4} = 0 & \frac{\partial f_1}{\partial i} = 0 \\ \frac{\partial f_2}{\partial x_1} = \frac{2KI^2}{M_1 x_1^3} + \frac{2KI^2}{M_1 (X_3 - X_1)^3} & \frac{\partial f_2}{\partial x_2} = -\frac{B}{M_1} & \frac{\partial f_2}{\partial x_3} = \frac{-2KI^2}{M_1 (X_3 - X_1)^3} & \frac{\partial f_2}{\partial x_4} = 0 & \frac{\partial f_2}{\partial i} = 0 \end{array}$$

$$\begin{array}{ccccc} \frac{\partial f_3}{\partial i} = \frac{2KI}{M_1} \left(\frac{-1}{X_1^2} + \frac{1}{(X_3 - X_1)^2} \right) & \frac{\partial f_3}{\partial x_1} = 0 & \frac{\partial f_3}{\partial x_2} = 0 & \frac{\partial f_3}{\partial x_3} = 0 & \frac{\partial f_3}{\partial x_4} = 1 & \frac{\partial f_3}{\partial i} = 0 \end{array}$$

$$\frac{\partial f_4}{\partial x_1} = \frac{-2KI^2}{M_2(X_3 - X_1)^3} \quad \frac{\partial f_4}{\partial x_2} = 0 \quad \frac{\partial f_4}{\partial x_3} = \frac{2KI^2}{M_2(X_3 - X_1)^3} \quad \frac{\partial f_4}{\partial x_4} = -\frac{B}{M_2} \quad \frac{\partial f_4}{\partial i} = \frac{-2KI}{M_2(X_3 - X_1)^2}$$

Linearized state equations: $M_1 = 2$, $M_2 = 1$, $g = 32.2$, $B = 0.1$, $K = 1$.

$$I = \left(\frac{32.2(1+2)}{1} \right)^{1/2} X_1 = \sqrt{96.6} X_1 = 9.8285 X_1 \quad X_1 = \frac{1}{9.8285} = 1$$

$$X_3 = (1 + \sqrt{1+2}) X_1 = 2.732 X_1 = Y_2 = 2.732 \quad X_3 - X_1 = 1.732$$

$$\mathbf{A}^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{2KI^2}{M_1} \left(\frac{1}{X_1^3} + \frac{1}{(X_3 - X_1)^3} \right) & -\frac{B}{M_1} & \frac{-2KI^2}{M_1(X_3 - X_1)^3} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-2KI^2}{M_2(X_3 - X_1)^3} & 0 & \frac{2KI^2}{M_2(X_3 - X_1)^3} & -\frac{B}{M_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 115.2 & -0.05 & -18.59 & 0 \\ 0 & 0 & 0 & 1 \\ -37.18 & 0 & 37.18 & -0.1 \end{bmatrix}$$

$$\mathbf{B}^* = \begin{bmatrix} 0 \\ \frac{2KI}{M_1} \left(\frac{-1}{X_1^2} + \frac{1}{(X_3 - X_1)^2} \right) \\ 0 \\ \frac{-2KI}{M_2(X_3 - X_1)^2} \end{bmatrix} = \begin{bmatrix} 0 \\ -6.552 \\ 0 \\ -6.552 \end{bmatrix}$$

6-25. The schematic diagram of a steel-rolling process is shown in Fig. 6P-25. The steel plate is fed through the rollers at a constant speed of V ft/sec. The distance between the rollers and the point where the thickness is measured is d ft. The rotary displacement of the motor, $\theta_m(t)$, is converted to the linear displacement $y(t)$ by the gear box and linear-actuator combination $y(t) = n\theta_m(t)$, where n is a positive constant in ft/rad. The equivalent inertia of the load that is reflected to the motor shaft is J_L .

(a) Draw a functional block diagram for the system.

(b) Derive the forward-path transfer function $Y(s)/E(s)$ and the closed-loop transfer function $Y(s)/R(s)$.

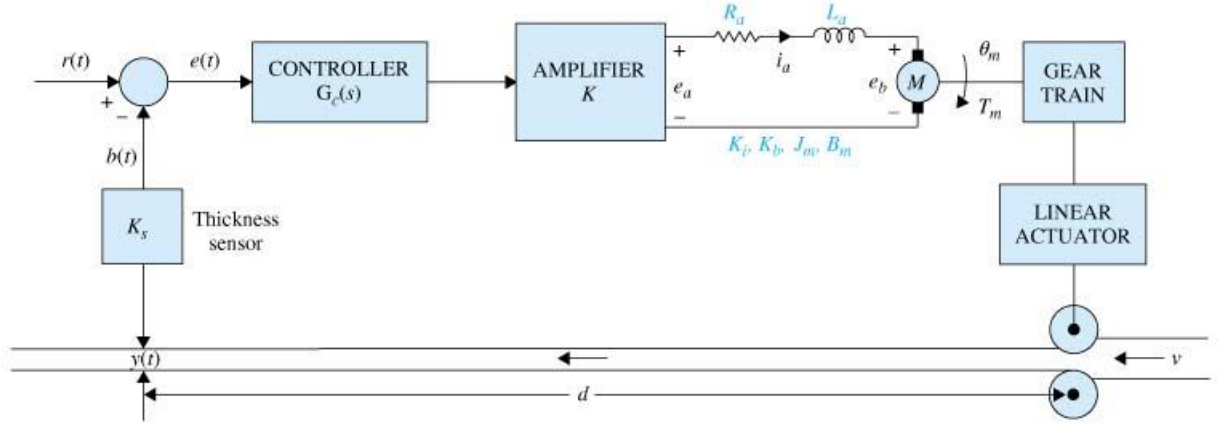


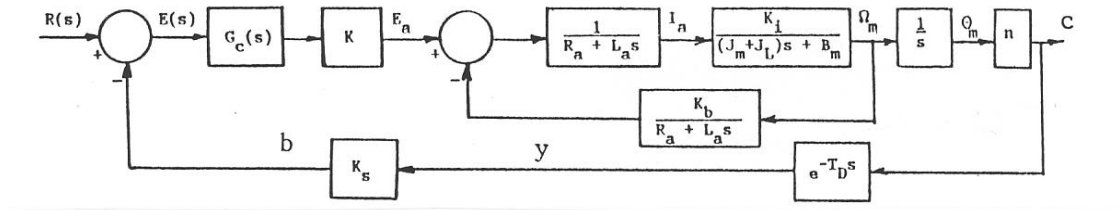
Figure 6P-25

(a) System equations:

$$T_m = K_i i_a = (J_m + J_L) \frac{d\omega_m}{dt} + B_m \omega_m \quad e_a = R_a i_a + L_a \frac{di_a}{dt} + K_b \omega_m \quad y = n\theta_m \quad y = y(t - T_D)$$

$$T_D = \frac{d}{V} \text{ (sec)} \quad e = r - b \quad b = K_s y \quad E_a(s) = KG_c(s)E(s)$$

Block diagram:



(b) Forward-path transfer function:

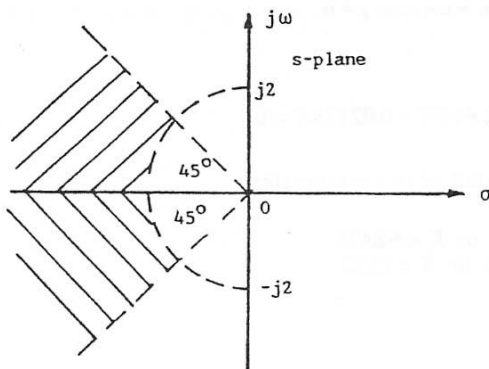
$$\frac{Y(s)}{E(s)} = \frac{KK_i n G_c(s) e^{-T_D s}}{s \{ (R_a + L_a s) [(J_m + J_L)s + B_m] + K_b K_i \}}$$

Closed-loop transfer function:

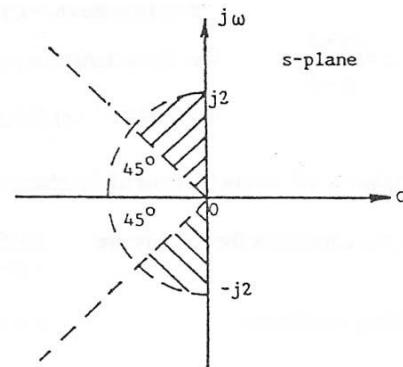
$$\frac{Y(s)}{R(s)} = \frac{KK_i n G_c(s) e^{-T_D s}}{s (R_a + L_a s) [(J_m + J_L)s + B_m] + K_b K_i s + KG_c(s) K_i n e^{-T_D s}}$$

Chapter 7

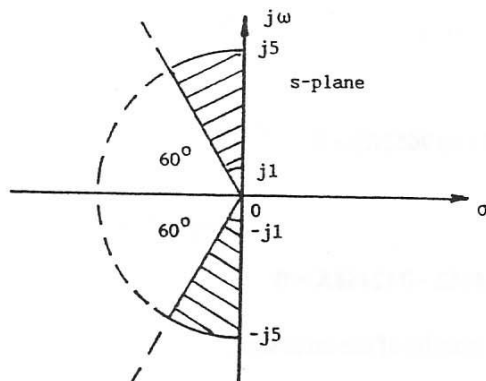
7-1 (a) $\zeta \geq 0.707$ $\omega_n \geq 2$ rad/sec



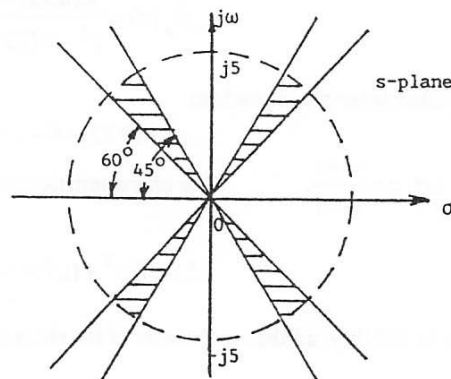
(b) $0 \leq \zeta \leq 0.707$ $\omega_n \leq 2$ rad/sec



(c) $\zeta \leq 0.5$ $1 \leq \omega_n \leq 5$ rad/sec



(d) $0.5 \leq \zeta \leq 0.707$ $\omega_n \leq 0.5$ rad/sec



7-2 (a) Type 0

(b) Type 0

(c) Type 1

(d) Type 2

(e) Type 3

(f) Type 3

(g) type 2

(h) type 1

7-3 (a) $K_p = \lim_{s \rightarrow 0} G(s) = 1000$

$K_v = \lim_{s \rightarrow 0} sG(s) = 0$

$K_a = \lim_{s \rightarrow 0} s^2 G(s) = 0$

(b) $K_p = \lim_{s \rightarrow 0} G(s) = \infty$

$K_v = \lim_{s \rightarrow 0} sG(s) = 1$

$K_a = \lim_{s \rightarrow 0} s^2 G(s) = 0$

$$(c) \quad K_p = \lim_{s \rightarrow 0} G(s) = \infty$$

$$K_v = \lim_{s \rightarrow 0} sG(s) = K$$

$$K_a = \lim_{s \rightarrow 0} s^2 G(s) = 0$$

$$(d) \quad K_p = \lim_{s \rightarrow 0} G(s) = \infty$$

$$K_v = \lim_{s \rightarrow 0} sG(s) = \infty$$

$$K_a = \lim_{s \rightarrow 0} s^2 G(s) = 1$$

$$(e) \quad K_p = \lim_{s \rightarrow 0} G(s) = \infty$$

$$K_v = \lim_{s \rightarrow 0} sG(s) = 1$$

$$K_a = \lim_{s \rightarrow 0} s^2 G(s) = 0$$

$$(f) \quad K_p = \lim_{s \rightarrow 0} G(s) = \infty$$

$$K_v = \lim_{s \rightarrow 0} sG(s) = \infty$$

$$K_a = \lim_{s \rightarrow 0} s^2 G(s) = K$$

7-4 (a) Input**Error Constants****Steady-state Error**

$u_s(t)$	$K_p = 1000$	1/1001
$tu_s(t)$	$K_v = 0$	∞
$t^2 u_s(t)/2$	$K_a = 0$	∞

(b)**Input****Error Constants****Steady-state Error**

$u_s(t)$	$K_p = \infty$	0
$tu_s(t)$	$K_v = 1$	1
$t^2 u_s(t)/2$	$K_a = 0$	∞

(c) Input**Error Constants****Steady-state Error**

$u_s(t)$	$K_p = \infty$	0
----------	----------------	---

$tu_s(t)$	$K_v = K$	$1/K$
$t^2 u_s(t)/2$	$K_a = 0$	∞

The above results are valid if the value of K corresponds to a stable closed-loop system.

(d) The closed-loop system is unstable. It is meaningless to conduct a steady-state error analysis.

(e)	Input	Error Constants	Steady-state Error
	$u_s(t)$	$K_p = \infty$	0
	$tu_s(t)$	$K_v = 1$	1
	$t^2 u_s(t)/2$	$K_a = 0$	∞

(f)	Input	Error Constants	Steady-state Error
	$u_s(t)$	$K_p = \infty$	0
	$tu_s(t)$	$K_v = \infty$	0
	$t^2 u_s(t)/2$	$K_a = K$	$1/K$

The closed-loop system is stable for all positive values of K . Thus the above results are valid.

7-5 (a) $K_H = H(0) = 1$

$$M(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{s+1}{s^3 + 2s^2 + 3s + 3}$$

$$a_0 = 3, \quad a_1 = 3, \quad a_2 = 2, \quad b_0 = 1, \quad b_1 = 1.$$

Unit-step Input:

$$e_{ss} = \frac{1}{K_H} \left(1 - \frac{b_0 K_H}{a_0} \right) = \frac{2}{3}$$

Unit-ramp input:

$$a_0 - b_0 K_H = 3 - 1 = 2 \neq 0. \text{ Thus } e_{ss} = \infty.$$

Unit-parabolic Input:

$$a_0 - b_0 K_H = 2 \neq 0 \text{ and } a_1 - b_1 K_H = 1 \neq 0. \text{ Thus } e_{ss} = \infty.$$

(b) $K_H = H(0) = 5$

$$M(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{1}{s^2 + 5s + 5} \quad a_0 = 5, \quad a_1 = 5, \quad b_0 = 1, \quad b_1 = 0.$$

Unit-step Input:

$$e_{ss} = \frac{1}{K_H} \left(1 - \frac{b_0 K_H}{a_0} \right) = \frac{1}{5} \left(1 - \frac{5}{5} \right) = 0$$

Unit-ramp Input:

$$i = 0: \quad a_0 - b_0 K_H = 0 \quad i = 1: \quad a_1 - b_1 K_H = 5 \neq 0$$

$$e_{ss} = \frac{a_1 - b_1 K_H}{a_0 K_H} = \frac{5}{25} = \frac{1}{5}$$

Unit-parabolic Input:

$$e_{ss} = \infty$$

(c) $K_H = H(0) = 1/5$

$$M(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{s + 5}{s^4 + 15s^3 + 50s^2 + s + 1} \quad \text{The system is stable.}$$

$$a_0 = 1, \quad a_1 = 1, \quad a_2 = 50, \quad a_3 = 15, \quad b_0 = 5, \quad b_1 = 1$$

Unit-step Input:

$$e_{ss} = \frac{1}{K_H} \left(1 - \frac{b_0 K_H}{a_0} \right) = 5 \left(1 - \frac{5/5}{1} \right) = 0$$

Unit-ramp Input:

$$i=0: a_0 - b_0 K_H = 0 \quad i=1: a_1 - b_1 K_H = 4/5 \neq 0$$

$$e_{ss} = \frac{a_1 - b_1 K_H}{a_0 K_H} = \frac{1 - 1/5}{1/5} = 4$$

Unit-parabolic Input:

$$e_{ss} = \infty$$

(d) $K_H = H(0) = 10$

$$M(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{1}{s^3 + 12s^2 + 5s + 10} \quad \text{The system is stable.}$$

$$a_0 = 10, \quad a_1 = 5, \quad a_2 = 12, \quad b_0 = 1, \quad b_1 = 0, \quad b_2 = 0$$

Unit-step Input:

$$e_{ss} = \frac{1}{K_H} \left(1 - \frac{b_0 K_H}{a_0} \right) = \frac{1}{10} \left(1 - \frac{10}{10} \right) = 0$$

Unit-ramp Input:

$$i=0: a_0 - b_0 K_H = 0 \quad i=1: a_1 - b_1 K_H = 5 \neq 0$$

$$e_{ss} = \frac{a_1 - b_1 K_H}{a_0 K_H} = \frac{5}{100} = 0.05$$

Unit-parabolic Input:

$$e_{ss} = \infty$$

7-6 (a) $M(s) = \frac{s+4}{s^4 + 16s^3 + 48s^2 + 4s + 4} \quad K_H = 1 \quad \text{The system is stable.}$

$$a_0 = 4, \quad a_1 = 4, \quad a_2 = 48, \quad a_3 = 16, \quad b_0 = 4, \quad b_1 = 1, \quad b_2 = 0, \quad b_3 = 0$$

Unit-step Input:

$$e_{ss} = \frac{1}{K_H} \left(1 - \frac{b_0 K_H}{a_0} \right) = \left(1 - \frac{4}{4} \right) = 0$$

Unit-ramp input:

$$i=0: a_0 - b_0 K_H = 0 \quad i=1: a_1 - b_1 K_H = 4 - 1 = 3 \neq 0$$

$$e_{ss} = \frac{a_1 - b_1 K_H}{a_0 K_H} = \frac{4-1}{4} = \frac{3}{4}$$

Unit-parabolic Input:

$$e_{ss} = \infty$$

(b) $M(s) = \frac{K(s+3)}{s^3 + 3s^2 + (K+2)s + 3K}$ $K_H = 1$ The system is stable for $K > 0$.

$$a_0 = 3K, \quad a_1 = K+2, \quad a_2 = 3, \quad b_0 = 3K, \quad b_1 = K$$

Unit-step Input:

$$e_{ss} = \frac{1}{K_H} \left(1 - \frac{b_0 K_H}{a_0} \right) = \left(1 - \frac{3K}{3K} \right) = 0$$

Unit-ramp Input:

$$i=0: \quad a_0 - b_0 K_H = 0 \quad i=1: \quad a_1 - b_1 K_H = K+2-K = 2 \neq 0$$

$$e_{ss} = \frac{a_1 - b_1 K_H}{a_0 K_H} = \frac{K+2-K}{3K} = \frac{2}{3K}$$

Unit-parabolic Input:

$$e_{ss} = \infty$$

The above results are valid for $K > 0$.

(c) $M(s) = \frac{s+5}{s^4 + 15s^3 + 50s^2 + 10s}$ $H(s) = \frac{10s}{s+5}$ $K_H = \lim_{s \rightarrow 0} \frac{H(s)}{s} = 2$

$$a_0 = 0, \quad a_1 = 10, \quad a_2 = 50, \quad a_3 = 15, \quad b_0 = 5, \quad b_1 = 1$$

Unit-step Input:

$$e_{ss} = \frac{1}{K_H} \left(\frac{a_2 - b_1 K_H}{a_1} \right) = \frac{1}{2} \left(\frac{50 - 1 \times 2}{10} \right) = 2.4$$

Unit-ramp Input:

$$e_{ss} = \infty$$

Unit-parabolic Input:

$$e_{ss} = \infty$$

(d) $M(s) = \frac{K(s+5)}{s^4 + 17s^3 + 60s^2 + 5Ks + 5K}$ $K_H = 1$ The system is stable for $0 < K < 204$.

$$a_0 = 5K, \quad a_1 = 5K, \quad a_2 = 60, \quad a_3 = 17, \quad b_0 = 5K, \quad b_1 = K$$

Unit-step Input:

$$e_{ss} = \frac{1}{K_H} \left(1 - \frac{b_0 K_H}{a_0} \right) = \left(1 - \frac{5K}{5K} \right) = 0$$

Unit-ramp Input:

$$i = 0: \quad a_0 - b_0 K_H = 0 \quad i = 1: \quad a_1 - b_1 K_H = 5K - K = 4K \neq 0$$

$$e_{ss} = \frac{a_1 - b_1 K_H}{a_0 K_H} = \frac{5K - K}{5K} = \frac{4}{5}$$

Unit-parabolic Input:

$$e_{ss} = \infty$$

The results are valid for $0 < K < 204$.

7-7)

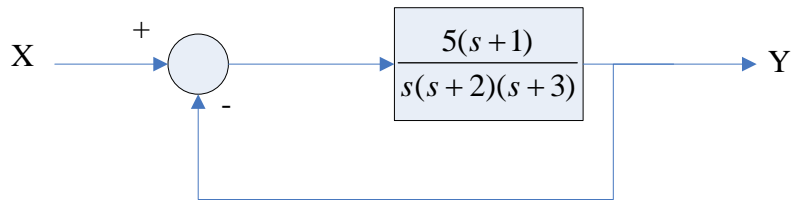
$$\frac{Y(s)}{X(s)} = \frac{\left(\left(\frac{s+1}{s+3} \right) \frac{s}{s(s+2)} \right)}{1 + \frac{5(s+1)}{s(s+2)(s+3)}} = \frac{5(s+1)}{s^3 + 5s^2 + 11s + 5}$$

⇒ Type of the system is zero

Pole: $s = -2.2013 + 1.8773i$, $s = -2.2013 - 1.8773i$, and $s = -0.5974$

Zero: $s = -1$

7-8)



$$G(s) = \frac{5(s+1)}{s(s+2)(s+3)}$$

- a) Position error: $K_p = \lim_{s \rightarrow 0} G(s) = \lim_{s \rightarrow 0} \frac{5(s+1)}{s(s+2)(s+3)} = \infty$
- b) Velocity error: $K_v = \lim_{s \rightarrow 0} sG(s) = \lim_{s \rightarrow 0} \frac{5(s+1)}{(s+2)(s+3)} = \frac{5}{6}$
- c) Acceleration error: $K_a = \lim_{s \rightarrow \infty} s^2 G(s) = \lim_{s \rightarrow \infty} \frac{5s(s+1)}{(s+2)(s+3)} = 0$

7-9) a) Steady state error for unit step input:

$$e_{ss} = \frac{1}{1+K_p}$$

Referring to the result of problem 7-8, $K_p = \infty \Rightarrow e_{ss} = 0$

b) Steady state error for ramp input:

$$e_{ss} = \frac{1}{K_v}$$

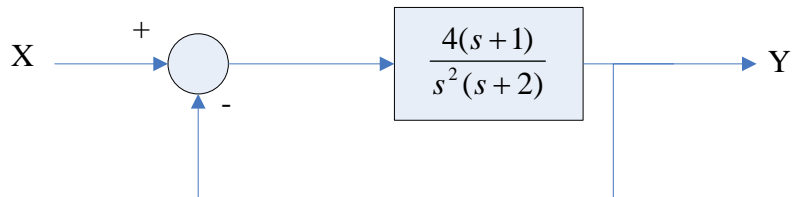
Regarding the result of problem 7-8, $K_v = \frac{5}{6} \rightarrow e(\infty) = \frac{6}{5}$

c) Steady state error for parabolic input:

$$e_{ss} = \frac{1}{K_a}$$

Regarding the result of problem 7-8, $K_a = 0 \rightarrow e(\infty) = \infty$

7-10)



a) Step error constant: $K_p = \lim_{s \rightarrow 0} \frac{4(s+1)}{s^2(s+2)} = \infty$

b) Ramp error constant: $K_v = \lim_{s \rightarrow 0} \frac{4(s+1)}{s(s+2)} = \infty$

c) Parabolic error constant: $K_a = \lim_{s \rightarrow 0} \frac{4(s+1)}{s+2} = 2$

7-11) $X = \frac{5}{2s} - \frac{3}{s^2} + \frac{4}{s^3} = \frac{5}{2}X_1(s) - 3X_2(s) + 4X_3(s)$

where x_1 is a unit step input, x_2 is a ramp input, and x_3 is a unit parabola input. Since the system is linear, then the effect of $X(s)$ is the summation of effect of each individual input.

That is: $e(\infty) = \frac{5}{2}e_1(\infty) - 3e_2(\infty) + 4e_3(\infty)$

So:

$$\begin{cases} e_{step} = \frac{1}{1+K_p} = 0 \\ e_{ramp} = \frac{1}{K_v} = 0 \\ e_{parabolic} = \frac{1}{K_a} = \frac{1}{2} \end{cases}$$

$$\Rightarrow e_{ss} = 4\left(\frac{1}{2}\right) = 2$$

7-12) The step input response of the system is:

$$Y(s) = G(s)U(s) = \frac{1-k}{s(s-k)} = \frac{1}{1+k} \left[\frac{1}{s} - \frac{1}{s-k} \right]$$

Therefore:

$$y(t) = \frac{1}{1+k} [e^{kt} + 1]u(t)$$

The rise time is the time that unit step response value reaches from 0.1 to 0.9. Then:

$$t_r = \frac{1}{1+k} [e^{0.9k} - e^{0.1k}]$$

It is obvious that $t_r > 0$, then:

$$\frac{1}{1+k} [e^{0.9k} - e^{0.1k}] > 0$$

As $|k| < 1$, then $\frac{1}{1+k} > 0$

Therefore $e^{0.9k} - e^{0.1k} > 0$ or $e^{0.9k} > e^{0.1k}$

which yields: $k > 0$

7-13)

$$G(s) = \frac{Y(s)}{E(s)} = \frac{KG_p(s)/20s}{1 + K_t G_p(s)} = \frac{100K}{20s(1 + 0.2s + 100K_t)} \quad \text{Type-1 system.}$$

Error constants: $K_p = \infty, \quad K_v = \frac{5K}{1 + 100K_t}, \quad K_a = 0$

(a) $r(t) = u_s(t): \quad e_{ss} = \frac{1}{1 + K_p} = 0$

(b) $r(t) = tu_s(t): \quad e_{ss} = \frac{1}{K_v} = \frac{1 + 100K_t}{5K}$

(c) $r(t) = t^2 u_s(t) / 2: \quad e_{ss} = \frac{1}{K_a} = \infty$

7-14

$$G_p(s) = \frac{100}{(1+0.1s)(1+0.5s)} \quad G(s) = \frac{Y(s)}{E(s)} = \frac{KG_p(s)}{20s[1+K_tG_p(s)]}$$

$$G(s) = \frac{100K}{20s[(1+0.1s)(1+0.5s)+100K_t]}$$

Error constants: $K_p = \infty, \quad K_v = \frac{5K}{1+100K_t}, \quad K_a = 0$

(a) $r(t) = u_s(t): \quad e_{ss} = \frac{1}{1+K_p} = 0$

(b) $r(t) = tu_s(t): \quad e_{ss} = \frac{1}{K_v} = \frac{1+100K_t}{5K}$

(c) $r(t) = t^2u_s(t)/2: \quad e_{ss} = \frac{1}{K_a} = \infty$

Since the system is of the third order, the values of K and K_t must be constrained so that the system is

stable. The characteristic equation is

$$s^3 + 12s^2 + (20 + 2000K_t)s + 100K = 0$$

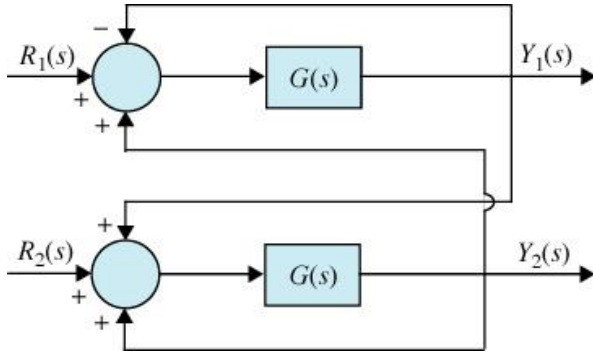
Routh Tabulation:

s^3	1	$20 + 2000K_t$
s^2	12	$100K$
s^1	$\frac{240 + 24000K_t - 100K}{12}$	
s^0	$100K$	

Stability Conditions: $K > 0 \quad 12(1+100K_t) - 5K > 0 \quad \text{or} \quad \frac{1+100K_t}{5K} > \frac{1}{12}$

Thus, the minimum steady-state error that can be obtained with a unit-ramp input is $1/12$.

7-15 (a) From Figure 3P-29,



$$\frac{\Theta_o(s)}{\Theta_r(s)} = \frac{1 + \frac{K_1 K_2}{R_a + L_a s} + \frac{K_i K_b + K K_1 K_i K_t}{(R_a + L_a s)(B_t + J_t s)}}{1 + \frac{K_1 K_2}{R_a + L_a s} + \frac{K_i K_b + K K_1 K_i K_t}{(R_a + L_a s)(B_t + J_t s)} + \frac{K K_s K_1 K_i N}{s(R_a + L_a s)(B_t + J_t s)}}$$

$$\frac{\Theta_o(s)}{\Theta_r(s)} = \frac{s[(R_a + L_a s)(B_t + J_t s) + K_1 K_2 (B_t + J_t s) + K_i K_b + K K_1 K_i K_t]}{L_a J_t s^3 + (L_a B_t + R_a J_t + K_1 K_2 J_t) s^2 + (R_a B_t + K_i K_b + K K_1 K_i K_t + K_1 K_2 B_t) s + K K_s K_1 K_i N}$$

$$\theta_r(t) = u_s(t), \quad \Theta_r(s) = \frac{1}{s} \quad \lim_{s \rightarrow 0} s \Theta_e(s) = 0$$

Provided that all the poles of $s \Theta_e(s)$ are all in the left-half s -plane.

(b) For a unit-ramp input, $\Theta_r(s) = 1/s^2$.

$$e_{ss} = \lim_{t \rightarrow \infty} \theta_e(t) = \lim_{s \rightarrow 0} s \Theta_e(s) = \frac{R_a B_t + K_1 K_2 B_t + K_i K_b + K K_1 K_i K_t}{K K_s K_1 K_i N}$$

if the limit is valid.

7-16 (a) Forward-path transfer function: $[n(t) = 0]$:

$$G(s) = \frac{Y(s)}{E(s)} = \frac{\frac{K(1+0.02s)}{s^2(s+25)}}{1 + \frac{KK_t s}{s^2(s+25)}} = \frac{K(1+0.02s)}{s(s^2 + 25s + KK_t)} \quad \text{Type-1 system.}$$

Error Constants: $K_p = \infty, \quad K_v = \frac{1}{K_t}, \quad K_a = 0$

For a unit-ramp input, $r(t) = tu_s(t)$, $R(s) = \frac{1}{s^2}$, $e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) = \frac{1}{K_v} = K_t$

Routh Tabulation:

s^3	1	$KK_t + 0.02K$
s^2	25	K
s^1	$\frac{25K(K_t + 0.02) - K}{25}$	
s^0	K	

Stability Conditions: $K > 0 \quad 25(K_t + 0.02) - K > 0 \quad \text{or} \quad K_t > 0.02$

(b) With $r(t) = 0$, $n(t) = u_s(t)$, $N(s) = 1/s$.

System Transfer Function with $N(s)$ as Input:

$$\frac{Y(s)}{N(s)} = \frac{\frac{K}{s^2(s+25)}}{1 + \frac{K(1+0.02s)}{s^2(s+25)} + \frac{KK_t s}{s^2(s+25)}} = \frac{K}{s^3 + 25s^2 + K(K_t + 0.02)s + K}$$

Steady-State Output due to $n(t)$:

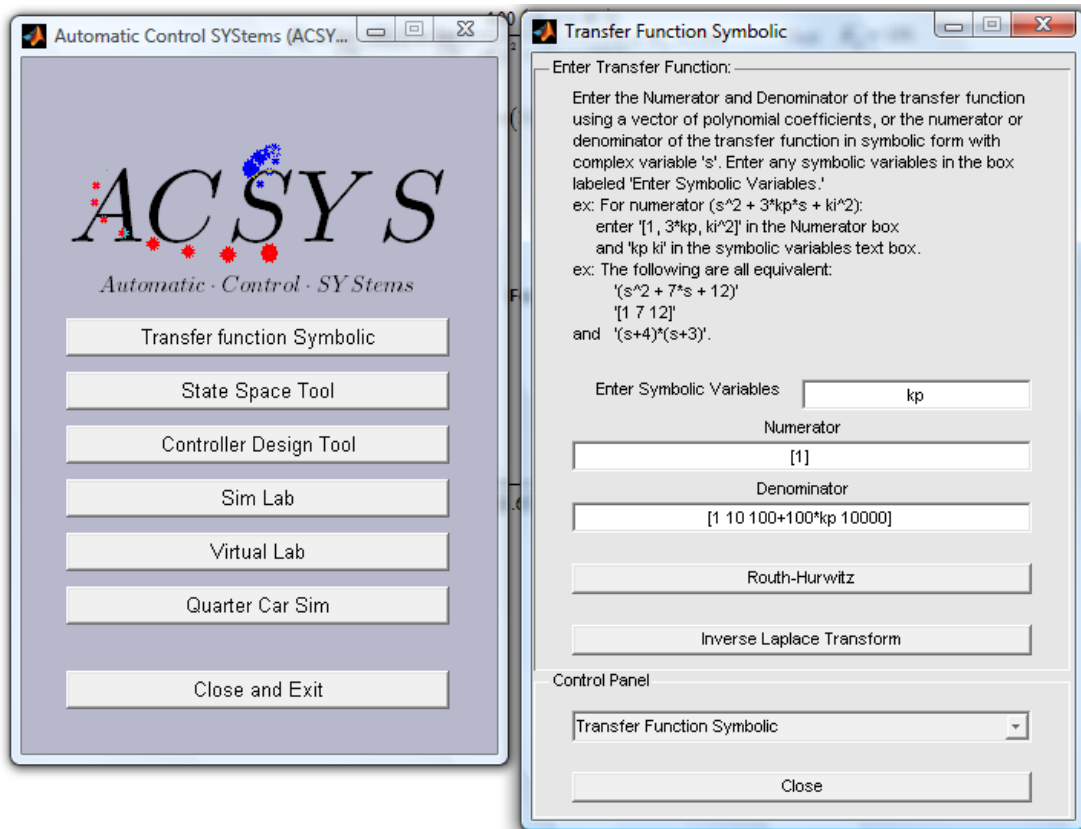
$$y_{ss} = \lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} sY(s) = 1 \quad \text{if the limit is valid.}$$

7-17 You may use MATLAB in all Routh Hurwitz calculations.

1. Activate MATLAB
2. Go to the directory containing the ACSYS software.
3. Type in

Acsys

4. Then press the “transfer function Symbolic” and enter the Characteristic equation
5. Then press the “Routh Hurwitz” button
6. For example look at below Figures



(a) $n(t) = 0, \quad r(t) = tu_s(t).$

Forward-path Transfer function:

$$G(s) = \left. \frac{Y(s)}{E(s)} \right|_{n=0} = \frac{K(s+\alpha)(s+3)}{s(s^2-1)} \quad \text{Type-1 system.}$$

Ramp-error constant: $K_v = \lim_{s \rightarrow 0} sG(s) = -3K\alpha$

Steady-state error: $e_{ss} = \frac{1}{K_v} = -\frac{1}{3K_v}$

Characteristic equation: $s^3 + Ks^2 + [K(3+\alpha)-1]s + 3\alpha K = 0$

Routh Tabulation:

s^3	1	$3K + \alpha K - 1$
s^2	K	$3\alpha K$
s^1	$\frac{K(3K + \alpha K - 1) - 3\alpha K}{K}$	
s^0	$3\alpha K$	

Stability Conditions: $3K + \alpha K - 1 - 3\alpha > 0 \quad \text{or} \quad K > \frac{1+3K}{3+\alpha}$
 $\alpha K > 0$

(b) When $r(t) = 0, \quad n(t) = u_s(t), \quad N(s) = 1/s.$

Transfer Function between $n(t)$ and $y(t)$: $\left. \frac{Y(s)}{N(s)} \right|_{r=0} = \frac{\frac{K(s+3)}{s^2-1}}{1 + \frac{K(s+\alpha)(s+3)}{s(s^2-1)}} = \frac{Ks(s+3)}{s^3 + Ks^2 + [K(s+\alpha)-1]s + 3\alpha K}$

Steady-State Output due to $n(t)$:

$$y_{ss} = \lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} sY(s) = 0 \quad \text{if the limit is valid.}$$

7-18

$$\text{Percent maximum overshoot} = 0.25 = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

Thus

$$\pi\zeta\sqrt{1-\zeta^2} = -\ln 0.25 = 1.386 \quad \pi^2\zeta^2 = 1.922(1-\zeta^2)$$

Solving for ζ from the last equation, we have $\zeta = 0.404$.

$$\text{Peak Time } t_{\max} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.01 \text{ sec.} \quad \text{Thus,} \quad \omega_n = \frac{\pi}{0.01\sqrt{1-(0.404)^2}} = 343.4 \text{ rad/sec}$$

Transfer Function of the Second-order Prototype System:

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{117916}{s^2 + 277.3s + 117916}$$

7-19 Closed-Loop Transfer Function:**Characteristic equation:**

$$\frac{Y(s)}{R(s)} = \frac{25K}{s^2 + (5 + 500K_t)s + 25K}$$

$$s^2 + (5 + 500K_t)s + 25K = 0$$

For a second-order prototype system, when the maximum overshoot is 4.3%, $\zeta = 0.707$.

$$\omega_n = \sqrt{25K}, \quad 2\zeta\omega_n = 5 + 500K_t = 1.414\sqrt{25K}$$

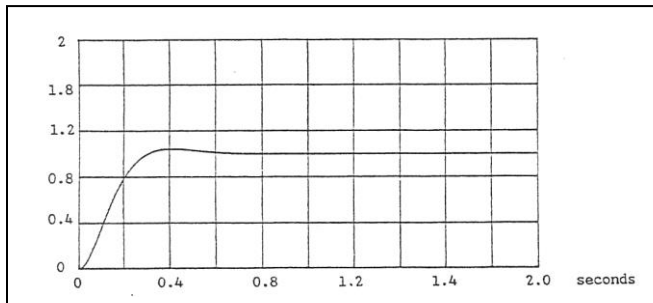
Rise Time:

$$t_r = \frac{1 - 0.4167\zeta + 2.917\zeta^2}{\omega_n} = \frac{2.164}{\omega_n} = 0.2 \text{ sec} \quad \text{Thus} \quad \omega_n = 10.82 \text{ rad/sec}$$

$$\text{Thus,} \quad K = \frac{\omega_n^2}{25} = \frac{(10.82)^2}{25} = 4.68 \quad 5 + 500K_t = 1.414\omega_n = 15.3 \quad \text{Thus} \quad K_t = \frac{10.3}{500} = 0.0206$$

With $K = 4.68$ and $K_t = 0.0206$, the system transfer function is

$$\frac{Y(s)}{R(s)} = \frac{117}{s^2 + 15.3s + 117}$$

Unit-step Response:

$$y = 0.1 \text{ at } t = 0.047 \text{ sec.}$$

$$y = 0.9 \text{ at } t = 0.244 \text{ sec.}$$

$$t_r = 0.244 - 0.047 = 0.197 \text{ sec.}$$

$$y_{\max} = 0.0432 \text{ (4.32\% max. overshoot)}$$

7-20 Closed-loop Transfer Function:**Characteristic Equation:**

$$\frac{Y(s)}{R(s)} = \frac{25K}{s^2 + (5 + 500K_t)s + 25K} \quad s^2 + (5 + 500K_t)s + 25K = 0$$

When Maximum overshoot = 10%, $\frac{\pi\zeta}{\sqrt{1-\zeta^2}} = -\ln 0.1 = 2.3 \quad \pi^2\zeta^2 = 5.3(1-\zeta^2)$

Solving for ζ , we get $\zeta = 0.59$.

The Natural undamped frequency is $\omega_n = \sqrt{25K}$ Thus, $5 + 500K_t = 2\zeta\omega_n = 1.18\omega_n$

Rise Time:

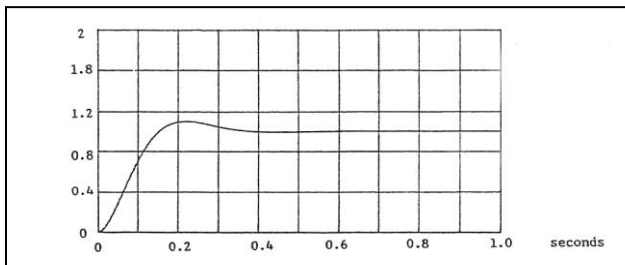
$$t_r = \frac{1 - 0.4167\zeta + 2.917\zeta^2}{\omega_n} = 0.1 = \frac{1.7696}{\omega_n} \text{ sec.} \quad \text{Thus } \omega_n = 17.7 \text{ rad/sec}$$

$$K = \frac{\omega_n^2}{25} = 12.58 \quad \text{Thus } K_t = \frac{15.88}{500} = 0.0318$$

With $K = 12.58$ and $K_t = 0.0318$, the system transfer function is

$$\frac{Y(s)}{R(s)} = \frac{313}{s^2 + 20.88s + 314.5}$$

Unit-step Response:



$y = 0.1$ when $t = 0.028$ sec.

$y = 0.9$ when $t = 0.131$ sec.

$t_r = 0.131 - 0.028 = 0.103$ sec.

$y_{\max} = 1.1$ (10% max. overshoot)

7-21 Closed-Loop Transfer Function:

Characteristic Equation:

$$\frac{Y(s)}{R(s)} = \frac{25K}{s^2 + (5 + 500K_t)s + 25K}$$

$$s^2 + (5 + 500K_t)s + 25K = 0$$

When Maximum overshoot = 20%, $\frac{\pi\zeta}{\sqrt{1-\zeta^2}} = -\ln 0.2 = 1.61$ $\pi^2\zeta^2 = 2.59(1-\zeta^2)$

Solving for ζ , we get $\zeta = 0.456$.

The Natural undamped frequency $\omega_n = \sqrt{25K}$ $5 + 500K_t = 2\zeta\omega_n = 0.912\omega_n$

Rise Time:

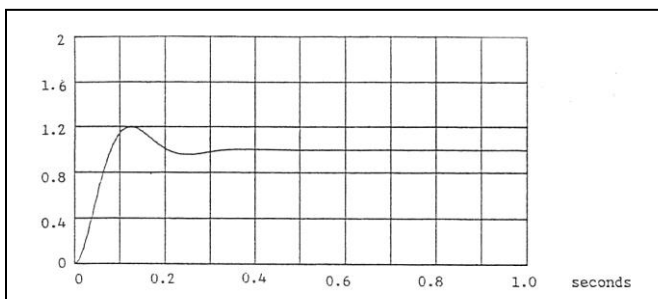
$$t_r = \frac{1 - 0.4167\zeta + 2.917\zeta^2}{\omega_n} = 0.05 = \frac{1.4165}{\omega_n} \text{ sec.} \quad \text{Thus, } \omega_n = \frac{1.4165}{0.05} = 28.33$$

$$K = \frac{\omega_n^2}{25} = 32.1 \quad 5 + 500K_t = 0.912\omega_n = 25.84 \quad \text{Thus, } K_t = 0.0417$$

With $K = 32.1$ and $K_t = 0.0417$, the system transfer function is

$$\frac{Y(s)}{R(s)} = \frac{802.59}{s^2 + 25.84s + 802.59}$$

Unit-step Response:



$$y = 0.1 \text{ when } t = 0.0178 \text{ sec.}$$

$$y = 0.9 \text{ when } t = 0.072 \text{ sec.}$$

$$t_r = 0.072 - 0.0178 = 0.0542 \text{ sec.}$$

$$y_{\max} = 1.2 \text{ (20\% max. overshoot)}$$

7-22 Closed-Loop Transfer Function:

Characteristic Equation:

$$\frac{Y(s)}{R(s)} = \frac{25K}{s^2 + (5 + 500K_t)s + 25K} \quad s^2 + (5 + 500K_t)s + 25K = 0$$

$$\text{Delay time } t_d \cong \frac{1.1 + 0.125\zeta + 0.469\zeta^2}{\omega_n} = 0.1 \text{ sec.}$$

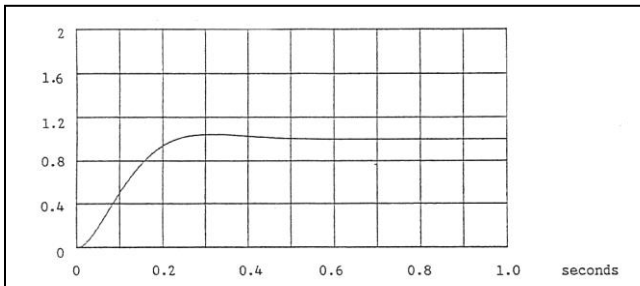
$$\text{When Maximum overshoot} = 4.3\%, \zeta = 0.707. \quad t_d = \frac{1.423}{\omega_n} = 0.1 \text{ sec.} \quad \text{Thus } \omega_n = 14.23 \text{ rad/sec.}$$

$$K = \left(\frac{\omega_n}{5}\right)^2 = \left(\frac{14.23}{5}\right)^2 = 8.1 \quad 5 + 500K_t = 2\zeta\omega_n = 1.414\omega_n = 20.12 \quad \text{Thus } K_t = \frac{15.12}{500} = 0.0302$$

With $K = 20.12$ and $K_t = 0.0302$, the system transfer function is

$$\frac{Y(s)}{R(s)} = \frac{202.5}{s^2 + 20.1s + 202.5}$$

Unit-Step Response:



When $y = 0.5$, $t = 0.1005$ sec.

Thus, $t_d = 0.1005$ sec.

$$y_{\max} = 1.043 \quad (4.3\% \text{ max. overshoot})$$

7-23 Closed-Loop Transfer Function:

$$\frac{Y(s)}{R(s)} = \frac{25K}{s^2 + (5 + 500K_t)s + 25K}$$

Characteristic Equation:

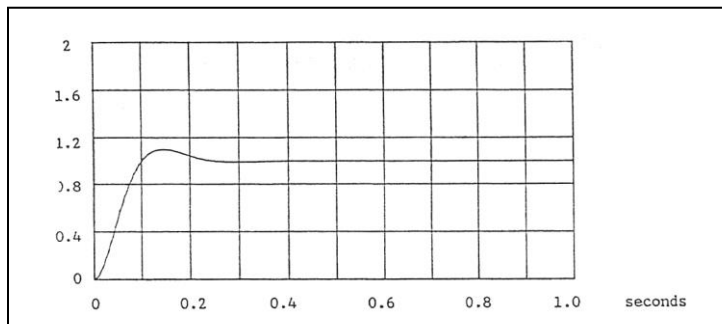
$$s^2 + (5 + 500K_t)s + 25K = 0$$

$$\text{Delay time } t_d \cong \frac{1.1 + 0.125\zeta + 0.469\zeta^2}{\omega_n} = 0.05 = \frac{1.337}{\omega_n} \quad \text{Thus, } \omega_n = \frac{1.337}{0.05} = 26.74$$

$$K = \left(\frac{\omega_n}{5}\right)^2 = \left(\frac{26.74}{5}\right)^2 = 28.6 \quad 5 + 500K_t = 2\zeta\omega_n = 2 \times 0.59 \times 26.74 = 31.55 \quad \text{Thus } K_t = 0.0531$$

With $K = 28.6$ and $K_t = 0.0531$, the system transfer function is

$$\frac{Y(s)}{R(s)} = \frac{715}{s^2 + 31.55s + 715}$$

Unit-Step Response:

$y = 0.5$ when $t = 0.0505$ sec.

Thus, $t_d = 0.0505$ sec.

$y_{\max} = 1.1007$ (10.07% max. overshoot)

7-24 Closed-Loop Transfer Function:

$$\frac{Y(s)}{R(s)} = \frac{25K}{s^2 + (5 + 500K_t)s + 25K}$$

Characteristic Equation:

$$s^2 + (5 + 500K_t)s + 25K = 0$$

For Maximum overshoot = 0.2, $\zeta = 0.456$.

$$\text{Delay time } t_d = \frac{1.1 + 0.125\zeta + 0.469\zeta^2}{\omega_n} = \frac{1.2545}{\omega_n} = 0.01 \text{ sec.}$$

$$\text{Natural Undamped Frequency } \omega_n = \frac{1.2545}{0.01} = 125.45 \text{ rad/sec. Thus, } K = \left(\frac{\omega_n}{5}\right)^2 = \frac{15737.7}{25} = 629.5$$

$$5 + 500K_t = 2\zeta\omega_n = 2 \times 0.456 \times 125.45 = 114.41 \quad \text{Thus, } K_t = 0.2188$$

With $K = 629.5$ and $K_t = 0.2188$, the system transfer function is

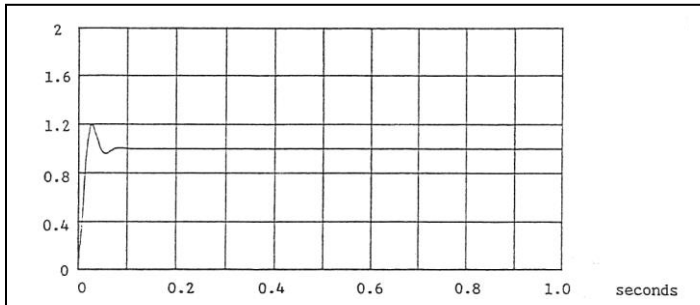
$$\frac{Y(s)}{R(s)} = \frac{15737.7}{s^2 + 114.41s + 15737.7}$$

Unit-step Response:

$$y = 0.5 \text{ when } t = 0.0101 \text{ sec.}$$

$$\text{Thus, } t_d = 0.0101 \text{ sec.}$$

$$y_{\max} = 1.2 \quad (20\% \text{ max. overshoot})$$



7-25 Closed-Loop Transfer Function:

$$\frac{Y(s)}{R(s)} = \frac{25K}{s^2 + (5 + 500K_t)s + 25K}$$

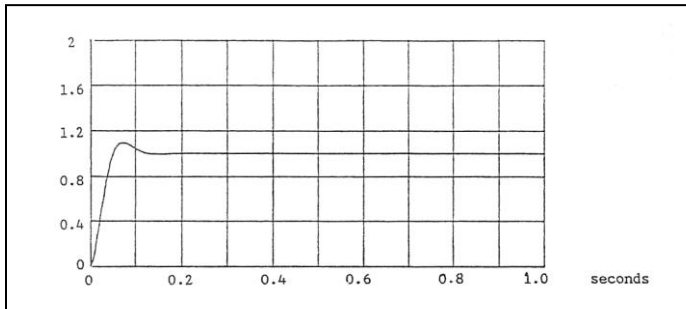
$$\zeta = 0.6 \quad 2\zeta\omega_n = 5 + 500K_t = 1.2\omega_n$$

$$\text{Settling time } t_s \cong \frac{3.2}{\zeta\omega_n} = \frac{3.2}{0.6\omega_n} = 0.1 \text{ sec. Thus, } \omega_n = \frac{3.2}{0.06} = 53.33 \text{ rad/sec}$$

$$K_t = \frac{1.2\omega_n - 5}{500} = 0.118 \quad K = \frac{\omega_n^2}{25} = 113.76$$

System Transfer Function:

$$\frac{Y(s)}{R(s)} = \frac{2844}{s^2 + 64s + 2844}$$

Unit-step Response:

$y(t)$ reaches 1.00 and never exceeds this

value at $t = 0.098$ sec.

Thus, $t_s = 0.098$ sec.

7-26 (a) Closed-Loop Transfer Function:**Characteristic Equation:**

$$\frac{Y(s)}{R(s)} = \frac{25K}{s^2 + (5 + 500K_t)s + 25K} \quad s^2 + (5 + 500K_t)s + 25K = 0$$

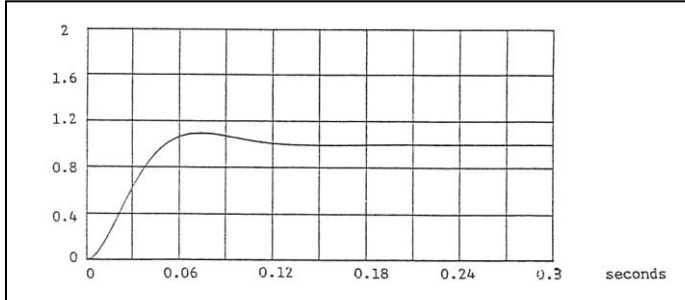
For maximum overshoot = 0.1, $\zeta = 0.59$. $5 + 500K_t = 2\zeta\omega_n = 2 \times 0.59\omega_n = 1.18\omega_n$

Settling time: $t_s = \frac{3.2}{\zeta\omega_n} = \frac{3.2}{0.59\omega_n} = 0.05 \text{ sec.}$ $\omega_n = \frac{3.2}{0.05 \times 0.59} = 108.47$

$$K_t = \frac{1.18\omega_n - 5}{500} = 0.246 \quad K = \frac{\omega_n^2}{25} = 470.63$$

System Transfer Function:

$$\frac{Y(s)}{R(s)} = \frac{11765.74}{s^2 + 128s + 11765.74}$$

Unit-Step Response:

$y(t)$ reaches 1.05 and never exceeds

this value at $t = 0.048 \text{ sec.}$

Thus, $t_s = 0.048 \text{ sec.}$

(b) For maximum overshoot = 0.2, $\zeta = 0.456$. $5 + 500K_t = 2\zeta\omega_n = 0.912\omega_n$

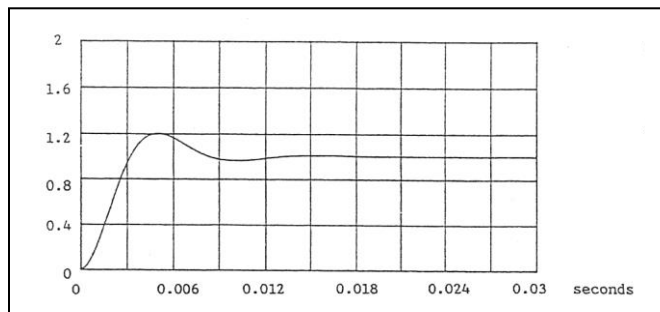
Settling time $t_s = \frac{3.2}{\zeta\omega_n} = \frac{3.2}{0.456\omega_n} = 0.01 \text{ sec.}$ $\omega_n = \frac{3.2}{0.456 \times 0.01} = 701.75 \text{ rad/sec}$

$$K_t = \frac{0.912\omega_n - 5}{500} = 1.27$$

System Transfer Function:

$$\frac{Y(s)}{R(s)} = \frac{492453}{s^2 + 640s + 492453}$$

Unit-Step Response:



$y(t)$ reaches 1.05 and never

exceeds this value at $t = 0.0074$ sec.

Thus, $t_s = 0.0074$ sec. This is less

than the calculated value of 0.01 sec.

7-27 Closed-Loop Transfer Function:

$$\frac{Y(s)}{R(s)} = \frac{25K}{s^2 + (5 + 500K_t)s + 25K}$$

Characteristic Equation:

$$s^2 + (5 + 500K_t)s + 25K = 0$$

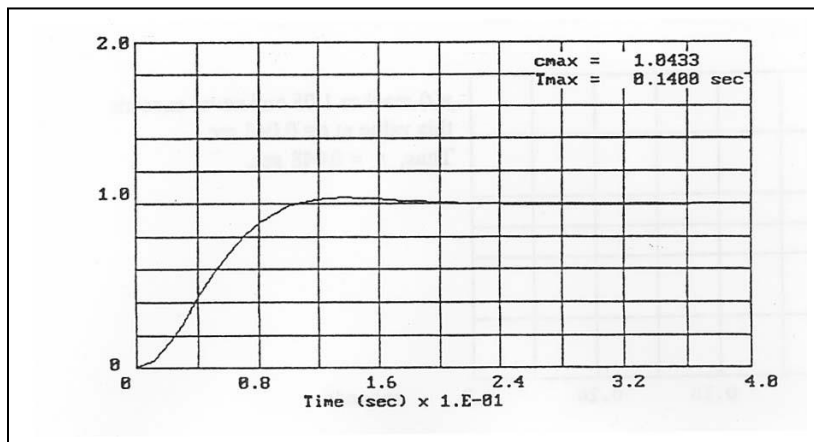
Damping ratio $\zeta = 0.707$. Settling time $t_s = \frac{4.5\zeta}{\omega_n} = \frac{3.1815}{\omega_n} = 0.1$ sec. Thus, $\omega_n = 31.815$ rad/sec.

$$5 + 500K_t = 2\zeta\omega_n = 44.986 \quad \text{Thus, } K_t = 0.08 \quad K = \frac{\omega_n^2}{2\zeta} = 40.488$$

System Transfer Function:

$$\frac{Y(s)}{R(s)} = \frac{1012.2}{s^2 + 44.986s + 1012.2}$$

Unit-Step Response: The unit-step response reaches 0.95 at $t = 0.092$ sec. which is the measured t_s .



7-28 (a) When $\zeta = 0.5$, the rise time is

$$t_r \cong \frac{1 - 0.4167\zeta + 2.917\zeta^2}{\omega_n} = \frac{1.521}{\omega_n} = 1 \text{ sec.} \quad \text{Thus } \omega_n = 1.521 \text{ rad/sec.}$$

The second-order term of the characteristic equation is written

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 1.521s + 2.313 = 0$$

The characteristic equation of the system is $s^3 + (a+30)s^2 + 30as + K = 0$

Dividing the characteristic equation by $s^2 + 1.521s + 2.313$, we have

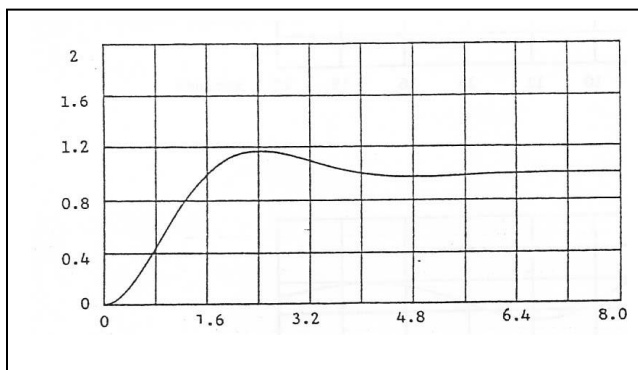
$$\begin{array}{r} s^2 + 1.521s + 2.313 \overline{) s^3 + (a+30)s^2 + 30as + K} \\ \underline{s^3 + 1.521s^2 + 2.313s} \\ (28.48 + a)s^2 + (30a - 2.323)s + K \\ \underline{(28.48 + a)s^2 + (1.521a + 43.32)s + 65.874 + 2.313a} \\ (28.48a - 45.63)s + K - 0.744 - 2.313a \end{array}$$

For zero remainders, $28.48a = 45.63$ Thus, $a = 1.6$ $K = 65.874 + 2.313a = 69.58$

Forward-Path Transfer Function:

$$G(s) = \frac{69.58}{s(s+1.6)(s+30)}$$

Unit-Step Response:



$y = 0.1$ when $t = 0.355$ sec.

$y = 0.9$ when $t = 1.43$ sec.

Rise Time:

$$t_r = 1.43 - 0.355 = 1.075 \text{ sec.}$$

(b) The system is type 1.

(i) For a unit-step input, $e_{ss} = 0$.

(ii) For a unit-ramp input, $K_v = \lim_{s \rightarrow 0} sG(s) = \frac{K}{30a} = \frac{60.58}{30 \times 1.6} = 1.45$ $e_{ss} = \frac{1}{K_v} = 0.69$

7-29 (a) Characteristic Equation:

$$s^3 + 3s^2 + (2 + K)s - K = 0$$

Apply the Routh-Hurwitz criterion to find the range of K for stability.

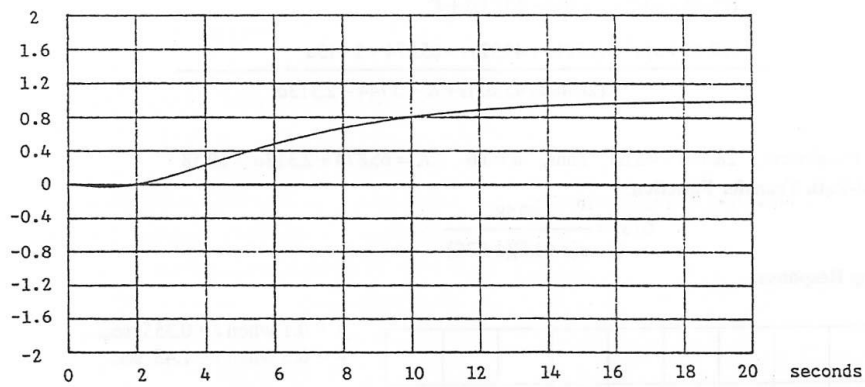
Routh Tabulation:

s^3	1	$2 + K$
s^2	3	$-K$
s^1	$\frac{6 + 4K}{3}$	
s^0	$-K$	

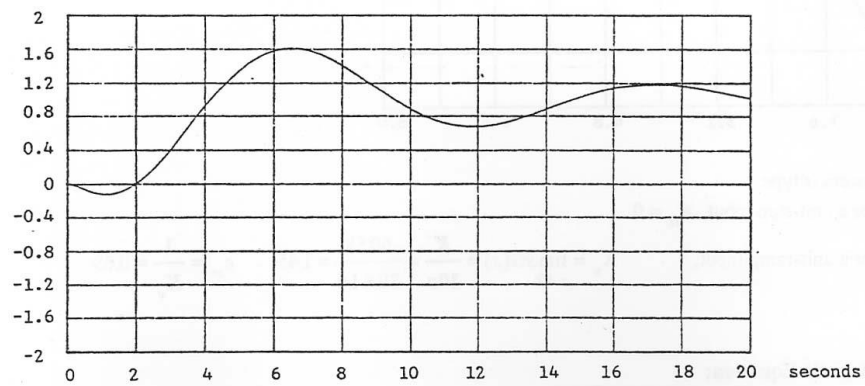
Stability Condition: $-1.5 < K < 0$ This simplifies the search for K for two equal roots.

When $K = -0.27806$, the characteristic equation roots are: -0.347 , -0.347 , and -2.3054 .

(b) Unit-Step Response: ($K = -0.27806$)



(c) Unit-Step Response ($K=-1$)



The step responses in (a) and (b) all have a negative undershoot for small values of t . This is due to the zero of $G(s)$ that lies in the right-half s -plane.

7-30 (a) The state equations of the closed-loop system are:

$$\frac{dx_1}{dt} = -x_1 + 5x_2 \quad \frac{dx_2}{dt} = -6x_1 - k_1x_1 - k_2x_2 + r$$

The characteristic equation of the closed-loop system is

$$\Delta = \begin{vmatrix} s+1 & -5 \\ 6+k_1 & s+k_2 \end{vmatrix} = s^2 + (1+k_2)s + (30+5k_1+k_2) = 0$$

For $\omega_n = 10$ rad/sec, $30+5k_1+k_2 = \omega_n^2 = 100$. Thus $5k_1+k_2 = 70$

(b) For $\zeta = 0.707$, $2\zeta\omega_n = 1+k_2$. Thus $\omega_n = 1 + \frac{k_2}{1.414}$.

$$\omega_n^2 = \frac{(1+k_2)^2}{2} = 30+5k_1+k_2 \quad \text{Thus} \quad k_2^2 = 59+10k_1$$

(c) For $\omega_n = 10$ rad/sec and $\zeta = 0.707$,

$$5k_1+k_2 = 100 \quad \text{and} \quad 1+k_2 = 2\zeta\omega_n = 14.14 \quad \text{Thus} \quad k_2 = 13.14$$

Solving for k_1 , we have $k_1 = 11.37$.

(d) The closed-loop transfer function is

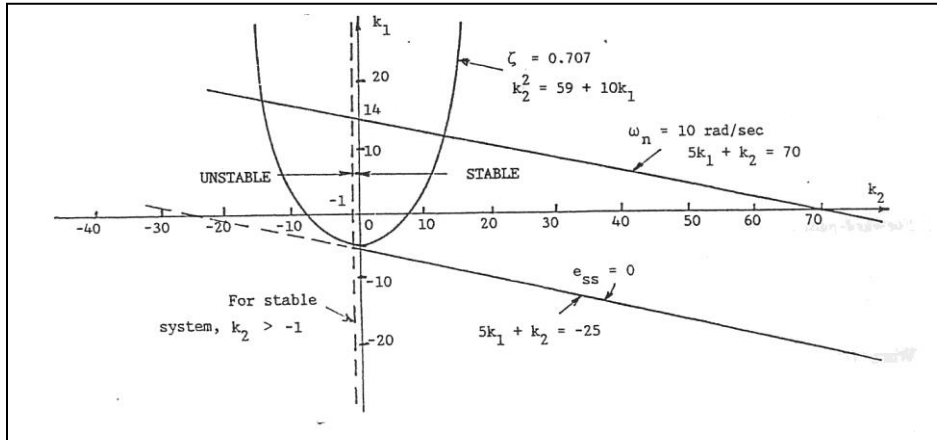
$$\frac{Y(s)}{R(s)} = \frac{5}{s^2 + (k_2+1)s + (30+5k_1+k_2)} = \frac{5}{s^2 + 14.14s + 100}$$

For a unit-step input, $\lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} sY(s) = \frac{5}{100} = 0.05$

(e) For zero steady-state error due to a unit-step input,

$$30+5k_1+k_2 = 5 \quad \text{Thus} \quad 5k_1+k_2 = -25$$

Parameter Plane k_1 versus k_2 :



7-31 (a) Closed-Loop Transfer Function

$$\frac{Y(s)}{R(s)} = \frac{100(K_p + K_D s)}{s^2 + 100K_D s + 100K_p}$$

(b) Characteristic Equation:

$$s^2 + 100K_D s + 100K_p = 0$$

The system is stable for $K_p > 0$ and $K_D > 0$.

(b) For $\zeta = 1$, $2\zeta\omega_n = 100K_D$.

$$\omega_n = 10\sqrt{K_p} \quad \text{Thus} \quad 2\omega_n = 100K_D = 20\sqrt{K_p} \quad K_D = 0.2\sqrt{K_p}$$

(c) See parameter plane in part (g).

(d) See parameter plane in part (g).

(e) Parabolic error constant $K_a = 1000 \text{ sec}^{-2}$

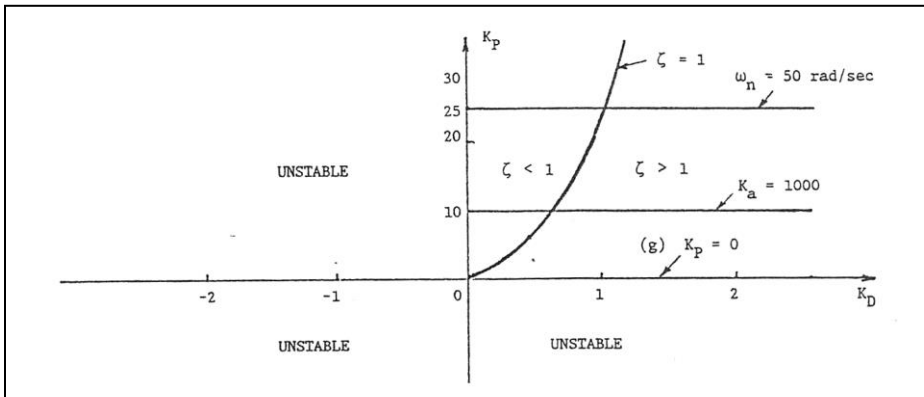
$$K_a = \lim_{s \rightarrow 0} s^2 G(s) = \lim_{s \rightarrow 0} 100(K_p + K_D s) = 100K_p = 1000 \quad \text{Thus } K_p = 10$$

(f) Natural undamped frequency $\omega_n = 50 \text{ rad/sec}$.

$$\omega_n = 10\sqrt{K_p} = 50 \quad \text{Thus } K_p = 25$$

(g) When $K_p = 0$,

$$G(s) = \frac{100K_D s}{s^2} = \frac{100K_D}{s} \quad (\text{pole-zero cancellation})$$



7-32 (a) Forward-path Transfer Function:

$$G(s) = \frac{Y(s)}{E(s)} = \frac{KK_i}{s[Js(1+Ts) + K_iK_t]} = \frac{10K}{s(0.001s^2 + 0.01s + 10K_t)}$$

$$\text{When } r(t) = tu_s(t), \quad K_v = \lim_{s \rightarrow 0} sG(s) = \frac{K}{K_t} \quad e_{ss} = \frac{1}{K_v} = \frac{K_t}{K}$$

(b) When $r(t) = 0$

$$\frac{Y(s)}{T_d(s)} = \frac{1+Ts}{s[Js(1+Ts) + K_iK_t] + KK_i} = \frac{1+0.1s}{s(0.001s^2 + 0.01s + 10K_t) + 10K}$$

$$\text{For } T_d(s) = \frac{1}{s} \quad \lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} sY(s) = \frac{1}{10K} \quad \text{if the system is stable.}$$

(c) The characteristic equation of the closed-loop system is

$$0.001s^3 + 0.01s^2 + 0.1s + 10K = 0$$

The system is unstable for $K > 0.1$. So we can set K to just less than 0.1. Then, the minimum value of the steady-state value of $y(t)$ is

$$\left. \frac{1}{10K} \right|_{K=0.1^-} = 1^+$$

However, with this value of K , the system response will be very oscillatory. The maximum overshoot will be nearly 100%.

(d) For $K = 0.1$, the characteristic equation is

$$0.001s^3 + 0.01s^2 + 10K_t s + 1 = 0 \quad \text{or} \quad s^3 + 10s^2 + 10^4 K_t s + 1000 = 0$$

For the two complex roots to have real parts of $-2/5$, we let the characteristic equation be written as

$$(s + a)(s^2 + 5s + b) = 0 \quad \text{or} \quad s^3 + (s + 5)s^2 + (5a + b)s + ab = 0$$

Then, $a + 5 = 10 \quad a = 5 \quad ab = 1000 \quad b = 200 \quad 5a + b = 10^4 K_t \quad K_t = 0.0225$

The three roots are: $s = -a = -5 \quad s = -a = -5 \quad s = -2.5 \pm j13.92$

7-33) Rise time: $t_r \cong \frac{0.8 + 2.5\xi}{\omega_n} = \frac{0.8 + 2.5 \cdot 0.6}{5} = 0.56 \text{ sec}$

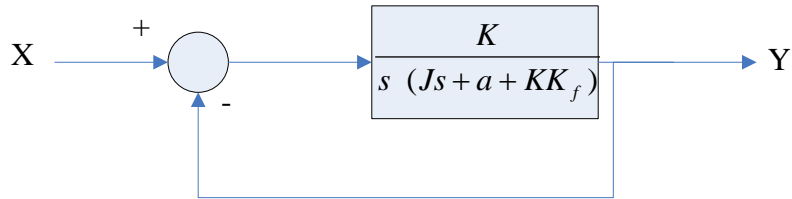
Peak time: $t_p = \frac{\pi}{\omega_n \sqrt{1 - \xi^2}} = \frac{3.14}{5\sqrt{0.64}} = 0.785 \text{ sec}$

Maximum overshoot: $M_p = e^{-\frac{\pi\xi}{\sqrt{1 - \xi^2}}} = e^{-\frac{0.6\pi}{0.8}} = 0.095$

Settling time: $t_s \cong \frac{3.2}{\xi\omega_n} \quad 0 < \xi < 0.69$

$\Rightarrow t_s \cong \frac{3.2}{0.615} \cong 1.067 \text{ sec}$

7-34)



$$\Rightarrow \frac{Y(s)}{X(s)} = \frac{K}{Js^2 + (a + KK_f)s + K}$$

$$\Rightarrow M_p = \exp\left(-\frac{\pi\xi}{\sqrt{1-\xi^2}}\right) = 0.2 \rightarrow \xi = 0.456$$

$$\Rightarrow t_p = \frac{\pi}{\omega_n \sqrt{1-\xi^2}} = 0.1 \rightarrow \omega_n = 0.353$$

$$\Rightarrow G(s) = \frac{\omega_n^2}{s(s + 2\xi\omega_n)} = \frac{\frac{K}{J}}{s\left(s + \frac{a + KK_f}{J}\right)}$$

$$\Rightarrow \begin{cases} \omega_n = \sqrt{\frac{K}{J}} \rightarrow K = 0.125 \\ 2\xi\omega_n = \frac{a + KK_f}{J} \rightarrow K_f = \frac{2J\xi\omega_n - a}{K} \cong -5.42 \end{cases}$$

$$\Rightarrow t_r = \frac{0.8 + 2.5f}{\omega_n} \cong 5.49 \text{ sec}$$

$$\Rightarrow t_s = \frac{3.2}{\xi\omega_n} \cong 19.88 \text{ sec}$$

7-35) a)

$$\begin{cases} \dot{x}_1 = -x_1 - x_2 + u_1 + u_2 \\ \dot{x}_2 = 6.5x_1 + u_1 \\ y_1 = x_1 \\ y_2 = x_2 \end{cases}$$

$$\begin{cases} sX_1(s) = -X_1(s) - X_2(s) + U_1(s) + U_2(s) & (1) \\ sX_2(s) = 6.5X_1(s) + U_1(s) & (2) \\ Y_1(s) = X_1(s) \\ Y_2(s) = X_2(s) \end{cases}$$

$$\Rightarrow (s+1)x_1(s) = -\frac{6.5}{s}X(s) + \frac{U_1(s)}{s} + U_1(s) + U_2(s)$$

$$\Rightarrow (s^2 + s + 6.5)X_1(s) = (s-1)U_1(s) + U_2(s)$$

$$\Rightarrow Y_1(s) = X_1(s) = \frac{s-1}{s^2+s+6.5}U_1(s) + \frac{5}{s^2+s+6.5}U_2(s)$$

Substituting into equation (2) gives:

$$Y_2(s) = X_2(s) = \frac{s+7.5}{s^2+s+6.5}U_1(s) + \frac{6.5}{s^2+s+6.5}U_2(s)$$

Since the system is multi input and multi output, there are 4 transfer functions as:

$$\left[\frac{Y_1(s)}{U_1(s)} \right]_{U_2=0}, \left[\frac{Y_1(s)}{U_2(s)} \right]_{U_1=0}, \left[\frac{Y_2(s)}{U_1(s)} \right]_{U_2=0}, \left[\frac{Y_2(s)}{U_2(s)} \right]_{U_1=0}$$

To find the unit step response of the system, let's consider

$$\frac{Y_2(s)}{U_2(s)} = \frac{6.5}{s^2+s+6.5} = \frac{\omega_n^2}{s(s^2+2\xi\omega_n+\omega_n^2)}$$

$$\text{where } \begin{cases} \omega_n^2 = 6.5 \rightarrow \omega_n = \sqrt{6.5} \\ 2\xi\omega_n = 1 \rightarrow \xi = \frac{1}{2\omega_n} = \frac{1}{2\sqrt{6.5}} \end{cases}$$

By looking at the Laplace transform function table:

$$y(t) = 1 - \frac{1}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin(\omega_n \sqrt{1-\xi^2} t + \theta)$$

where $\theta = \cos^{-1} \xi$

b)

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - x_2 + u \\ y = x_1 \end{cases}$$

Therefore:

$$\begin{cases} sX_1(s) = X_2(s) \\ sX_2(s) = -X_1(s) - X_2(s) + U(s) \\ Y(s) = X_1(s) \end{cases}$$

As a result:

$$s^2 X_1(s) = -X_1(s) - sX_1(s) + U(s)$$

which means:

$$X_1(s) = \frac{1}{s^2 + s + 1} U(s)$$

The unit step response is:

$$Y(s) = X_1(s) = \frac{1}{s(s^2 + s + 1)}$$

Therefore as a result:

$$y(t) = 1 - \frac{1}{\sqrt{1 - \xi^2}} e^{-\xi \omega_n t} \sin \omega_n \sqrt{1 - \xi^2} t$$

where $\omega_n = 1$ and $\xi \omega_n = 1/2$

c)

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - x_2 + 4 \\ \dot{x}_3 = x_1 \\ y = x_3 \end{cases}$$

Therefore:

$$\begin{cases} sX_1(s) = X_2(s) \\ (s+1)X_2(s) = -X_1(s) + U(s) \rightarrow s(s+1)X_1(s) = -X_1(s) + U(s) \\ sX_3(s) = X_1(s) \\ Y(s) = X_3(s) \rightarrow Y(s) = \frac{X_1(s)}{s} \end{cases}$$

As a result, the step response of the system is:

$$Y(s) = \frac{1}{s^2(s^2 + s + 1)}$$

By looking up at the Laplace transfer function table:

$$Y(s) = \frac{\omega_n^2}{s^2(s^2 + 2\xi\omega_n s + \omega_n^2)}$$

where $\omega_n = 1$, and $2\xi\omega_n = 1 \rightarrow \xi = 1/2$

$$y(t) = t - \frac{2\xi}{\omega_n} + \frac{1}{\omega_n\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin(\omega_n\sqrt{1-\xi^2} t + \theta)$$

where $\theta = \cos^{-1}(2\xi^2 - 1) = \cos^{-1}(0.5)$, therefore,

$$y(t) = t - 1 + \frac{2}{\sqrt{3}} e^{-\xi\omega_n t} \sin\left(\frac{\sqrt{3}}{2} t + \theta\right)$$

7-36) MATLAB CODE

(a)

```
clear all
Amat=[-1 -1;6.5 0]
Bmat=[1 1;1 0]
Cmat=[1 0;0 1]
Dmat=[0 0;0 0]
disp(' State-Space Model is:')
Statemodel=ss(Amat,Bmat,Cmat,Dmat)
[mA,nA]=size(Amat);
rankA=rank(Amat);
disp(' Characteristic Polynomial:')
chareq=poly(Amat);

%p = poly(A) where A is an n-by-n matrix returns an n+1 element
%row vector whose elements are the coefficients of the characteristic
%polynomialdet(sI-A). The coefficients are ordered in descending powers.

[mchareq,nchareq]=size(chareq);
syms 's';

poly2sym(chareq,s)
disp(' Equivalent Transfer Function Model is:')

Hmat=Cmat*inv(s*eye(2)-Amat)*Bmat+Dmat
```

Since the system is multi input and multi output, there are 4 transfer functions as:

$$\left[\frac{Y_1(s)}{U_1(s)}\right]_{U_2=0}, \left[\frac{Y_1(s)}{U_2(s)}\right]_{U_1=0}, \left[\frac{Y_2(s)}{U_1(s)}\right]_{U_2=0}, \left[\frac{Y_2(s)}{U_2(s)}\right]_{U_1=0}$$

To find the unit step response of the system, let's consider

$$\frac{Y_2(s)}{U_2(s)} = \frac{6.5}{s^2 + s + 6.5}$$

Let's obtain this term and find $Y_2(s)$ time response for a step input.

```
H22=Hmat(2,2)
ilaplace(H22/s)
Pretty(H22)
H22poly=tf([13/2],chareq)
step(H22poly)
```

H22 =

$13/(2s^2+2s+13)$

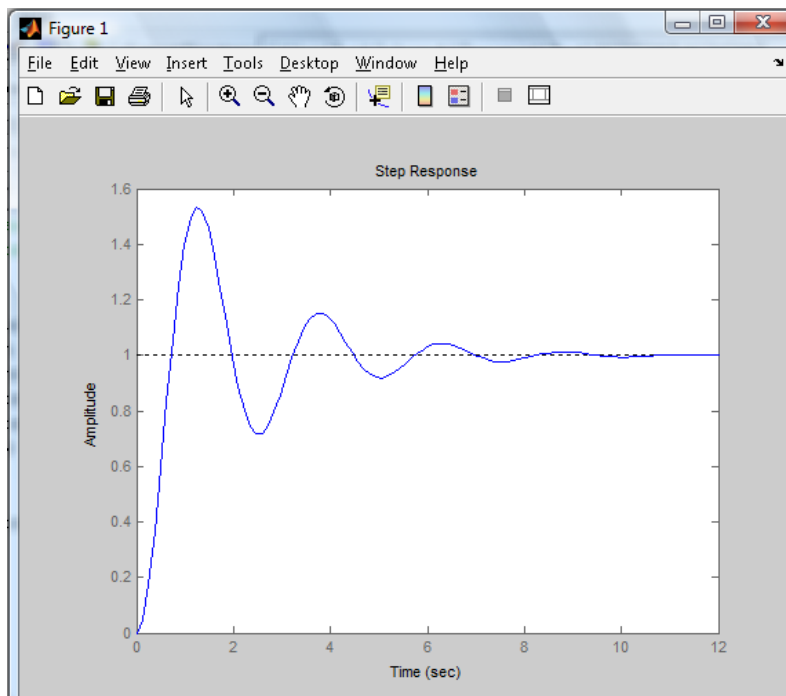
ans =

$1 - 1/5 \exp(-1/2t) * (5 \cos(5/2t) + \sin(5/2t))$

$$\frac{13}{2s^2 + 2s + 13}$$

Transfer function:

$$\frac{6.5}{s^2 + s + 6.5}$$



To find the step response H11, H12, and H21 follow the same procedure.

Other parts are the same.

7-37) Impulse response:

a) $Y(s) = \frac{6.5}{s^2 + s + 6.5}$ and $U(s) = 1$, therefore,

$$y(t) = \frac{\omega_n}{\sqrt{1 - \xi^2}} e^{-\xi \omega_n t} \sin(\omega_n \sqrt{1 - \xi^2} t)$$

b) $Y(s) = \frac{1}{s^2 + s + 1}$

$$y(t) = \frac{\omega_n}{\sqrt{1 - \xi^2}} e^{-\xi \omega_n t} \sin(\omega_n \sqrt{1 - \xi^2} t)$$

c) $Y(s) = \frac{1}{s(s^2 + s + 1)}$

$$y(t) = 1 - \frac{1}{\sqrt{1 - \xi^2}} e^{-\xi \omega_n t} \sin(\omega_n \sqrt{1 - \xi^2} t + \alpha)$$

where $\alpha = \cos^{-1} \xi$

7-38) Use the approach in 7-36 except:

```
H22=Hmat(2,2)
ilaplace(H22)
Pretty(H22)
H22poly=tf([13/2],chareq)
impulse(H22poly)

H22 =

13/(2*s^2+2*s+13)

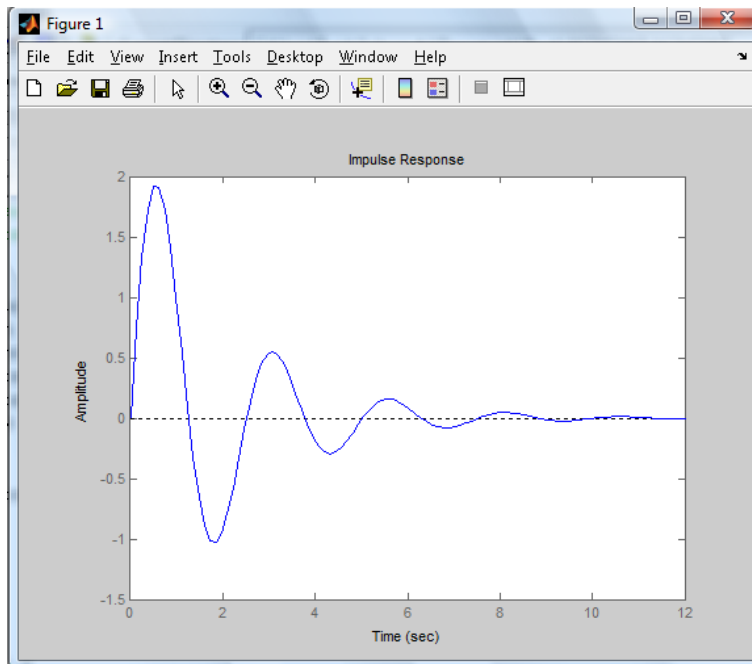
ans =

13/5*exp(-1/2*t)*sin(5/2*t)
```

$$\frac{13}{2s^2 + 2s + 13}$$

Transfer function:

$$\frac{6.5}{s^2 + s + 6.5}$$



Other parts are the same.

7-39) a) The displacement of the bar is:

$$x = L \sin \theta$$

Then the equation of motion is:

$$B \left(\frac{dy}{dt} - \frac{dx}{dt} \right) - K\tau = 0$$

As x is a function of θ and changing with time, then

$$\frac{dx}{dt} = L \frac{d\theta}{dt} \cos \theta$$

If θ is small enough, then $\sin \theta \approx \theta$ and $\cos \theta \approx 1$. Therefore, the equation of motion is rewritten as:

$$B(\dot{y} - L\dot{\theta}) - KL\theta = 0$$

$$L(B\dot{\theta} + K\theta) = B\dot{y}$$

$$L(Bs + K)\theta(s) = BsY(s)$$

$$\frac{\theta(s)}{Y(s)} = \frac{Bs}{L(Bs + K)}$$

(b) To find the unit step response, you can use the symbolic approach shown in Toolbox 2-1-1:

```
clear all
% s=tf('s');
syms s B L K
Theta=B*s/s/L/(B*s+K)
ilaplace(Theta)

Theta =
B/L/(B*s+K)

ans =
1/L*exp(-K*t/B)
```

Alternatively, assign values to B L K and find the step response – see solution to problem 7-36.

7-40 (a) $K_t = 10000 \text{ oz-in/rad}$

The Forward-Path Transfer Function:

$$G(s) = \frac{9 \times 10^{12} K}{s(s^4 + 5000s^3 + 1.067 \times 10^7 s^2 + 50.5 \times 10^9 s + 5.724 \times 10^{12})}$$

$$= \frac{9 \times 10^{12} K}{s(s+116)(s+4883)(s+41.68 + j3178.3)(s+41.68 - j3178.3)}$$

Routh Tabulation:

s^5	1	1.067×10^7	5.724×10^{12}
s^4	5000	50.5×10^9	$9 \times 10^{12} K$
s^3	5.7×10^5	$5.72 \times 10^{12} - 1.8 \times 10^9 K$	0
s^2	$2.895 \times 10^8 + 1.579 \times 10^7 K$	$9 \times 10^{12} K$	
s^1	$\frac{16.6 \times 10^{13} + 8.473 \times 10^{12} K - 2.8422 \times 10^9 K^2}{29 + 1.579 K}$		
s^0	$9 \times 10^{12} K$		

From the s^1 row, the condition of stability is $165710 + 8473K - 2.8422K^2 > 0$

$$\text{or } K^2 - 2981.14K - 58303.427 < 0 \quad \text{or} \quad (K + 19.43)(K - 3000.57) < 0$$

Stability Condition: $0 < K < 3000.56$

The critical value of K for stability is 3000.56. With this value of K , the roots of the characteristic equation are: -4916.9 , $-41.57 + j3113.3$, $-41.57 + j3113.3$, $-j752.68$, and $j752.68$

(b) $K_L = 1000$ oz-in/rad. The forward-path transfer function is

$$\begin{aligned} G(s) &= \frac{9 \times 10^{11} K}{s(s^4 + 5000s^3 + 1.582 \times 10^6 s^2 + 5.05 \times 10^9 s + 5.724 \times 10^{11})} \\ &= \frac{9 \times 10^{11} K}{s(1 + 116.06)(s + 4882.8)(s + 56.248 + j1005)(s + 56.248 - j1005)} \end{aligned}$$

(c) Characteristic Equation of the Closed-Loop System:

$$s^5 + 5000s^4 + 1.582 \times 10^6 s^3 + 5.05 \times 10^9 s^2 + 5.724 \times 10^{11} s + 9 \times 10^{11} K = 0$$

Routh Tabulation:

s^5	1	1.582×10^6	5.724×10^{11}
s^4	5000	5.05×10^9	$9 \times 10^{11} K$
s^3	5.72×10^5	$5.724 \times 10^{11} - 1.8 \times 10^8 K$	0
s^2	$4.6503 \times 10^7 + 1.5734 \times 10^6 K$	$9 \times 10^{11} K$	
s^1	$\frac{26.618 \times 10^{18} + 377.43 \times 10^{15} K - 2.832 \times 10^{14} K^2}{4.6503 \times 10^7 + 1.5734 \times 10^6 K}$		
s^0	$9 \times 10^{11} K$		

From the s^1 row, the condition of stability is $26.618 \times 10^4 + 3774.3K - 2.832K^2 > 0$

Or, $K^2 - 1332.73K - 93990 < 0$ or $(K - 1400)(K + 67.14) < 0$

Stability Condition: $0 < K < 1400$

The critical value of K for stability is 1400. With this value of K , the characteristic equation root are:

$$-4885.1, \quad -57.465 + j676, \quad -57.465 - j676, \quad j748.44, \quad \text{and} \quad -j748.44$$

(c) $K_L = \infty$.

Forward-Path Transfer Function:

$$G(s) = \frac{nK_s K_i K}{s \left[L_a J_T s^2 + (R_a J_T + R_m L_a) s + R_a B_m + K_i K_b \right]} \quad J_T = J_m + n^2 J_L$$

$$= \frac{891100K}{s(s^2 + 5000s + 566700)} = \frac{891100K}{s(s + 116)(s + 4884)}$$

Characteristic Equation of the Closed-Loop System:

$$s^3 + 5000s^2 + 566700s + 891100K = 0$$

Routh Tabulation:

s^3	1	566700
s^2	5000	891100K
s^1	$566700 - 178.22K$	
s^0	$891100K$	

From the s^1 row, the condition of K for stability is $566700 - 178.22K > 0$.

Stability Condition: $0 < K < 3179.78$

The critical value of K for stability is 3179.78. With $K = 3179.78$, the characteristic equation roots are

$$-5000, \quad j752.79, \quad \text{and} \quad -j752.79.$$

When the motor shaft is flexible, K_L is finite, two of the open-loop poles are complex. As the shaft becomes stiffer, K_L increases, and the imaginary parts of the open-loop poles also increase. When $K_L = \infty$, the shaft is rigid, the poles of the forward-path transfer function are all real. Similar effects are observed for the roots of the characteristic equation with respect to the value of K_L .

7-41 (a)

$$G_c(s) = 1 \quad G(s) = \frac{100(s+2)}{s^2-1} \quad K_p = \lim_{s \rightarrow 0} G(s) = -200$$

When $d(t) = 0$, the steady-state error due to a unit-step input is

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1-200} = -\frac{1}{199} = -0.005025$$

(b)

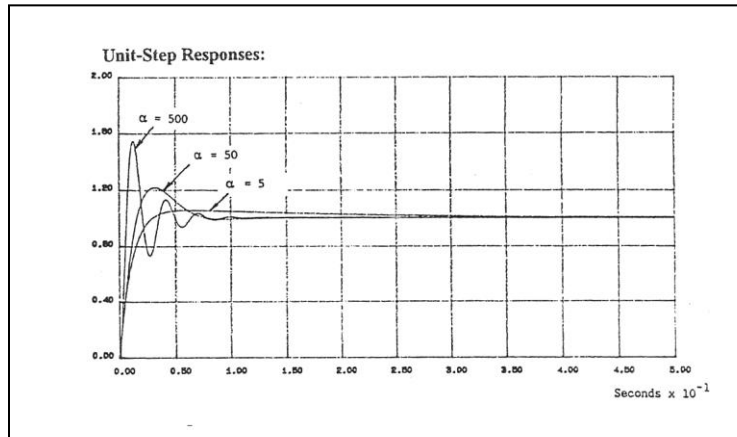
$$G_c(s) = \frac{s+\alpha}{s} \quad G(s) = \frac{100(s+2)(s+\alpha)}{s(s^2-1)} \quad K_p = \infty \quad e_{ss} = 0$$

(c)

$$\begin{aligned} \alpha = 5 & \quad \text{maximum overshoot} = 5.6\% \\ \alpha = 50 & \quad \text{maximum overshoot} = 22\% \\ \alpha = 500 & \quad \text{maximum overshoot} = 54.6\% \end{aligned}$$

As the value of α increases, the maximum overshoot increases because the damping effect of the zero at $s = -\alpha$ becomes less effective.

Unit-Step Responses:



(d) $r(t) = 0$ and $G_c(s) = 1$. $d(t) = u_s(t)$ $D(s) = \frac{1}{s}$

System Transfer Function: ($r = 0$)

$$\left. \frac{Y(s)}{D(s)} \right|_{r=0} = \frac{100(s+2)}{s^3 + 100s^2 + (199 + 100\alpha)s + 200\alpha}$$

Output Due to Unit-Step Input:

$$Y(s) = \frac{100(s+2)}{s[s^3 + 100s^2 + (199 + 100\alpha)s + 200\alpha]}$$

$$y_{ss} = \lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} sY(s) = \frac{200}{200\alpha} = \frac{1}{\alpha}$$

(e) $r(t) = 0$, $d(t) = u_s(t)$

$$G_c(s) = \frac{s + \alpha}{s}$$

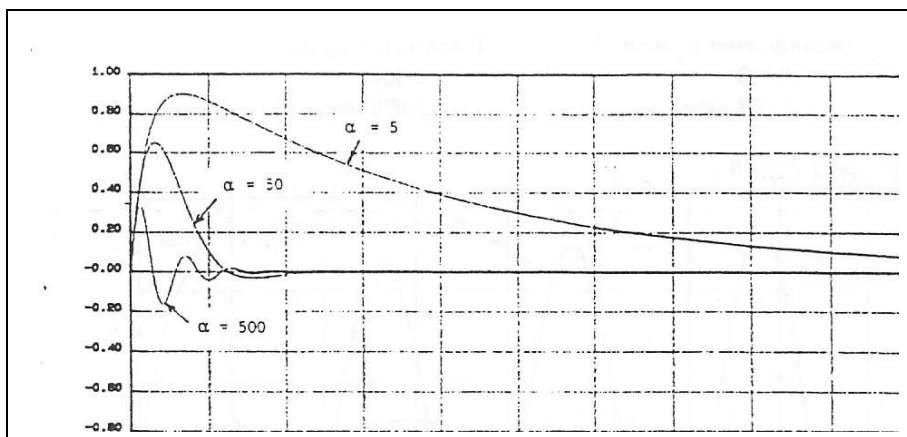
System Transfer Function $[r(t) = 0]$

$$\left. \frac{Y(s)}{D(s)} \right|_{r=0} = \frac{100s(s+20)}{s^3 + 100s^2 + (199+100\alpha)s + 200\alpha} \quad D(s) = \frac{1}{s}$$

$$y_{ss} = \lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} sY(s) = 0$$

(f)

$$\begin{aligned} \alpha = 5 \quad \left. \frac{Y(s)}{D(s)} \right|_{r=0} &= \frac{100s(s+2)}{s^3 + 100s^2 + 699s + 1000} \\ \alpha = 50 \quad \left. \frac{Y(s)}{D(s)} \right|_{r=0} &= \frac{100s(s+2)}{s^3 + 100s^2 + 5199s + 10000} \\ \alpha = 5000 \quad \left. \frac{Y(s)}{D(s)} \right|_{r=0} &= \frac{100s(s+2)}{s^3 + 100s^2 + 50199s + 100000} \end{aligned}$$

Unit-Step Responses:

- (g)** As the value of α increases, the output response $y(t)$ due to $r(t)$ becomes more oscillatory, and the overshoot is larger. As the value of α increases, the amplitude of the output response $y(t)$ due to $d(t)$ becomes smaller and more oscillatory.

7-42 (a) Forward-Path Transfer function:**Characteristic Equation:**

$$G(s) = \frac{H(s)}{E(s)} = \frac{10N}{s(s+1)(s+10)} \cong \frac{N}{s(s+1)}$$

$$s^2 + s + N = 0$$

N=1: Characteristic Equation: $s^2 + s + 1 = 0$

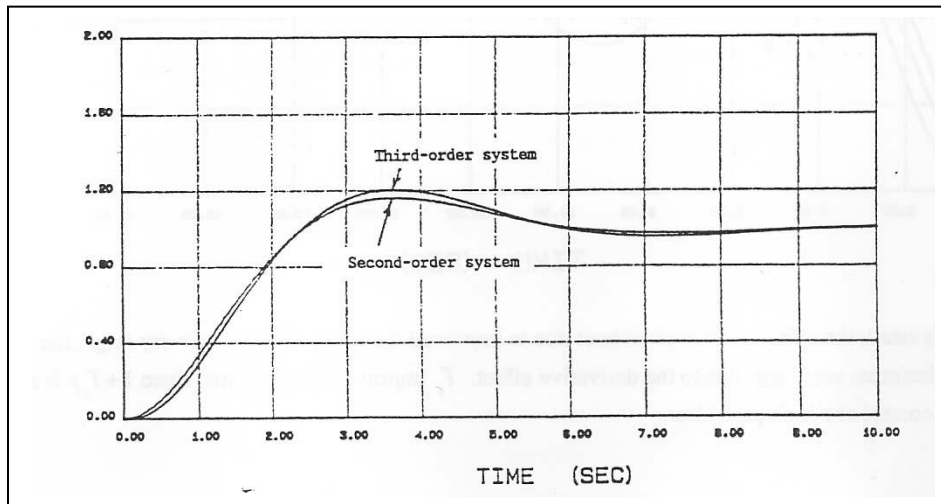
$$\zeta = 0.5 \quad \omega_n = 1 \text{ rad/sec.}$$

$$\text{Maximum overshoot} = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} = 0.163 \text{ (16.3\%)} \quad \text{Peak time } t_{\max} = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} = 3.628 \text{ sec.}$$

N=10: Characteristic Equation: $s^2 + s + 10 = 0$

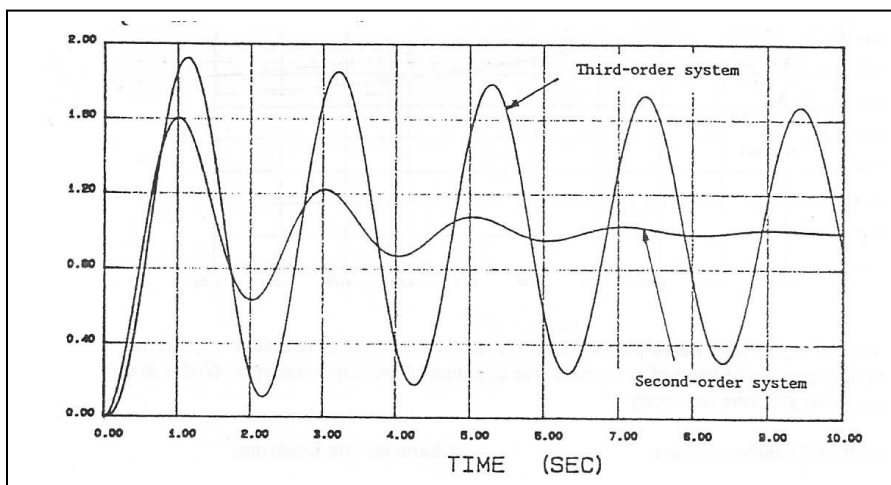
$$\zeta = 0.158 \quad \omega_n = 10 \text{ rad/sec.}$$

$$\text{Maximum overshoot} = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} = 0.605 \text{ (60.5\%)} \quad \text{Peak time } t_{\max} = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} = 1.006 \text{ sec.}$$

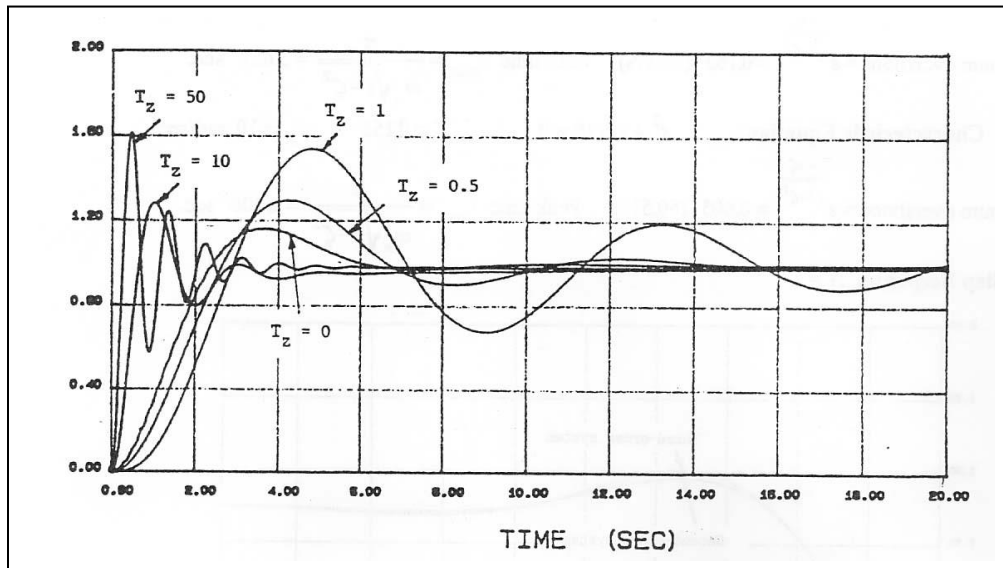
(b) Unit-Step Response: $N = 1$ 

	Second-order System	Third-order System
Maximum overshoot	0.163	0.206
Peak time	3.628 sec.	3.628 sec.

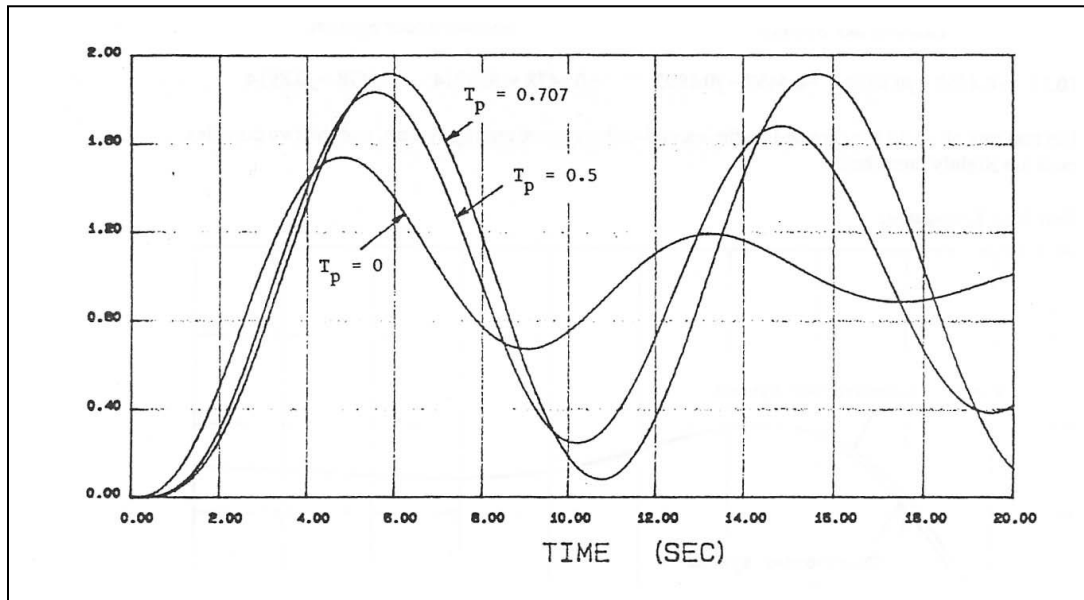
Unit-Step Response: $N = 10$



	Second-order System	Third-order System
Maximum overshoot	0.605	0.926
Peak time	1.006 sec.	1.13 sec.

7-43 Unit-Step Responses:

When T_z is small, the effect is lower overshoot due to improved damping. When T_z is very large, the overshoot becomes very large due to the derivative effect. T_z improves the rise time, since $1 + T_z s$ is a derivative control or a high-pass filter.

7-44 Unit-Step Responses

The effect of adding the pole at $s = -\frac{1}{T_p}$ to $G(s)$ is to increase the rise time and the overshoot. The system is

less stable. When $T_p > 0.707$, the closed-loop system is stable.

7-45) You may use the ACSYS software developed for this book. For description refer to Chapter 9. **We use a MATLAB code similar to toolboxes in Chapter 7 to solve this problem.**

(a) **Using MATLAB**

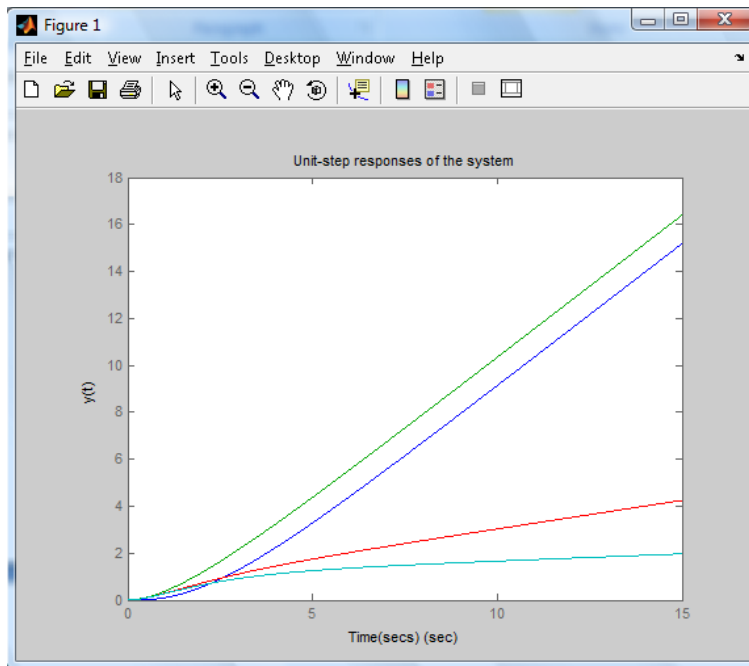
```
clear all
num = [];
den = [0 -0.55 -1.5];
G=zpk(num,den,1)
t=0:0.001:15;
step(G,t);
hold on;
for Tz=[1 5 20];
t=0:0.001:15;
num = [-1/Tz];
den = [0 -0.55 -1.5];
G=zpk(num,den,1)
step(G,t);
hold on;
end
xlabel('Time(secs)')
ylabel('y(t)')
title('Unit-step responses of the system')
```

```
Zero/pole/gain:
      1
-----
s (s+0.55) (s+1.5)
```

```
Zero/pole/gain:
      (s+1)
-----
s (s+0.55) (s+1.5)
```

```
Zero/pole/gain:
      (s+0.2)
-----
s (s+0.55) (s+1.5)
```

```
Zero/pole/gain:
      (s+0.05)
-----
s (s+0.55) (s+1.5)
```



(b)

```

clear all
for Tz=[0 1 5 20];
t=0:0.001:15;
num = [Tz 1];
den = [1 2 2];
G=tf(num,den)
step(G,t);
hold on;
end
xlabel('Time(secs) ')
ylabel('y(t) ')
title('Unit-step responses of the system')

```

Transfer function:

$$\frac{1}{s^2 + 2s + 2}$$

Transfer function:

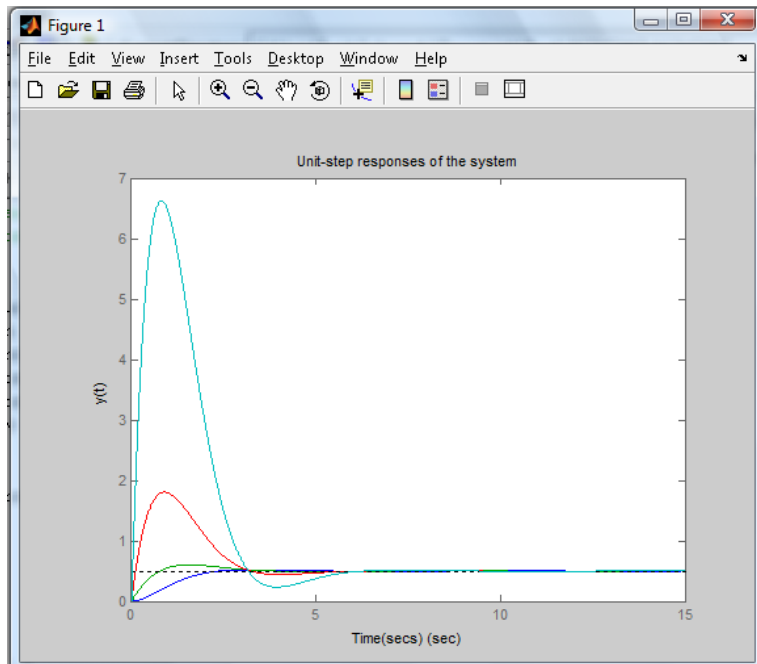
$$\frac{s + 1}{s^2 + 2s + 2}$$

Transfer function:

$$\frac{5s + 1}{s^2 + 2s + 2}$$

Transfer function:

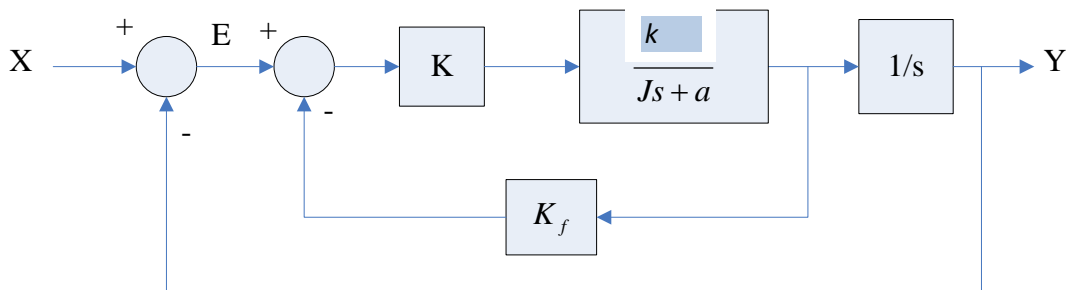
$$\frac{20s + 1}{s^2 + 2s + 2}$$



Follow the same procedure for other parts.

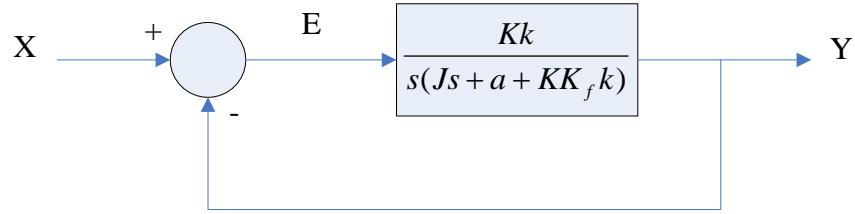
7-46) Since the system is linear we use superposition to find Y , for inputs X and D

First, consider $D = 0$



Then

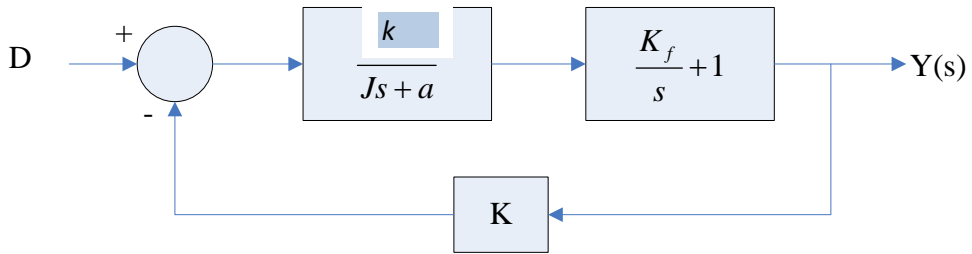
$$\frac{Y}{X} = G(s) = \frac{G_1}{1 + G_1} = \frac{Kk}{s(Js + a + KkK_f) + Kk} ; G_1 = \frac{Kk}{s(Js + a + KkK_f)}$$



According to above block diagram:

$$E(s) = X(s) - Y(s) = X(s) - \frac{X(s)G_1(s)}{1 + G_1(s)} = \frac{1}{1 + G_1(s)}X(s)$$

Now consider $X = 0$, then:



Accordingly,

$$G_2(s) = \frac{k(K_f + s)}{s(Js + a)}$$

and

$$Y(s) = \frac{G_2(s)}{1 + KG_2(s)}D(s) = \frac{k(K_f + s)}{s(Js + a) + k(K_f + s)K}D(s)$$

In this case, $E(s) = D - KY(s)$

$$E(s) = D(s) - \frac{KG_2(s)}{1 + KG_2(s)}D(s) = -\frac{1}{1 + KG_2(s)}D(s)$$

Now the steady state error can be easily calculated by:

$$\left\{ \begin{array}{l} \text{for unit step input } X = 1/s: e_{ss} = \lim_{s \rightarrow 0} \frac{1}{1 + G_1(s)} = \frac{Kk}{s(Js + a + KkK_f)} \\ \\ \lim_{s \rightarrow 0} \frac{1}{1 + \frac{Kk}{s(Js + a + KkK_f)}} = \lim_{s \rightarrow 0} \frac{s(Js + a + KkK_f)}{s(Js + a + KkK_f) + Kk} = 0 \\ \\ \text{for ramp input } D = \frac{1}{s^2}: e_{ss} = \lim_{s \rightarrow 0} -\frac{1}{s(1 + KG_2(s))} \\ \\ = \lim_{s \rightarrow 0} \frac{-1}{s \left(1 + K \frac{k(K_f + s)}{s(Js + a)} \right)} \\ \\ = \lim_{s \rightarrow 0} \frac{-s(Js + a)}{s(s(Js + a) + Kk(K_f + s))} = -\frac{a}{KkK_f} \end{array} \right.$$

(c) The overall response is obtained through superposition

$$Y(s) = Y(s)|_{D=0} + Y(s)|_{X=0}$$

$$y(t) = y(t)|_{d(t)=0} + y(t)|_{x(t)=0}$$

$$Y(s) = \frac{Kk}{s(Js + a + KkK_f) + Kk} X(s) + \frac{k(K_f + s)}{s(Js + a) + k(K_f + s)K} D(s)$$

MATLAB

```
clear all
syms s K k J a Kf
X=1/s;
D=1/s^2
Y=K*k*X/(s*(J*s+a+K*k*Kf)+K*k)+k*(Kf+s)*D/(s*(J*s+a)+k*(Kf+s)*K)
ilaplace(Y)
D =
1/s^2
Y =
K*k/s/(s*(J*s+a+K*k*Kf)+K*k)+k*(Kf+s)/s^2/(s*(J*s+a)+k*(Kf+s)*K)
ans =

1+t/K+1/k/K^2/Kf/(a^2+2*a*K*k+K^2*k^2-
4*J*K*k*Kf)^(1/2)*sinh(1/2*t/J*(a^2+2*a*K*k+K^2*k^2-4*J*K*k*Kf)^(1/2))*exp(-
1/2*(a+K*k)/J*t)*(a^2+a*K*k-2*J*K*k*Kf)-cosh(1/2*t/J*(a^2+2*a*K*k*Kf+K^2*k^2*Kf^2-
4*J*K*k)^(1/2))*exp(-1/2*(a+K*k*Kf)/J*t)-(a+K*k*Kf)/(a^2+2*a*K*k*Kf+K^2*k^2*Kf^2-
4*J*K*k)^(1/2)*sinh(1/2*t/J*(a^2+2*a*K*k*Kf+K^2*k^2*Kf^2-4*J*K*k)^(1/2))*exp(-
1/2*(a+K*k*Kf)/J*t)+1/k/K^2/Kf*a*(-1+exp(-
1/2*(a+K*k)/J*t)*cosh(1/2*t/J*(a^2+2*a*K*k+K^2*k^2-4*J*K*k*Kf)^(1/2)))
```

7-47) (a) Find the $\int_0^\infty e(t)dt$ when $e(t)$ is the error in the unit step response.

As the system is stable then $\int_0^\infty e(t)dt$ will converge to a constant value:

$$\int_0^\infty e(t)dt = \lim_{s \rightarrow 0} s \frac{E(s)}{s} = \lim_{s \rightarrow 0} E(s)$$

$$\frac{Y(s)}{X(s)} = \frac{G(s)}{1+G(s)} = \frac{(A_1s+1)(A_2s+1)\dots(A_ns+1)}{(B_1s+1)(B_2s+1)\dots(B_ms+1)} \quad n \leq m$$

$$\begin{aligned} E(s) &= X(s) - Y(s) = X(s) - \frac{G(s)}{1+G(s)} X(s) = \frac{1}{1+G(s)} X(s) \\ &= \left(1 - \frac{(A_1s+1)(A_2s+1)\dots(A_ns+1)}{(B_1s+1)(B_2s+1)\dots(B_ms+1)} \right) X(s) = \left(\frac{(B_1s+1)(B_2s+1)\dots(B_ms+1) - (A_1s+1)(A_2s+1)\dots(A_ns+1)}{(B_1s+1)(B_2s+1)\dots(B_ms+1)} \right) X(s) \\ \lim_{s \rightarrow 0} E(s) &= \lim_{s \rightarrow 0} \frac{1}{s} \left(\frac{(B_1s+1)(B_2s+1)\dots(B_ms+1) - (A_1s+1)(A_2s+1)\dots(A_ns+1)}{(B_1s+1)(B_2s+1)\dots(B_ms+1)} \right) \\ &= (B_1 + B_2 + \dots + B_m) - (A_1 + A_2 + \dots + A_n) \end{aligned}$$

$$G(s) = \left(\frac{(B_1s+1)(B_2s+1)\dots(B_ms+1)}{(B_1s+1)(B_2s+1)\dots(B_ms+1) - (A_1s+1)(A_2s+1)\dots(A_ns+1)} \right) - 1$$

$$G(s) = \left(\frac{(A_1s+1)(A_2s+1)\dots(A_ns+1)}{(B_1s+1)(B_2s+1)\dots(B_ms+1) - (A_1s+1)(A_2s+1)\dots(A_ns+1)} \right)$$

$$e_{ss} = \lim_{s \rightarrow 0} s \frac{1}{s} \left(1 - \frac{(A_1s+1)(A_2s+1)\dots(A_ns+1)}{(B_1s+1)(B_2s+1)\dots(B_ms+1)} \right) = 0$$

(b) Calculate $\frac{1}{K} = \frac{1}{\lim_{s \rightarrow 0} sG(s)}$

Recall

$$E(s) = X(s) - Y(s) = X(s) - \frac{G(s)}{1+G(s)} X(s) = \frac{1}{1+G(s)} X(s)$$

Hence

$$\lim_{s \rightarrow 0} E(s) = \lim_{s \rightarrow 0} \frac{1}{1+G(s)} X(s) = \lim_{s \rightarrow 0} \frac{1}{s + sG(s)} = \lim_{s \rightarrow 0} \frac{1}{sG(s)} = \frac{1}{K_v}$$

Ramp Error Constant

7-48)

$$\frac{C(s)}{R(s)} = \frac{10(s+K)}{(s+p)(s+25) + (s+K)10} = \frac{10(s+K)}{s^2 + (35+p)s + (25p+10K)}$$

Comparing with the second order prototype system and matching denominators:

$$\begin{cases} 25p + 10K = \omega_n^2 \\ 35 + p = 2\xi\omega_n \end{cases}$$

$$\begin{cases} M_p = \exp\left(-\frac{\pi\xi}{\sqrt{1-\xi^2}}\right) \geq 0.25 \rightarrow \frac{\pi\xi}{\sqrt{1-\xi^2}} \geq 1.386 \rightarrow \xi \geq 0.210 \\ t_s = \frac{3.2}{\xi\omega_n} \leq 0.1 \rightarrow \omega_n \leq 80, \text{ when } 0 < \xi < 0.69 \end{cases}$$

Let $\xi = 0.4$ and $\omega_n = 80$

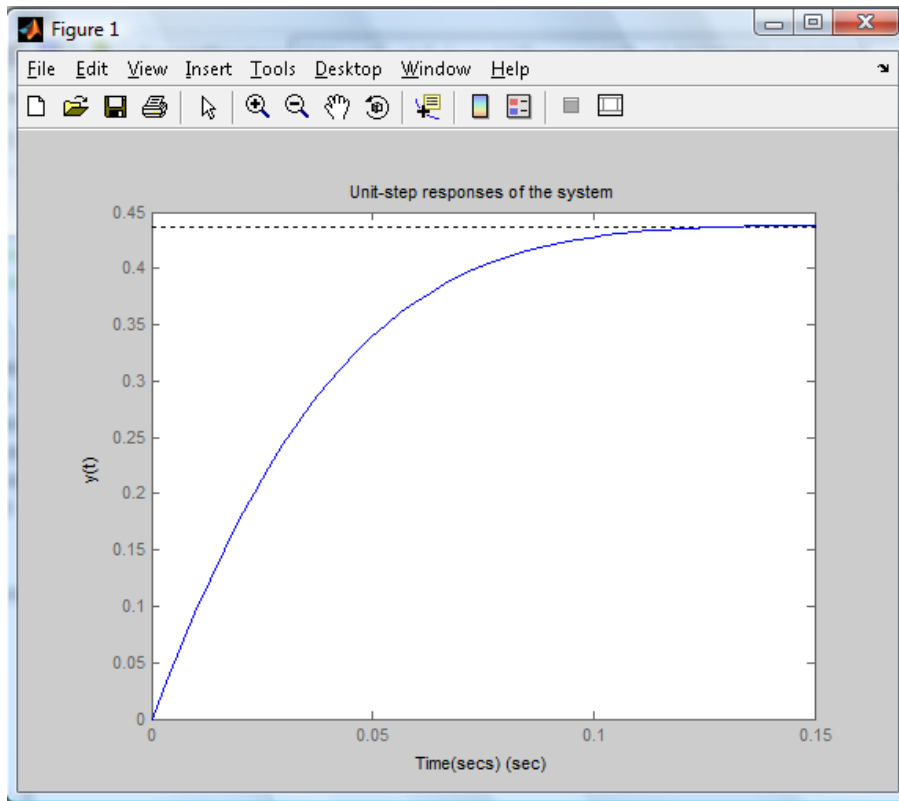
Then

$$\begin{cases} p = (2\xi\omega_n)(35) = 29 \\ K = \frac{\omega_n^2 - 25p}{100} = 56.25 \end{cases}$$

```
clear all
p=29;
K=56.25;
num = [10 10*K];
den = [1 35+p 25*p+10*K];
G=tf(num,den)
step(G);
xlabel('Time(secs)')
ylabel('y(t)')
title('Unit-step responses of the system')
```

Transfer function:

$$\frac{10s + 562.5}{s^2 + 64s + 1288}$$

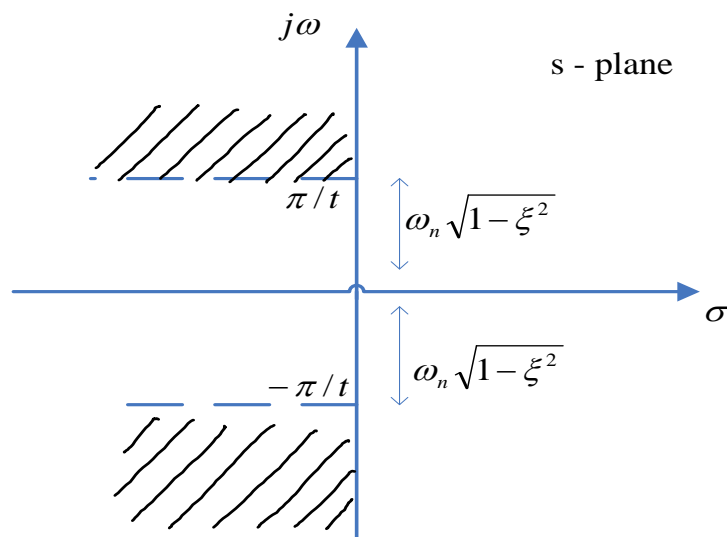


7-49) According to the maximum overshoot:

$$t_{max} = \frac{\pi}{\omega_n \sqrt{1-\xi^2}}$$

which should be less than t , then

$$\frac{\pi}{\omega_n \sqrt{1-\xi^2}} < t \text{ or } \omega_n \sqrt{1-\xi^2} > \frac{\pi}{t}$$



- 7-50)** Using a 2nd order prototype system format, from Figure 7-9, ω_n is the radial distance from the complex conjugate roots to the origin of the s-plane, then ω_n with respect to the origin of the shown region is $\omega_n \approx 3.6$.

Therefore the natural frequency range in the region shown is around $2.6 \leq \omega_n \leq 4.6$

On the other hand, the damping ratio ζ at the two dashed radial lines is obtained from:

$$\begin{cases} \zeta_1 = \cos(\pi/2 - \alpha_1) = \sin \alpha_1 \\ \zeta_2 = \cos(\pi/2 - \alpha_2) = \sin \alpha_2 \end{cases}$$

The approximation from the figure gives:

$$\begin{cases} \zeta_1 \approx 0.56 \\ \zeta_2 \approx 0.91 \end{cases}$$

Therefore $0.56 \leq \zeta \leq 0.91$

b)

$$\frac{C(s)}{R(s)} = \frac{KK_p(s + K_I)}{s^2 + (p + KK_p)s + KK_pK_I}$$

As $K_p=2$, then:

$$\frac{C(s)}{R(s)} = \frac{2K(s + K_I)}{s^2 + 2(K + 1)s + 2KK_I}$$

If the roots of the characteristic equations are assumed to be lied in the centre of the shown region:

$$\begin{cases} P_1 = s - 3 - 2j \\ P_2 = s - 3 + 2j \end{cases} \Rightarrow s^2 + 6s + 13 = 0$$

Comparing with the characteristic equation:

$$\begin{cases} 2(K + 1) = 6 \rightarrow K = 2 \\ 2KK_I = 13 \rightarrow K_I = 3.25 \end{cases}$$

c) The characteristic equation

$$s^2 + 2(p + KK_p)s + KK_pK_I = 0$$

is a second order polynomial with two roots. These two roots can be determined by two terms $2(p + KK_p)$ and KK_pK_I which includes four parameters. Regardless of the p and K_p values, we can always choose K and K_I so that to place the roots in a desired location.

7-51) a) $J_m s^2 \theta_m(s) + \left(B + \frac{K_1 K_2}{R}\right) s \theta(s) = \frac{K_1}{R} V(s)$

$$\frac{\theta_m(s)}{V(s)} = \frac{\frac{K_1}{R}}{s \left(s + \left(B + \frac{K_1 K_2}{R} \right) \right)}$$

By substituting the values:

$$\frac{\theta_m(s)}{V(s)} = \frac{0.2}{s(s + 0.109)}$$

b) Speed of the motor is $\frac{d\theta}{dt} = \omega$

$$\frac{\omega(s)}{V(s)} = \frac{s \theta(s)}{V(s)} = \frac{0.2}{s + 0.109}$$

$$e_{ss} = V \lim_{s \rightarrow 0} G(s) = 10 \frac{0.2}{0.109} = 19.23$$

(c)

$$\frac{\theta_m(s)}{V(s)} = \frac{0.2}{s(s + 0.109)}$$

d)

$$\begin{aligned} \theta_m(s) &= \frac{0.2}{s(s + 0.109)} V(s) \\ &= \frac{0.2}{s(s + 0.109)} K (\theta_p(s) - \theta_m(s)) \end{aligned}$$

As a result:

$$\frac{\theta_m(s)}{\theta_p(s)} = \frac{0.2K}{s^2 + 0.109s + 0.2K}$$

e) As $M_p = \exp\left(-\frac{\pi\xi}{\sqrt{1-\xi^2}}\right) = 0.2s$, then, $\xi = 0.404$

According to the transfer function, $2\xi\omega_n = 0.109$, then $\omega_n = \frac{0.109}{(2)(0.404)} \approx 0.14 \text{ rad/sec}$

where $\omega_n^2 = 0.2K \Rightarrow K < 0.0845$

f) As $t_r = \frac{0.8+2.5\xi}{\omega_n} \approx \frac{1.8}{\omega_n}$, then $\omega_n \geq 0.6$,
as $\omega_n^2 = 0.2K$, therefore; $K \geq 1$

g) MATLAB

```
clear all
for K=[0.5 1 2];
t=0:0.001:15;
num = [0.2*K];
den = [1 0.109 0.2*K];
G=tf(num,den)
step(G,t);
hold on;
end
xlabel('Time(secs)')
ylabel('y(t)')
title('Unit-step responses of the system')
```

Transfer function:
0.1

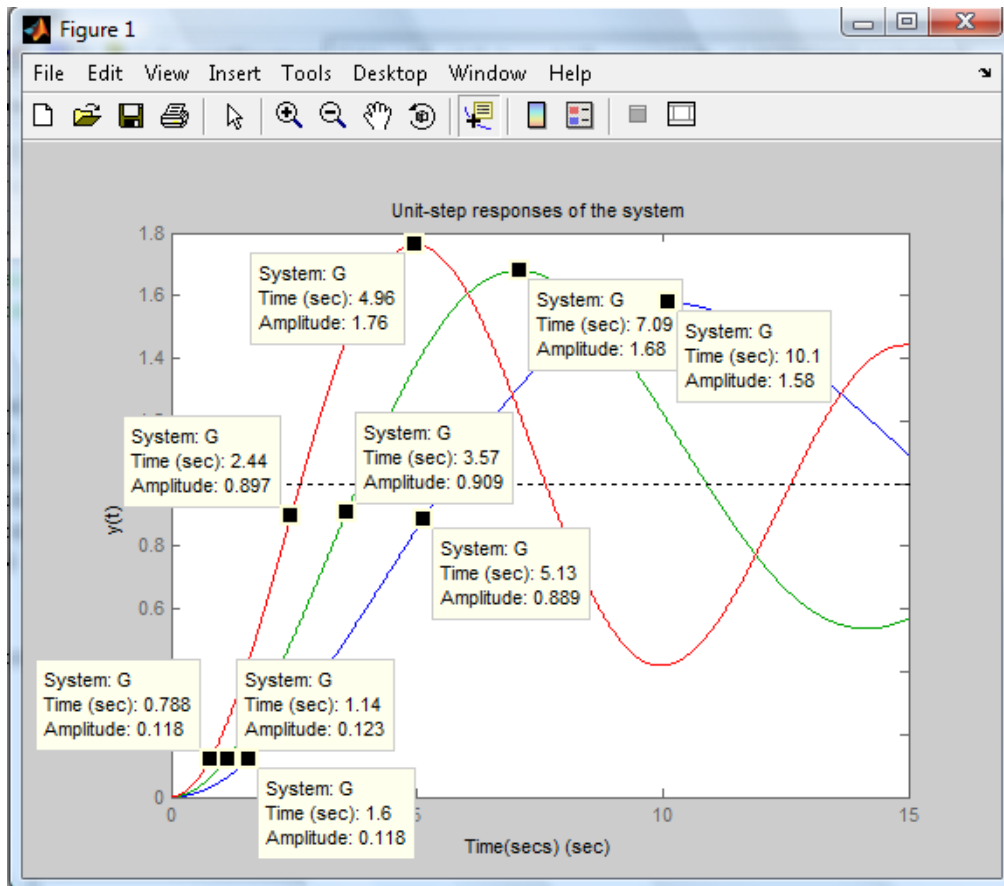
 $s^2 + 0.109 s + 0.1$

Transfer function:
0.2

 $s^2 + 0.109 s + 0.2$

Transfer function:
0.4

 $s^2 + 0.109 s + 0.4$



Rise time decreases with K increasing.

Overshoot increases with K .

7-52) $M_p = \exp\left(-\frac{\pi\xi}{\sqrt{1-\xi^2}}\right) = 0.1$, therefore, $\xi = 0.59$

As $t_s = \frac{3.2}{\xi\omega_n}$, then, $\omega_n = \frac{3.2}{(0.59)(1.5)} \approx 3.62$

$$\begin{aligned}\frac{Y(s)}{X(s)} &= \frac{G(s)H(s)}{1 + G(s)H(s)} = \frac{K(s+a)}{s(s+3)(s+b) + K(s+a)} \\ &= \frac{K(s+a)}{s^3 + (3+b)s^2 + (3b+K)s + Ka}\end{aligned}$$

Therefore:

$$s^3 + (3+b)s^2 + (3b+K)s + Ka = (s+p)(s^2 + 2\xi\omega_n s + \omega_n^2)$$

If p is a non-dominant pole; therefore comparing both sides of above equation and:

$$\begin{cases} 2\xi\omega_n + p = 3+b \\ 2\xi\omega_n p + \omega_n^2 = 3b+K \\ \omega_n^2 p = Ka \end{cases}$$

If we consider $p = 10a$ (non-dominant pole), $\xi = 0.6$ and $\omega_n = 4$, then:

$$\begin{cases} 4.8 + 10a = 3+b \\ 48a + 16 = 3b+K \\ 160a = Ka \rightarrow K = 160 \end{cases}$$

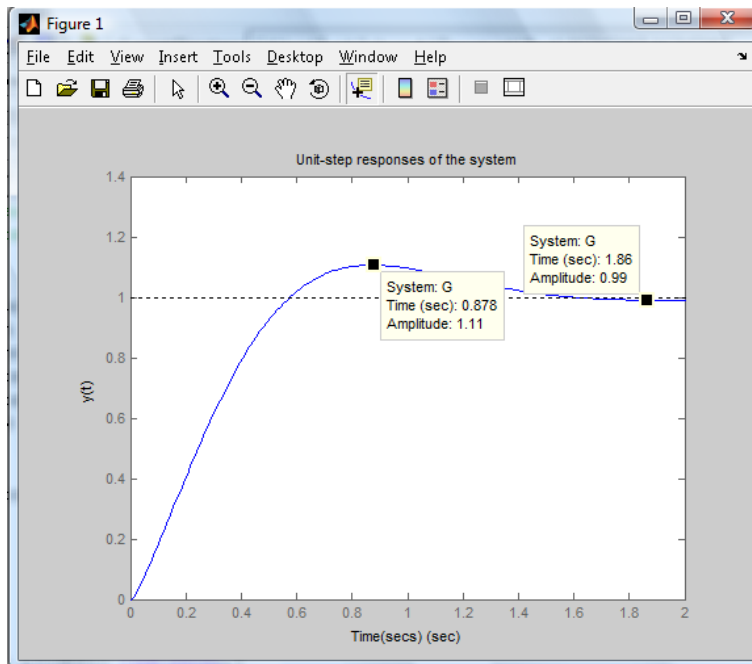
$$\begin{cases} a = 8.3 \\ b = 89.8 \\ p = 83 \end{cases}$$

```
clear all
K=160;
a=8.3;
b=89.8;
p=83;
num = [K K*a];
den = [1 3+b 3*b+K K*a];
G=tf(num,den)
step(G);
xlabel('Time(secs)')
ylabel('y(t)')
title('Unit-step responses of the system')
```

Transfer function:

$$160 s + 1328$$

$$s^3 + 92.8 s^2 + 429.4 s + 1328$$



Both Overshoot and settling time values are met. No need to adjust parameters.

7-53) For the controller and the plant to be in series and using a unity feedback loop we have:

MATLAB

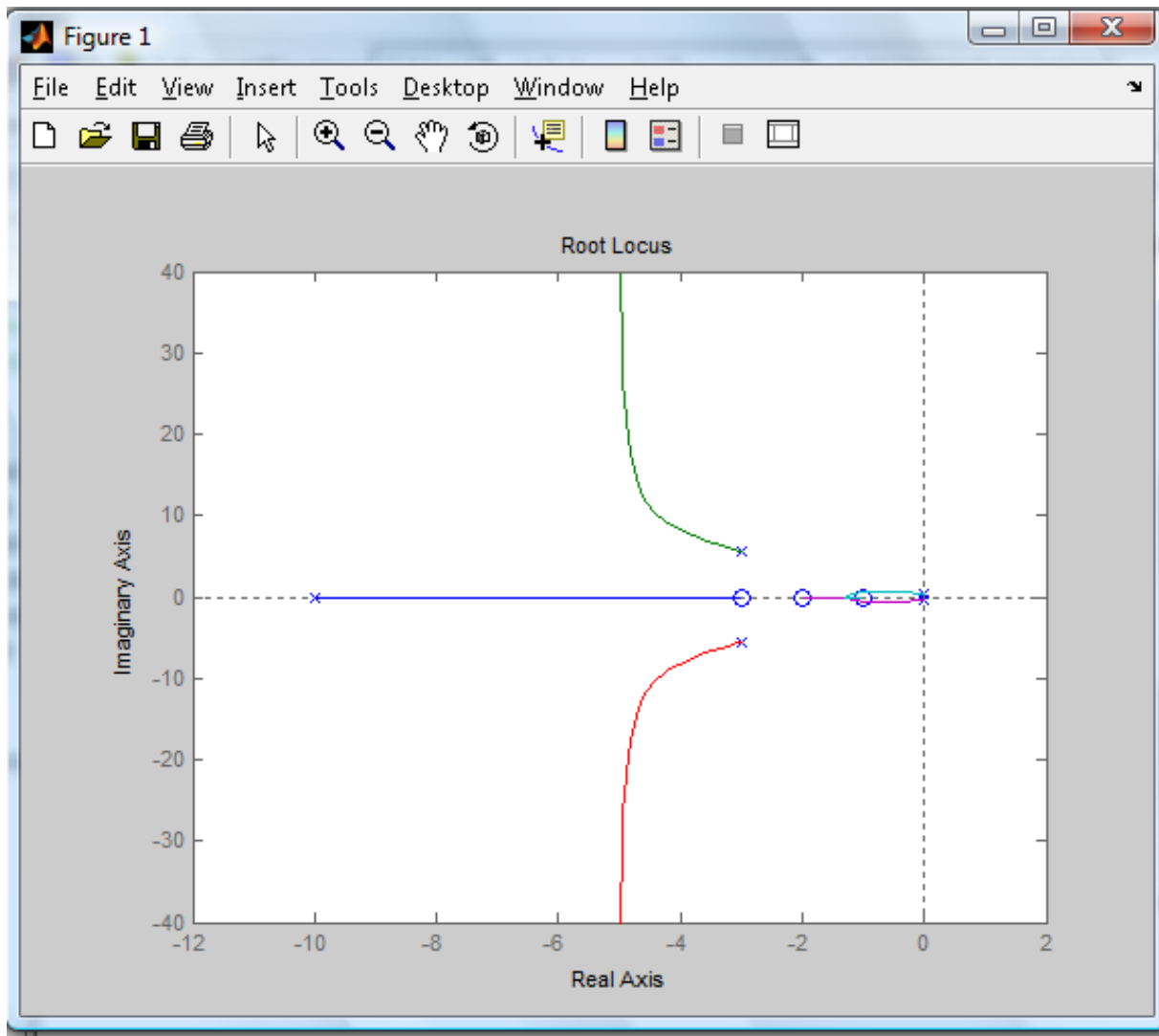
USE toolbox

```
clear all
num=[-1 -2 -3];
denom=[-3+sqrt(9-40) -3-sqrt(9-40) -0.02+sqrt(.004-.07) -0.02-sqrt(.004-.07) -10];
G=zpk(num,denom,60)
rlocus(G)
```

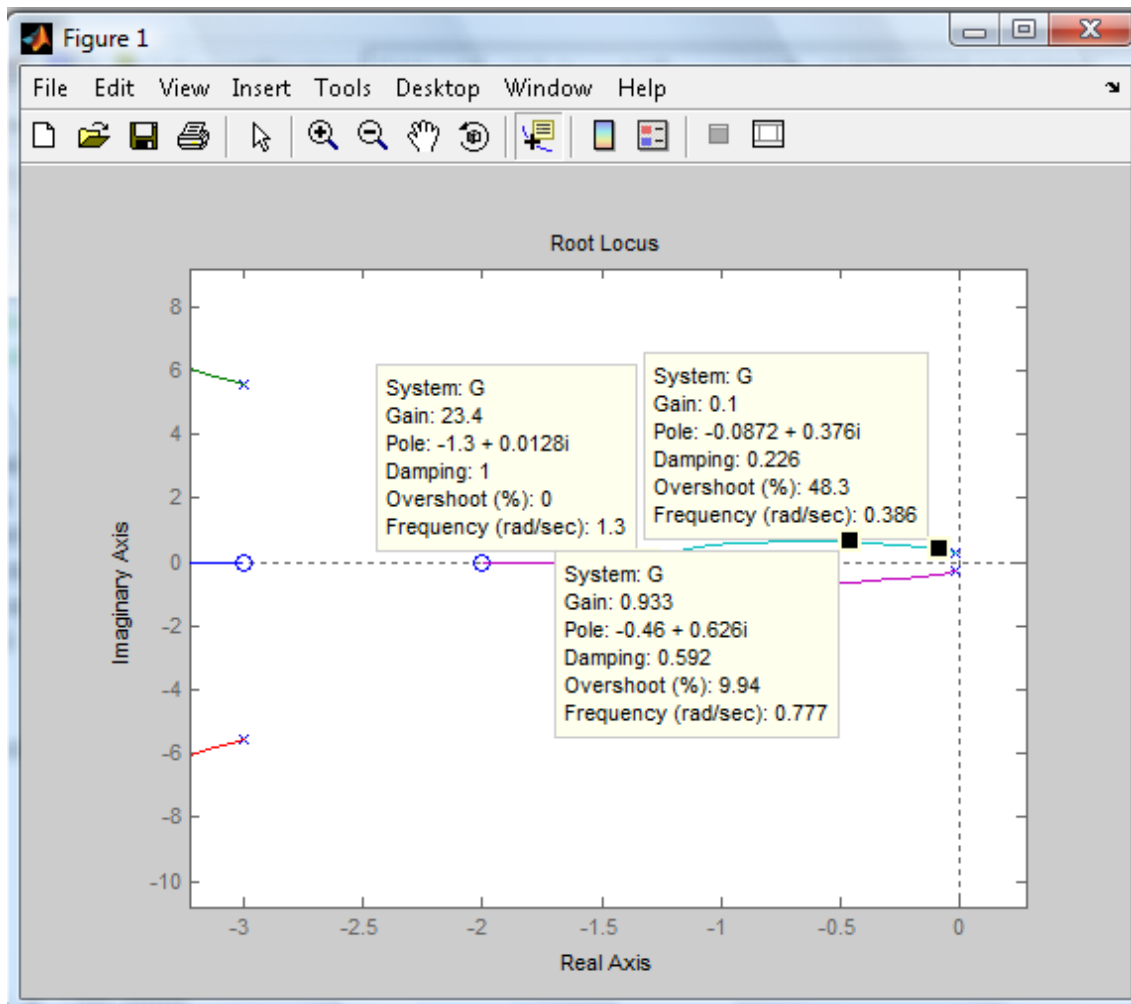
Zero/pole/gain:

$$60 (s+1) (s+2) (s+3)$$

$$(s+10) (s^2 + 0.04s + 0.0664) (s^2 + 6s + 40)$$



Note the system has two dominant complex poles close to the imaginary axis. Lets zoom in the root locus diagram and use the cursor to find the parameter values.



As shown for $K=0.933$ the dominant closed loop poles are at $-0.46 \pm j 0.626$ AND OVERSHOOT IS ALMOST 10%.

Increasing K will push the poles closer towards less dominant zeros and poles. As a process the design process becomes less trivial and more difficult.

To confirm use

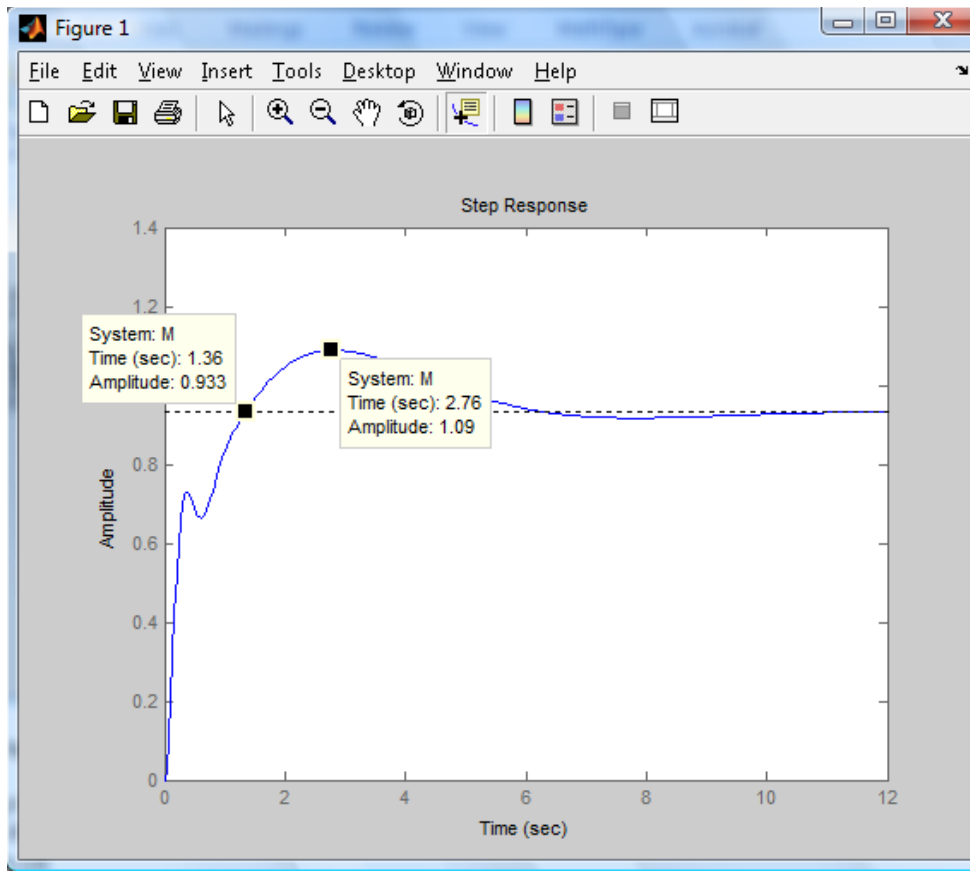
`M=feedback(G*.933,1) %See toolbox in problem 7-53`

`step(M)`

Zero/pole/gain:

55.98 (s+3) (s+2) (s+1)

(s+7.048) (s² + 0.9195s + 0.603) (s² + 8.072s + 85.29)



To reduce rise time, the poles have to move to left to make the secondary poles more dominant. As a result the little bump in the left hand side of the above graph should rise. Try $K=3$:

```
>> M=feedback(G*3,1)
```

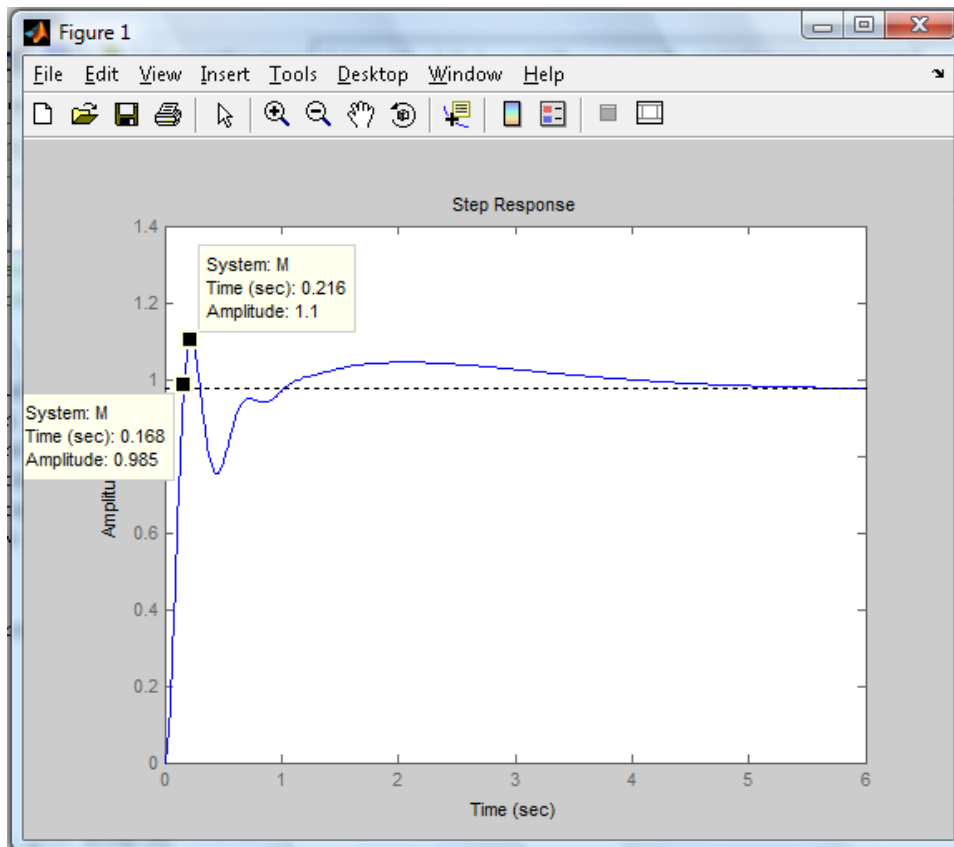
Zero/pole/gain:

$$180 (s+3) (s+2) (s+1)$$

$$(s+5.01) (s^2 + 1.655s + 1.058) (s^2 + 9.375s + 208.9)$$

```
>> step(M)
```

****Try a higher K value, but looking at the root locus and the time plots, it appears that the overshoot and rise time criteria will never be met simultaneously.**



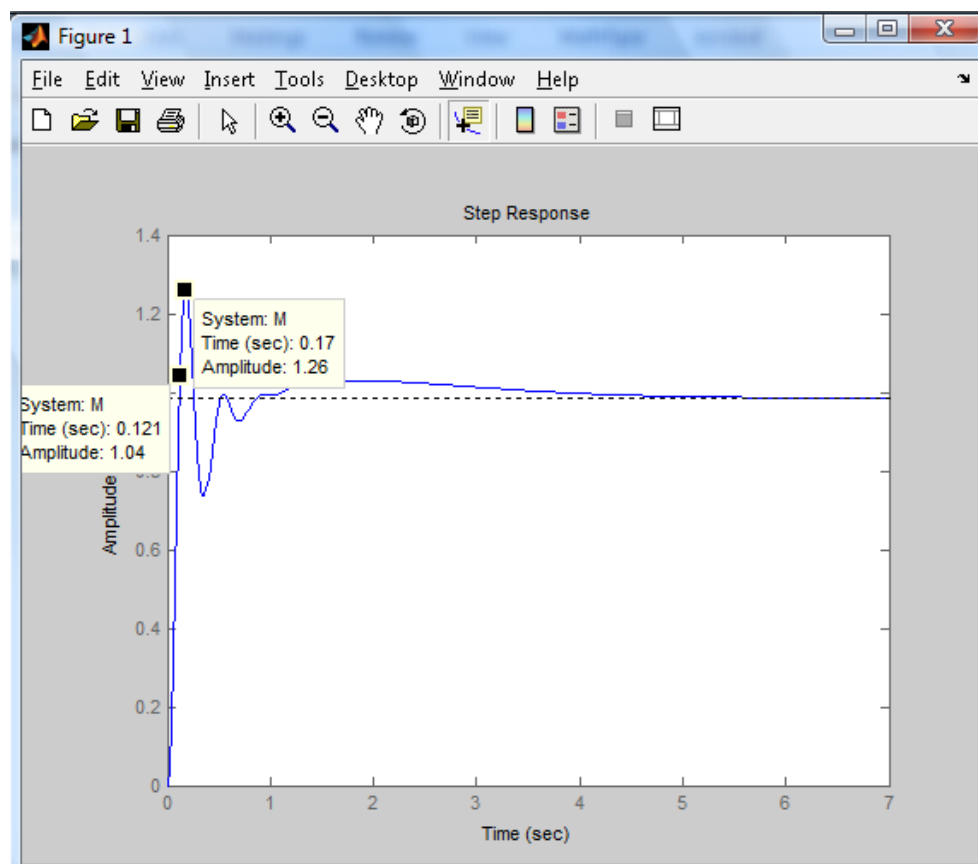
K=5

`M=feedback(G*5,1) %See toolbox in problem 7-53`
`step(M)`

Zero/pole/gain:

$$300 (s+3) (s+2) (s+1)$$

$$(s+4.434) (s^2 + 1.958s + 1.252) (s^2 + 9.648s + 329.1)$$



7-54) Forward-path Transfer Function:

$$G(s) = \frac{M(s)}{1 - M(s)} = \frac{K}{s^3 + (20 + a)s^2 + (200 + 20a)s + 200a - K}$$

For type 1 system, $200a - K = 0$ Thus $K = 200a$

Ramp-error constant:

$$K_v = \lim_{s \rightarrow 0} sG(s) = \frac{K}{200 + 20a} = \frac{200a}{200 + 20a} = 5 \quad \text{Thus} \quad a = 10 \quad K = 2000$$

MATLAB Symbolic tool can be used to solve above. We use it to find the roots for the next part:

```
>> syms s a K

>> solve(5*200+5*20*a-200a)

ans =

10

>> D=(s^2+20*s+200)*(s+a))

D =

(s^2+20*s+200)*(s+a)

>> expand(D)

ans =

s^3+s^2*a+20*s^2+20*s*a+200*s+200*a

>> solve(ans,s)

ans =

-a

-10+10*i

-10-10*i
```

The forward-path transfer function is

The controller transfer function is

$$G(s) = \frac{2000}{s(s^2 + 30s + 400)}$$

$$G_c(s) = \frac{G(s)}{G_p(s)} = \frac{20(s^2 + 10s + 100)}{(s^2 + 30s + 400)}$$

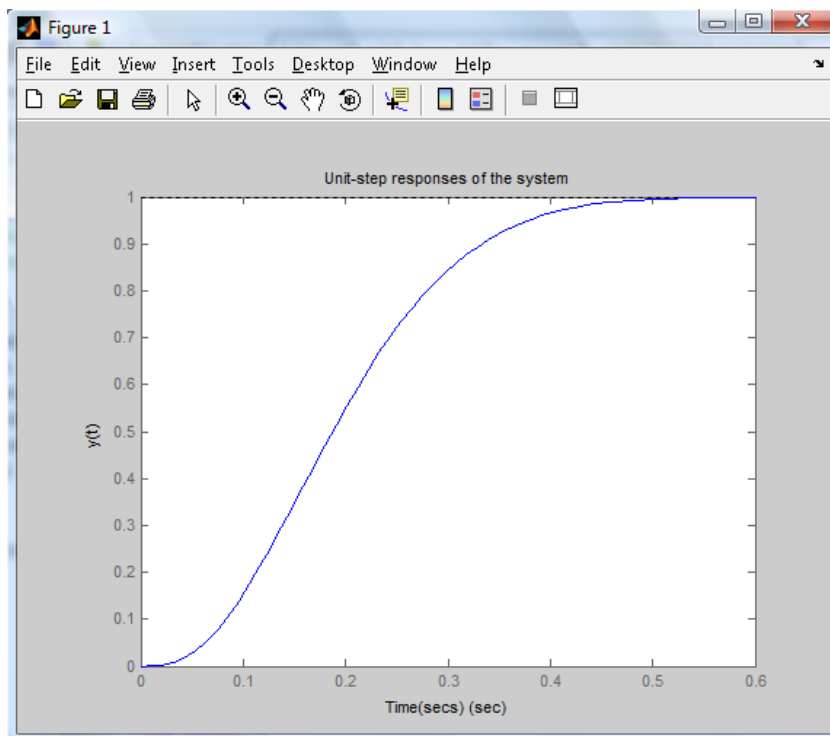
The maximum overshoot of the unit-step response is 0 percent.

MATLAB

```
clear all
K=2000;
a=10;
num = [];
den = [-10+10i -10-10i -a];
G=zpk(num,den,K)
step(G);
xlabel('Time(secs)')
ylabel('y(t)')
title('Unit-step responses of the system')
```

Zero/pole/gain:
2000

(s+10) (s^2 + 20s + 200)



Clearly PO=0.

7-55)

Forward-path Transfer Function:

$$G(s) = \frac{M(s)}{1 - M(s)} = \frac{K}{s^3 + (20 + a)s^2 + (200 + 20a)s + 200a - K}$$

For type 1 system, $200a - K = 0$ Thus $K = 200a$

Ramp-error constant:

$$K_v = \lim_{s \rightarrow 0} sG(s) = \frac{K}{200 + 20a} = \frac{200a}{200 + 20a} = 9 \quad \text{Thus} \quad a = 90 \quad K = 18000$$

MATLAB Symbolic tool can be used to solve above. We use it to find the roots for the next part:

```
>> syms s a K
solve(9*200+9*20*a-200*a)

ans =

90

>> D=(s^2+20*s+200)*(s+a)

D =

(s^2+20*s+200)*(s+a)

>> expand(D)

ans =

s^3+s^2*a+20*s^2+20*s*a+200*s+200*a

>> solve(ans,s)

ans =

-a

-10+10*i

-10-10*i
```

The forward-path transfer function is

The controller transfer function is

$$G(s) = \frac{18000}{s(s^2 + 110s + 2000)}$$

$$G_c(s) = \frac{G(s)}{G_p(s)} = \frac{180(s^2 + 10s + 100)}{(s^2 + 110s + 2000)}$$

The maximum overshoot of the unit-step response is 4.3 percent.

From the expression for the ramp-error constant, we see that as a or K goes to infinity, K_v approaches 10.

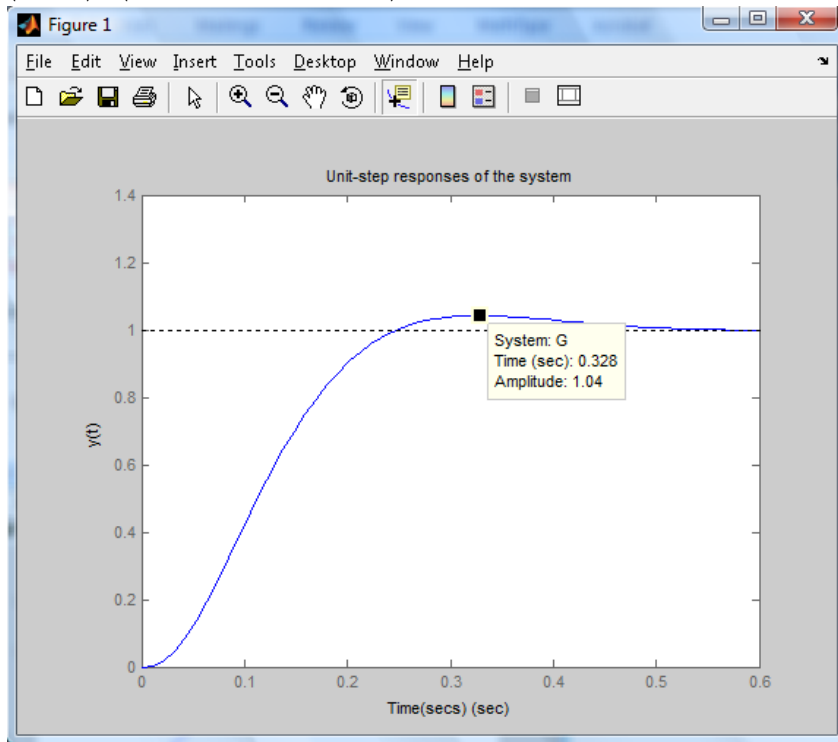
Thus the maximum value of K_v that can be realized is 10. The difficulties with very large values of K and

a are that a high-gain amplifier is needed and unrealistic circuit parameters are needed for the controller.

```
clear all
K=18000;
a=90;
num = [];
den = [-10+10i -10-10i -a];
G=zpk(num,den,K)
step(G);
xlabel('Time(secs)')
ylabel('y(t)')
title('Unit-step responses of the system')
```

```
Zero/pole/gain:
          18000
```

```
-----
(s+90) (s^2 + 20s + 200)
```



PO is less than 4.

7-56) (a) Ramp-error Constant:**MATLAB**

```

clear all
syms s Kp Kd kv
Gnum=(Kp+Kd*s)*1000
Gden= (s*(s+10))
G=Gnum/Gden
Kv=s*G
s=0
eval(Kv)

Gnum =
1000*Kp+1000*Kd*s

Gden =
s*(s+10)

G =
(1000*Kp+1000*Kd*s)/s/(s+10)

Kv =
(1000*Kp+1000*Kd*s)/(s+10)

s =
0

ans =
100*Kp

```

$$K_v = \lim_{s \rightarrow 0} s \frac{1000(K_p + K_d s)}{s(s+10)} = \frac{1000K_p}{10} = 100K_p = 1000 \quad \text{Thus} \quad K_p = 10$$

Kp=10

clear s

syms s

Mnum=(Kp+Kd*s)*1000/s/(s+10)

Mden=1+(Kp+Kd*s)*1000/s/(s+10)

Kp =

10

Mnum =

(10000+1000*Kd*s)/s/(s+10)

Mden =

1+(10000+1000*Kd*s)/s/(s+10)

ans =

(s^2+10*s+10000+1000*Kd*s)/s/(s+10)

Characteristic Equation: $s^2 + (10 + 1000K_D)s + 1000K_P = 0$

Match with a 2nd order prototype system

$$\omega_n = \sqrt{1000K_P} = \sqrt{10000} = 100 \text{ rad/sec} \quad 2\zeta\omega_n = 10 + 1000K_D = 2 \times 0.5 \times 100 = 100$$

solve(10+1000*Kd-100)

ans =

9/100

Thus $K_D = \frac{90}{1000} = 0.09$

Use the same procedure for other parts.

(b) For $K_v = 1000$ and $\zeta = 0.707$, and from part (a), $\omega_n = 100$ rad/sec,

$$2\zeta\omega_n = 10 + 1000K_D = 2 \times 0.707 \times 100 = 141.4 \quad \text{Thus} \quad K_D = \frac{131.4}{1000} = 0.1314$$

(c) For $K_v = 1000$ and $\zeta = 1.0$, and from part (a), $\omega_n = 100$ rad/sec,

$$2\zeta\omega_n = 10 + 1000K_D = 2 \times 1 \times 100 = 200 \quad \text{Thus} \quad K_D = \frac{190}{1000} = 0.19$$

7-57) The ramp-error constant:

$$K_v = \lim_{s \rightarrow 0} s \frac{1000(K_p + K_D s)}{s(s+10)} = 100K_p = 10,000 \quad \text{Thus } K_p = 100$$

The forward-path transfer function is:
$$G(s) = \frac{1000(100 + K_D s)}{s(s+10)}$$

```
clear all
for KD=0.2:0.2:1.0;
num = [-100/KD];
den = [0 -10];
G=zpk(num,den,1000);
M=feedback(G,1)
step(M);
hold on;
end
xlabel('Time(secs)')
ylabel('y(t)')
title('Unit-step responses of the system')
```

Zero/pole/gain:

1000 (s+500)

(s^2 + 1010s + 5e005)

Zero/pole/gain:

1000 (s+250)

(s+434.1) (s+575.9)

Zero/pole/gain:

1000 (s+166.7)

(s+207.7) (s+802.3)

Zero/pole/gain:

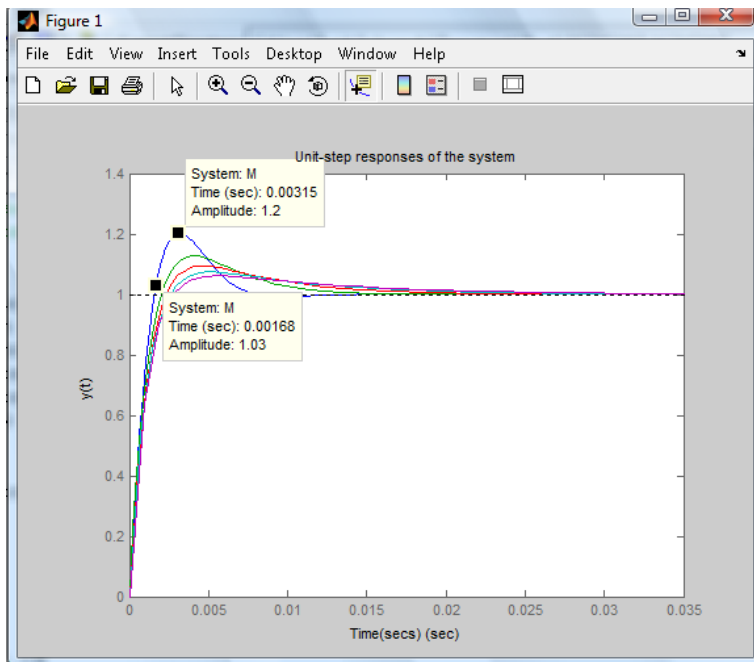
$$1000 (s+125)$$

$$(s+144.4) (s+865.6)$$

Zero/pole/gain:

$$1000 (s+100)$$

$$(s+111.3) (s+898.7)$$



Use the cursor to obtain the PO and tr values.

For part b the maximum value of K_D results in the minimum overshoot.

7-58) (a) Forward-path Transfer Function:

$$G(s) = G_c(s)G_p(s) = \frac{4500K(K_p + K_D s)}{s(s + 361.2)}$$

Ramp Error Constant: $K_v = \lim_{s \rightarrow 0} sG(s) = \frac{4500KK_p}{361.2} = 12.458KK_p$

$$e_{ss} = \frac{1}{K_v} = \frac{0.0802}{KK_p} \leq 0.001 \quad \text{Thus} \quad KK_p \geq 80.2 \quad \text{Let} \quad K_p = 1 \quad \text{and} \quad K = 80.2$$

```
clear all
KP=1;
K=80.2;
figure(1)
num = [-KP];
den = [0 -361.2];
G=zpk(num,den,4500*K)
M=feedback(G,1)
step(M)
hold on;
for KD=0.0005:0.0005:0.002;
num = [-KP/KD];
den = [0 -361.2];
G=zpk(num,den,4500*K*KD)
M=feedback(G,1)
step(M)
end
xlabel('Time(secs)')
ylabel('y(t)')
title('Unit-step responses of the system')
```

Zero/pole/gain:

360900 (s+1)

s (s+361.2)

Zero/pole/gain:

360900 (s+1)

(s+0.999) (s+3.613e005)

Zero/pole/gain:

180.45 (s+2000)

s (s+361.2)

Zero/pole/gain:

180.45 (s+2000)

(s² + 541.6s + 3.609e005)

Zero/pole/gain:
360.9 (s+1000)

s (s+361.2)

Zero/pole/gain:
360.9 (s+1000)

(s² + 722.1s + 3.609e005)

Zero/pole/gain:
541.35 (s+666.7)

s (s+361.2)

Zero/pole/gain:
541.35 (s+666.7)

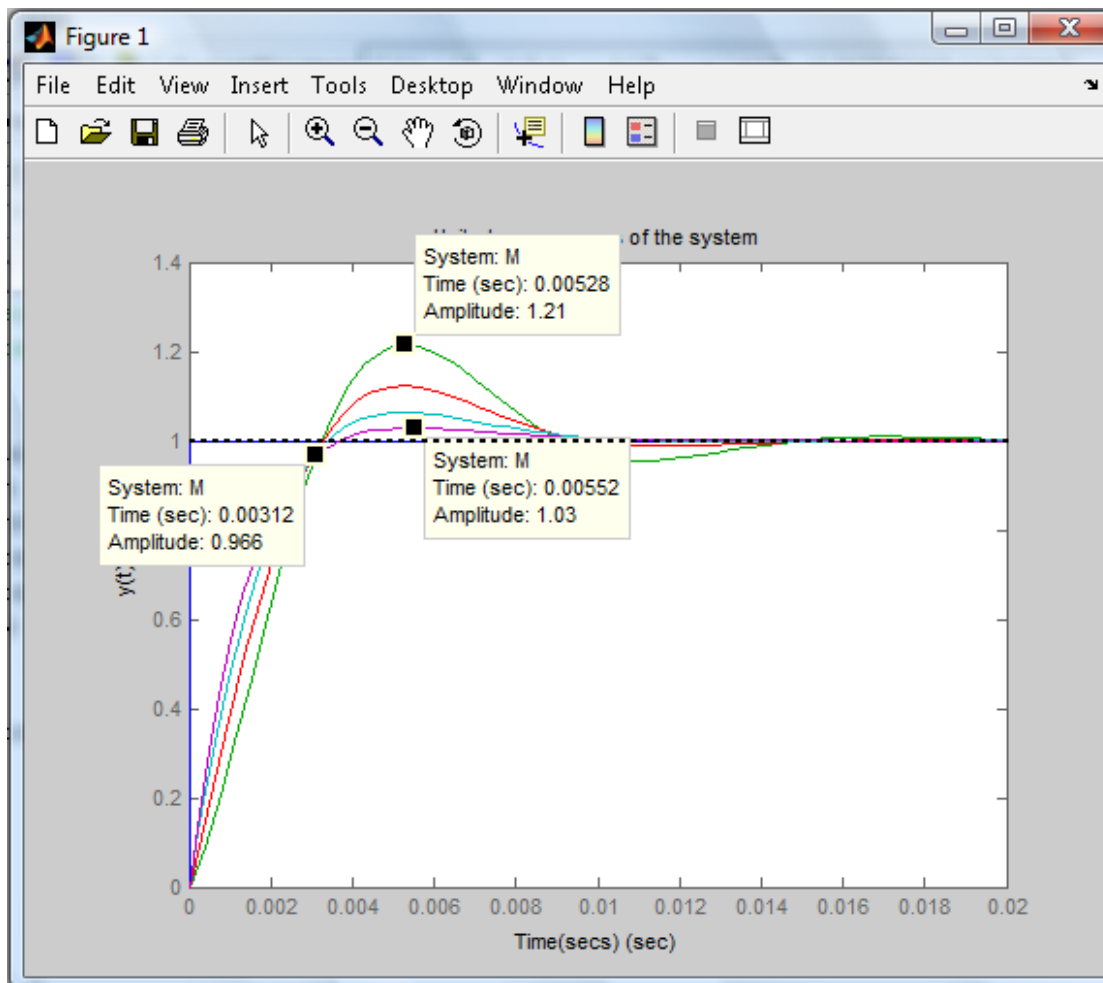
(s² + 902.5s + 3.609e005)

Zero/pole/gain:
721.8 (s+500)

s (s+361.2)

Zero/pole/gain:
721.8 (s+500)

(s² + 1083s + 3.609e005)



K_D	t_r (sec)	t_s (sec)	Max Overshoot (%)
0	0.00221	0.0166	37.1
0.0005	0.00242	0.00812	21.5
0.0010	0.00245	0.00775	12.2
0.0015	0.0024	0.0065	6.4
0.0016	0.00239	0.00597	5.6
0.0017	0.00238	0.00287	4.8
0.0018	0.00236	0.0029	4.0
0.0020	0.00233	0.00283	2.8

7-59) The forward-path Transfer Function: $N = 20$

$$G(s) = \frac{200(K_p + K_D s)}{s(s+1)(s+10)}$$

To stabilize the system, we can reduce the forward-path gain. Since the system is type 1, reducing the gain does not affect the steady-state liquid level to a step input. Let $K_p = 0.05$

$$G(s) = \frac{200(0.05 + K_D s)}{s(s+1)(s+10)}$$

ALSO try other K_p values and compare your results.

```
clear all
figure(1)
KD=0
num = [];
den = [0 -1 -10];
G=zpk(num,den,200*0.05)
M=feedback(G,1)
step(M)
hold on;
for KD=0.01:0.01:0.1;
KD
num = [-0.05/KD];
G=zpk(num,den,200*KD)
M=feedback(G,1)
step(M)
end
xlabel('Time(secs)')
ylabel('y(t)')
title('Unit-step responses of the system')
```

KD =
0

Zero/pole/gain:
10

s (s+1) (s+10)

Zero/pole/gain:
10

(s+10.11) (s^2 + 0.8914s + 0.9893)

KD =
0.0100

Zero/pole/gain:

$$\frac{2 (s+5)}{s (s+1) (s+10)}$$

Zero/pole/gain:

$$\frac{2 (s+5)}{(s+9.889) (s^2 + 1.111s + 1.011)}$$

KD =

$$0.0200$$

Zero/pole/gain:

$$\frac{4 (s+2.5)}{s (s+1) (s+10)}$$

Zero/pole/gain:

$$\frac{4 (s+2.5)}{(s+9.658) (s^2 + 1.342s + 1.035)}$$

KD =

$$0.0300$$

Zero/pole/gain:

$$\frac{6 (s+1.667)}{s (s+1) (s+10)}$$

Zero/pole/gain:

$$\frac{6 (s+1.667)}{(s+9.413) (s^2 + 1.587s + 1.062)}$$

KD =

$$0.0400$$

Zero/pole/gain:

$$\frac{8 (s+1.25)}{s (s+1) (s+10)}$$

Zero/pole/gain:

$$\frac{8 (s+1.25)}{(s+9.153) (s^2 + 1.847s + 1.093)}$$

$$K_D = 0.0500$$

$$\text{Zero/pole/gain:} \\ 10 (s+1)$$

$$\frac{\text{-----}}{s (s+1) (s+10)}$$

$$\text{Zero/pole/gain:} \\ 10 (s+1) \\ \text{-----} \\ (s+8.873) (s+1.127) (s+1)$$

$$K_D = 0.0600$$

$$\text{Zero/pole/gain:} \\ 12 (s+0.8333)$$

$$\frac{\text{-----}}{s (s+1) (s+10)}$$

$$\text{Zero/pole/gain:} \\ 12 (s+0.8333) \\ \text{-----} \\ (s+8.569) (s+1.773) (s+0.6582)$$

$$K_D = 0.0700$$

$$\text{Zero/pole/gain:} \\ 14 (s+0.7143)$$

$$\frac{\text{-----}}{s (s+1) (s+10)}$$

$$\text{Zero/pole/gain:} \\ 14 (s+0.7143) \\ \text{-----} \\ (s+8.232) (s+2.221) (s+0.547)$$

$$K_D = 0.0800$$

$$\text{Zero/pole/gain:} \\ 16 (s+0.625)$$

$$\frac{\text{-----}}{s (s+1) (s+10)}$$

$$\text{Zero/pole/gain:} \\ 16 (s+0.625) \\ \text{-----} \\ (s+7.85) (s+2.673) (s+0.4765)$$

$$K_D = 0.0900$$

Zero/pole/gain:
18 (s+0.5556)

$$\frac{\text{-----}}{s (s+1) (s+10)}$$

Zero/pole/gain:
18 (s+0.5556)

$$\frac{\text{-----}}{(s+7.398) (s+3.177) (s+0.4255)}$$

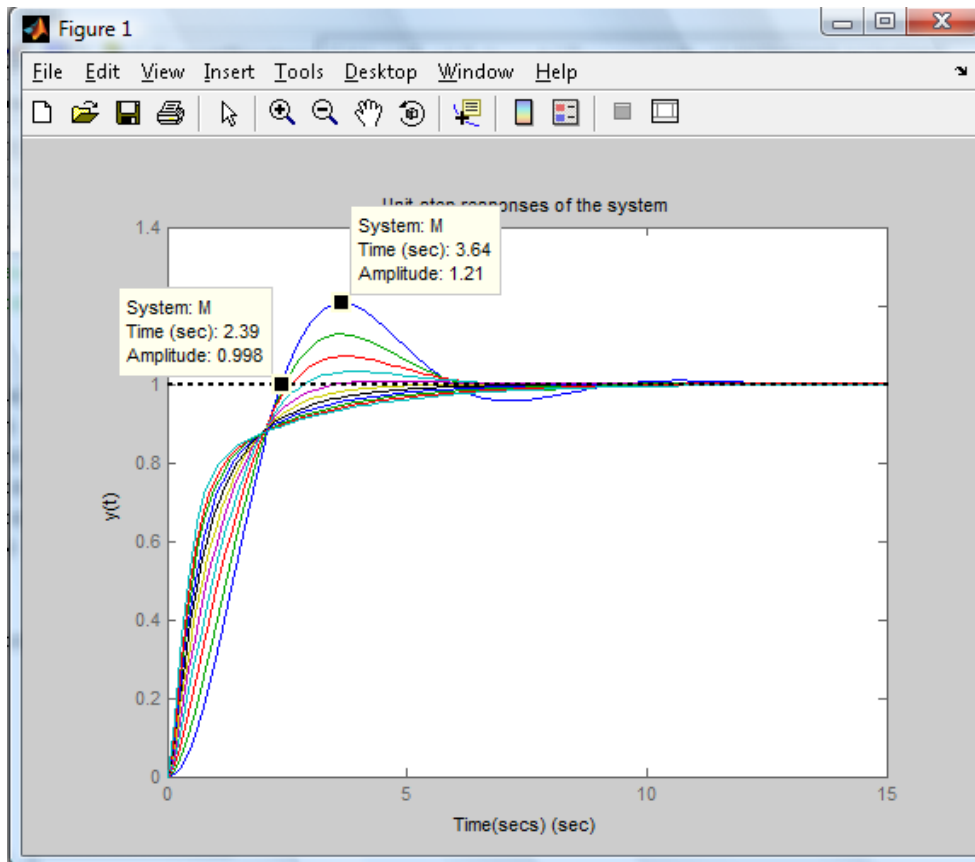
$$K_D = 0.1000$$

Zero/pole/gain:
20 (s+0.5)

$$\frac{\text{-----}}{s (s+1) (s+10)}$$

Zero/pole/gain:
20 (s+0.5)

$$\frac{\text{-----}}{(s+0.3861) (s+3.803) (s+6.811)}$$



Unit-step Response Attributes:

K_D	t_s (sec)	Max Overshoot (%)
0.01	5.159	12.7
0.02	4.57	7.1
0.03	2.35	3.2
0.04	2.526	0.8
0.05	2.721	0
0.06	3.039	0
0.10	4.317	0

When $K_D = 0.05$ the rise time is 2.721 sec, and the step response has no overshoot.

7-60) (a) For $e_{ss} = 1$,

$$K_v = \lim_{s \rightarrow 0} sG(s) = \lim_{s \rightarrow 0} s \frac{200(K_p + K_D s)}{s(s+1)(s+10)} = 20K_p = 1 \quad \text{Thus } K_p = 0.05$$

Forward-path Transfer Function:

$$G(s) = \frac{200(0.05 + K_D s)}{s(s+1)(s+10)}$$

Because of the choice of K_p this is the same as previous part.

7-61)

(a) Forward-path Transfer Function:

$$G(s) = \frac{100\left(K_p + \frac{K_I}{s}\right)}{s^2 + 10s + 100} \quad \text{For } K_v = 10, \quad K_v = \lim_{s \rightarrow 0} sG(s) = \lim_{s \rightarrow 0} s \frac{100(K_p s + K_I)}{s(s^2 + 10s + 100)} = K_I = 10$$

Thus $K_I = 10$.

$$G_{cl}(s) = \frac{100(K_p s + K_I)}{s^3 + 10s^2 + 100s + 100(K_p s + K_I)} = \frac{100(K_p s + 10)}{s^3 + 10s^2 + 100(1 + K_p)s + 1000}$$

(b) Let the complex roots of the characteristic equation be written as $s = -\sigma + j15$ and $s = -\sigma - j15$.

The quadratic portion of the characteristic equation is $s^2 + 2\sigma s + (\sigma^2 + 225) = 0$

The characteristic equation of the system is $s^3 + 10s^2 + (100 + 100K_p)s + 1000 = 0$

The quadratic equation must satisfy the characteristic equation. Using long division and solve for zero remainder condition.

$$\begin{aligned}
 & \frac{s + (10 - 2\sigma)}{s^2 + 2\sigma s + \sigma^2 + 225} \left| \frac{s^3 + 10s^2 + (100 + 100K_p)s + 1000}{s^3 + 2\sigma s^2 + (\sigma^2 + 225)s} \right. \\
 & \left. \frac{(10 - 2\sigma)s^2 + (100K_p - \sigma^2 - 125)s + 1000}{(10 - 2\sigma)s^2 + (20\sigma - 4\sigma^2)s + (10 - 2\sigma)(s^2 + 225)} \right. \\
 & \left. \frac{(100K_p + 3\sigma^2 - 20\sigma - 125)s + 2\sigma^3 - 10\sigma^2 + 450\sigma - 1250}{(100K_p + 3\sigma^2 - 20\sigma - 125)s + 2\sigma^3 - 10\sigma^2 + 450\sigma - 1250} \right.
 \end{aligned}$$

For zero remainder, $2\sigma^3 - 10\sigma^2 + 450\sigma - 1250 = 0$ (1)

and $100K_p + 3\sigma^2 - 20\sigma - 125 = 0$ (2)

The real solution of Eq. (1) is $\sigma = 2.8555$. From Eq. (2),

$$K_p = \frac{125 + 20\sigma - 3\sigma^2}{100} = 1.5765$$

The characteristic equation roots are: $s = -2.8555 + j15$, $-2.8555 - j15$, and $s = -10 + 2\sigma = -4.289$

(c) Root Contours:

Dividing both sides of $s^3 + 10s^2 + (100 + 100K_p)s + 1000 = 0$ by the terms that do not contain K_p we have:

$$1 + \frac{100K_p s}{s^3 + 10s^2 + 100s + 1000} = 1 + G_{eq}$$

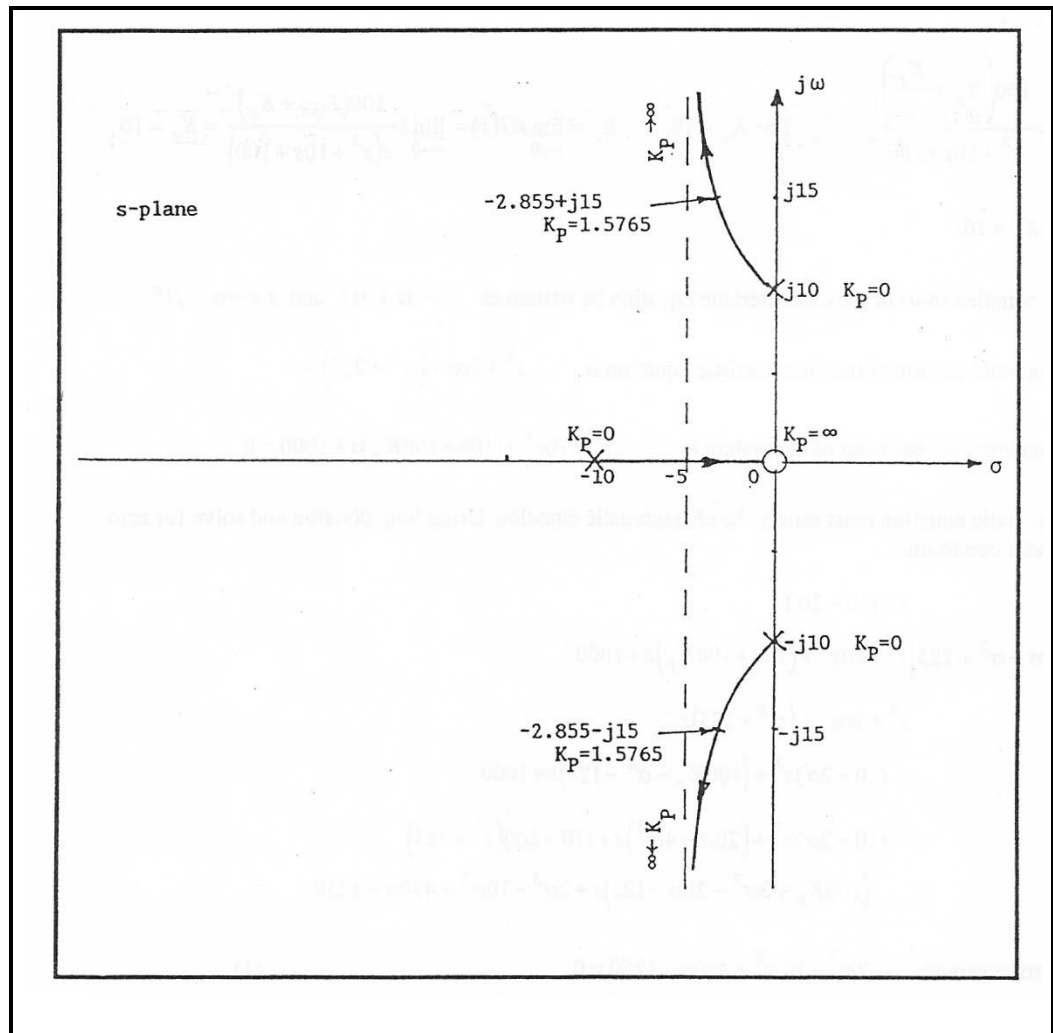
$$G_{eq}(s) = \frac{100K_p s}{s^3 + 10s^2 + 100s + 1000} = \frac{100K_p s}{(s + 10)(s^2 + 100)}$$

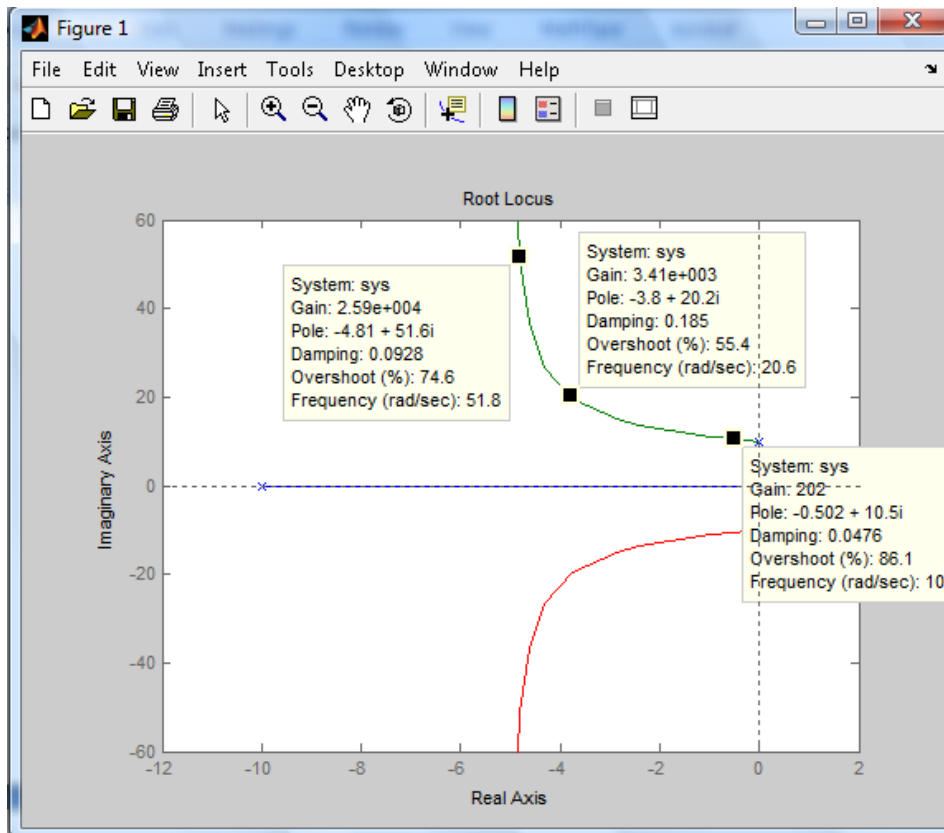
Root Contours: See Chapter 10 for more information

```

clear all
Kp = .001;
num = [100*Kp 0];
den = [1 10 100 1000];
rlocus(num,den)

```





7-62) (a) Forward-path Transfer Function:

$$G(s) = \frac{100 \left(K_p + \frac{K_I}{s} \right)}{s^2 + 10s + 100} \quad \text{For } K_v = 10, \quad K_v = \lim_{s \rightarrow 0} sG(s) = \lim_{s \rightarrow 0} s \frac{100(K_p s + K_I)}{s(s^2 + 10s + 100)} = K_I = 10$$

Thus the forward-path transfer function becomes

$$G(s) = \frac{100(10 + K_p s)}{s(s^2 + 10s + 100)}$$

$$G_{cl}(s) = \frac{100(K_p s + K_I)}{s^3 + 10s^2 + 100s + 100(K_p s + K_I)} = \frac{100(K_p s + 10)}{s^3 + 10s^2 + 100(1 + K_p)s + 1000}$$

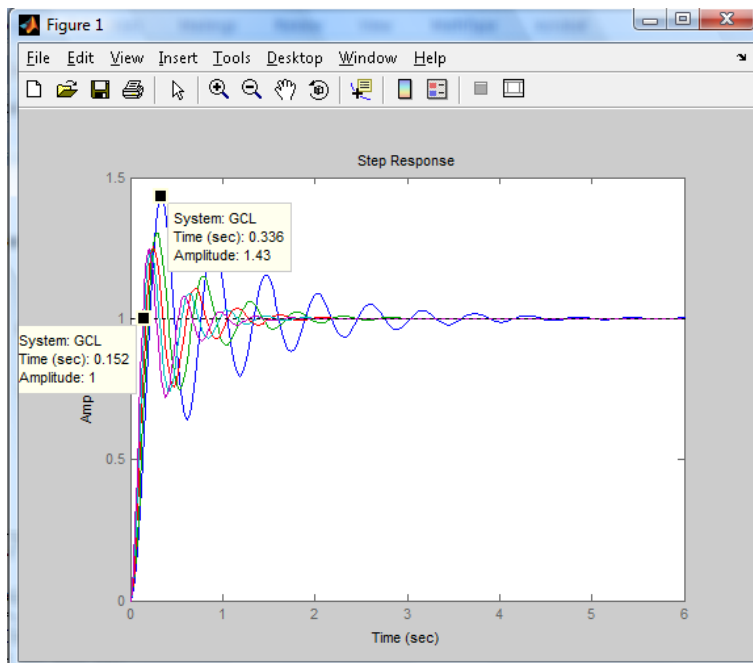
```
clear all
for Kp=.4:0.4:2;
num = [100*Kp 1000];
den = [1 10 100 0];
[numCL,denCL]=cloop(num,den);
GCL=tf(numCL,denCL);
step(GCL)
hold on;
end
```

Use the cursor to find the maximum overshoot and rise time. For example when $K_p = 2$, $PO=43$ and $tr_{100\%}=0.152$ sec.

Transfer function:

$$200s + 1000$$

$$s^3 + 10s^2 + 300s + 1000$$



7-63)**(a) Forward-path Transfer Function:**

$$G(s) = \frac{100(K_p s + K_I)}{s(s^2 + 10s + 100)}$$

For $K_v = 100$,

$$K_v = \lim_{s \rightarrow 0} sG(s) = \lim_{s \rightarrow 0} s \frac{100(K_p s + K_I)}{s(s^2 + 10s + 100)} = K_I = 100 \quad \text{Thus } K_I = 100.$$

(b) The characteristic equation is $s^3 + 10s^2 + (100 + 100K_p)s + 100K_I = 0$ **Routh Tabulation:**

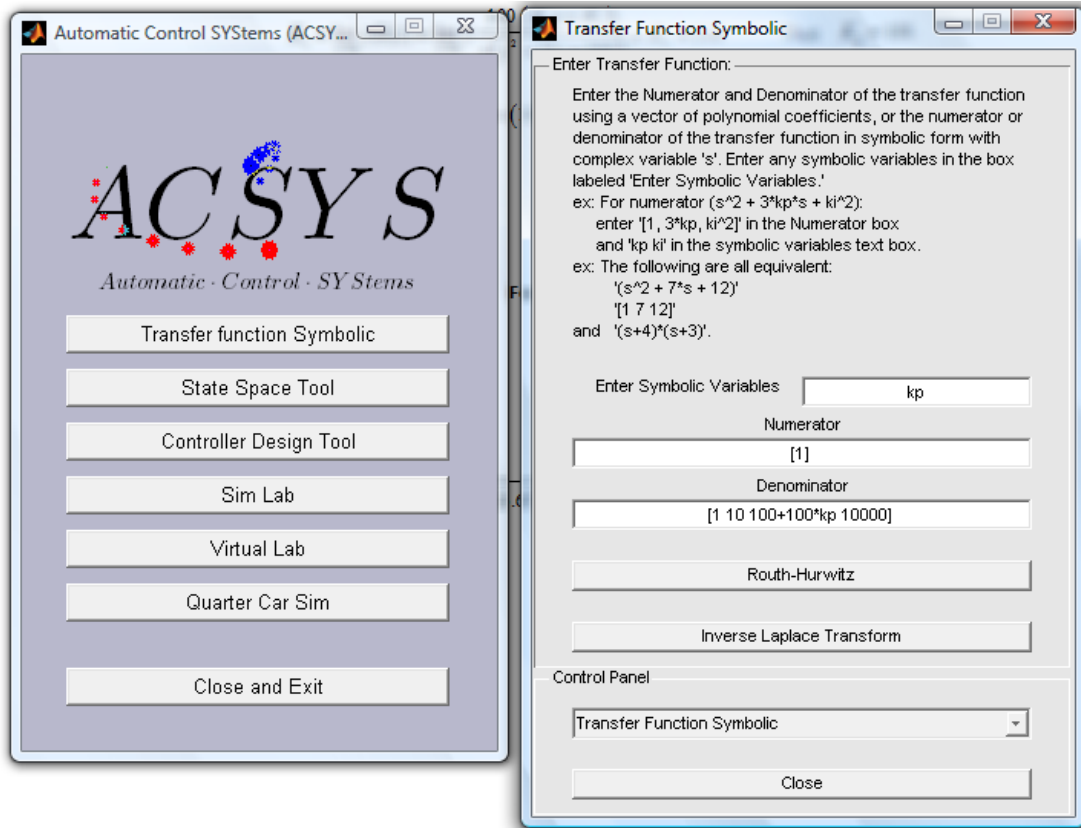
s^3	1	$100 + 100K_p$
s^2	10	10,000
s^1	$100K_p - 900$	0
s^0	10,000	

For stability, $100K_p - 900 > 0$ Thus $K_p > 9$

7. Activate MATLAB
8. Go to the directory containing the ACSYS software.
9. Type in

Acsys

10. Then press the “transfer function Symbolic” and enter the Characteristic equation
11. Then press the “Routh Hurwitz” button



RH =

[1, 100+100*kp]

[10, 10000]

[-900+100*kp, 0]

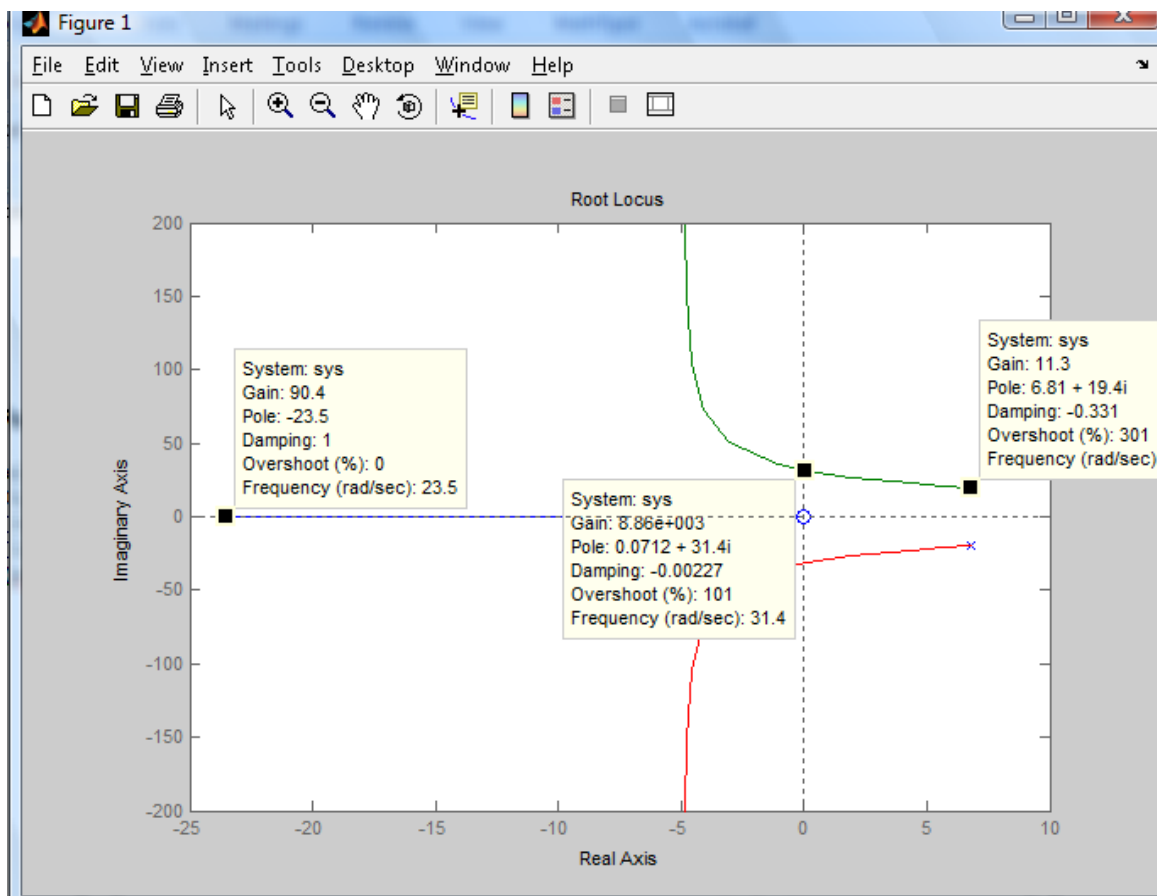
[(-9000000+1000000*kp)/(-900+100*kp), 0]

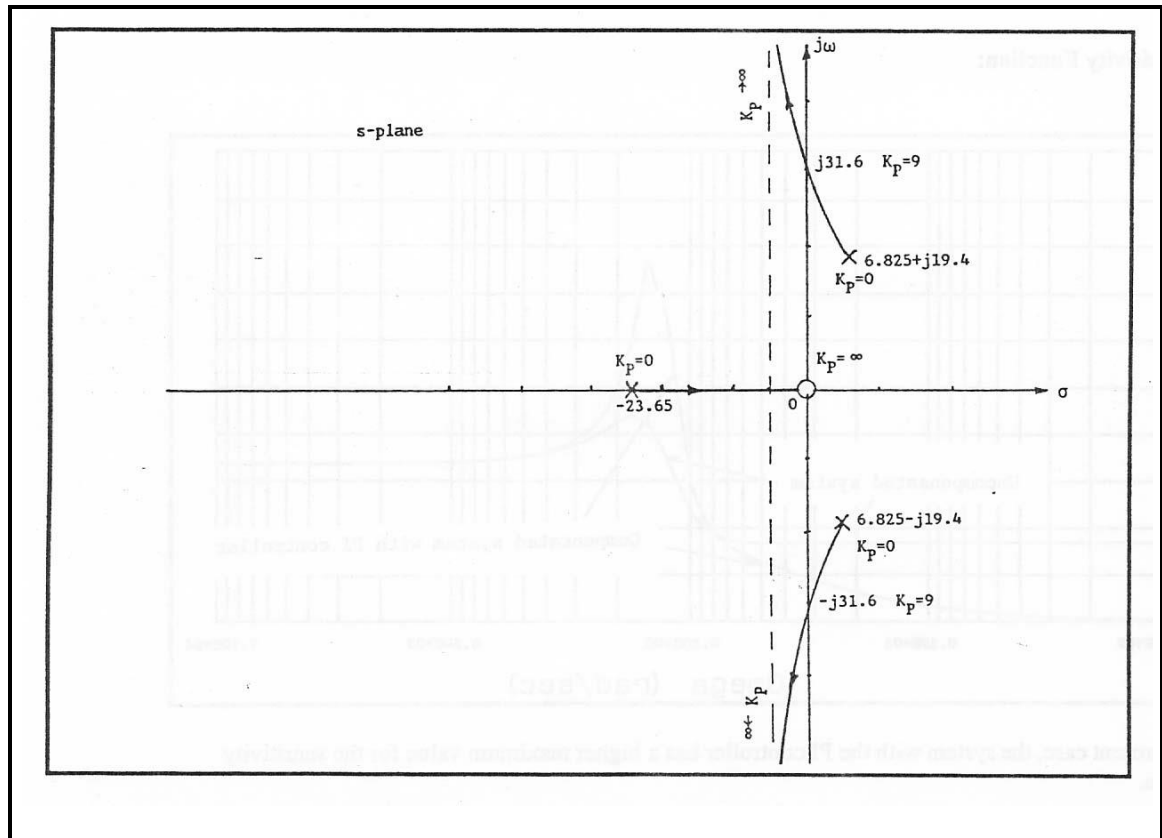
Root Contours:

$$G_{eq}(s) = \frac{100K_p s}{s^3 + 10s^2 + 100s + 10,000} = \frac{100K_p s}{(s + 23.65)(s - 6.825 + j19.4)(s - 6.825 - j19.4)}$$

Root Contours: See Chapter 10 for more information

```
clear all
Kp = .001;
num = [100*Kp 0];
den = [1 10 100 10000];
rlocus(num,den)
```





(c) $K_I = 100$

$$G(s) = \frac{100(K_p s + 100)}{s(s^2 + 10s + 100)}$$

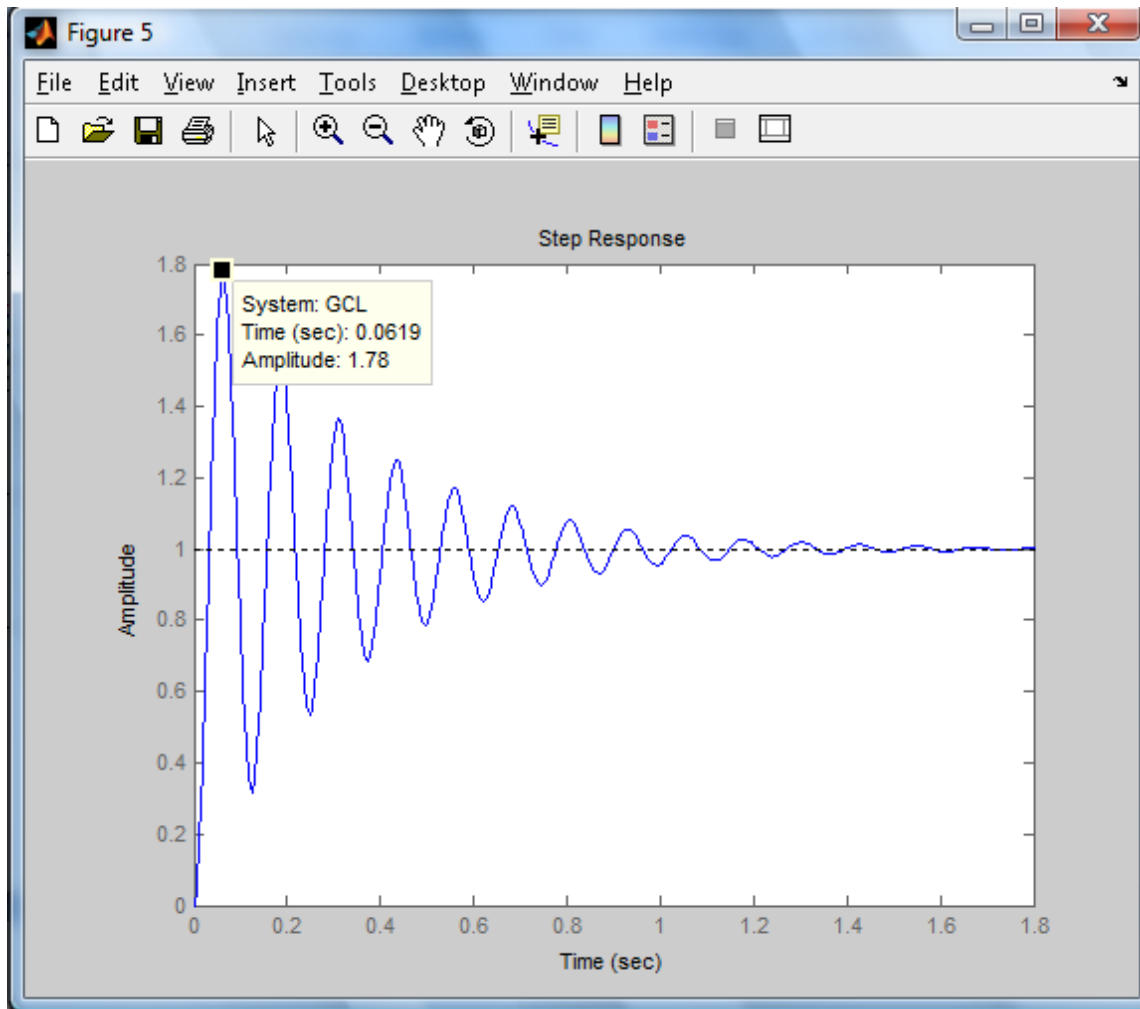
The following maximum overshoots of the system are computed for various values of K_p .

```
clear all
Kp=[15 20 22 24 25 26 30 40 100 1000];
[N,M]=size(Kp);
for i=1:M
    num = [100*Kp(i) 10000];
    den = [1 10 100 0];
    [numCL,denCL]=cloop(num,den);
    GCL=tf(numCL,denCL);
    figure(i)
    step(GCL)
end
```

K_p	15	20	22	24	25	26	30	40	100	1000

y_{\max}	1.794	1.779	1.7788	1.7785	1.7756	1.779	1.782	1.795	1.844	1.859
------------	-------	-------	--------	--------	--------	-------	-------	-------	-------	-------

When $K_p = 25$, minimum $y_{\max} = 1.7756$



Use: **close all** to close all the figure windows.

7-64) MATLAB solution is the same as 7-63.

(a) Forward-path Transfer Function:

$$G(s) = \frac{100(K_p s + K_I)}{s(s^2 + 10s + 100)} \quad \text{For } K_v = \frac{100K_I}{100} = 10, \quad K_I = 10$$

(b) Characteristic Equation: $s^3 + 10s^2 + 100(K_p + 1)s + 1000 = 0$

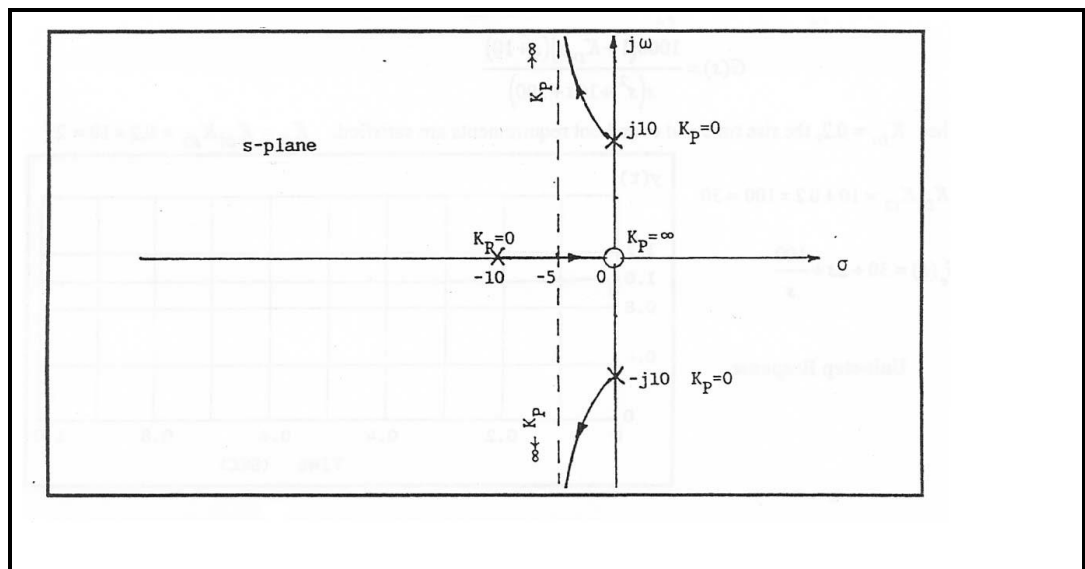
Routh Tabulation:

s^3	1	$100 + 100K_p$
s^2	10	1000
s^1	$100K_p$	0
s^0	1000	

For stability, $K_p > 0$

Root Contours:

$$G_{eq}(s) = \frac{100K_p s}{s^3 + 10s^2 + 100s + 1000}$$



- (c) The maximum overshoots of the system for different values of K_p ranging from 0.5 to 20 are computed and tabulated below.

K_p	0.5	1.0	1.6	1.7	1.8	1.9	2.0	3.0	5.0	10	20
y_{\max}	1.393	1.275	1.2317	1.2416	1.2424	1.2441	1.246	1.28	1.372	1.514	1.642

When $K_p = 1.7$, maximum $y_{\max} = 1.2416$

7-65)

$$G_c(s) = K_p + K_D s + \frac{K_I}{s} = \frac{K_D s^2 + K_p s + K_I}{s} = (1 + K_{D1} s) \left(K_{P2} + \frac{K_{I2}}{s} \right)$$

where

$$K_P = K_{P2} + K_{D1} K_{I2} \quad K_D = K_{D1} K_{P2} \quad K_I = K_{I2}$$

Forward-path Transfer Function:

$$G(s) = G_c(s)G_p(s) = \frac{100(K_D s^2 + K_P s + K_I)}{s(s^2 + 10s + 100)}$$

And rename the ratios: $K_D / K_P = A$, $K_I / K_P = B$

Thus

$$K_v = \lim_{s \rightarrow 0} sG(s) = 100 \frac{K_I}{100} = 100$$

$$K_I = 100$$

For K_D being sufficiently small:

Forward-path Transfer Function:

$$G(s) = \frac{100(K_p s + 100)}{s(s^2 + 10s + 100)}$$

Characteristic Equation:

$$s^3 + 10s^2 + (100 + 100K_p)s + 10,000 = 0$$

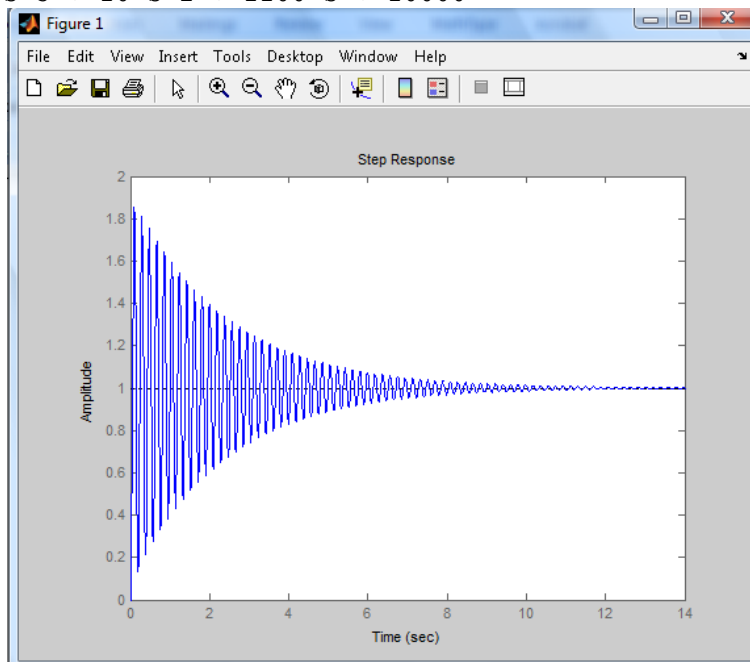
For stability, $K_p > 9$. Select $K_p = 10$ and observe the response.

```
clear all
Kp=10;
num = [100*Kp 10000];
den =[1 10 100 0];
[numCL,denCL]=cloop(num,den);
GCL=tf(numCL,denCL)
step(GCL)
```

Transfer function:

$$1000 s + 10000$$

$$s^3 + 10 s^2 + 1100 s + 10000$$



Obviously by increasing K_p more oscillations will occur. Add K_D to reduce oscillations.

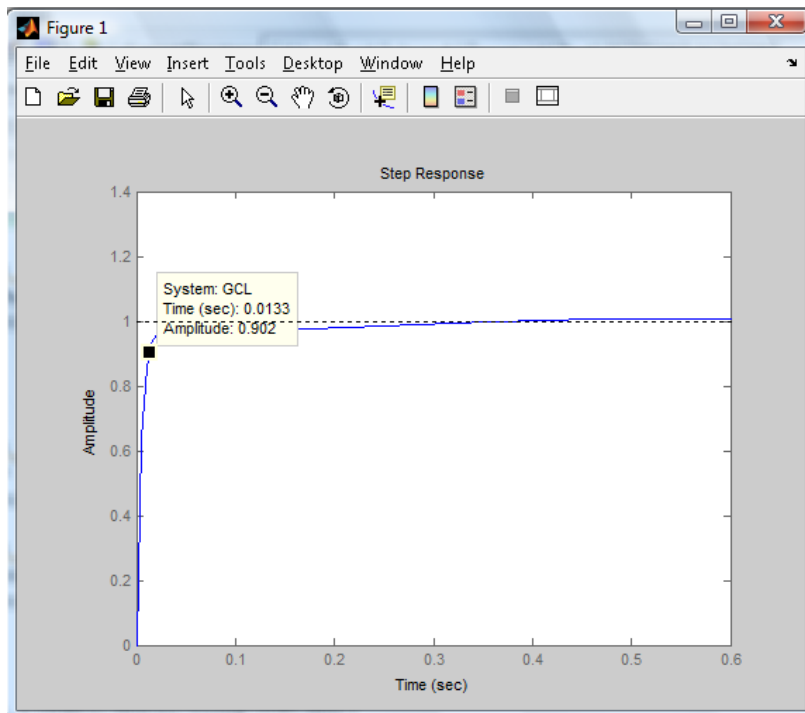
```
clear all
Kp=10;
Kd=2;
num = [100*Kd 100*Kp 10000];
den =[1 10 100 0];
[numCL,denCL]=cloop(num,den);
GCL=tf(numCL,denCL)
step(GCL)
```

Transfer function:

$$200 s^2 + 1000 s + 10000$$

$$s^3 + 210 s^2 + 1100 s + 10000$$

Unit-step Response



The rise time seems reasonable. But we need to increase K_p to improve approach to steady state.

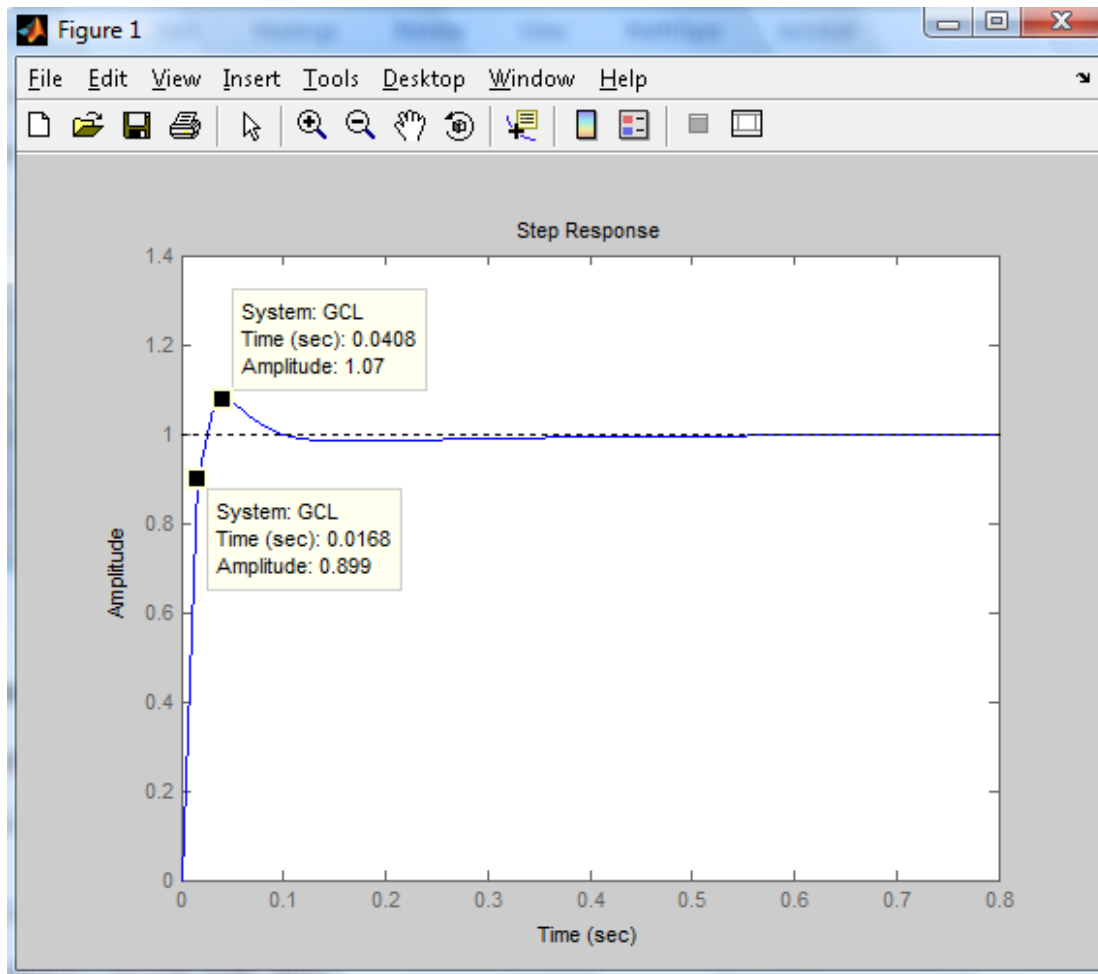
Increase K_p to $K_p=30$.

```
clear all
Kp=30;
Kd=1;
num = [100*Kd 100*Kp 10000];
den = [1 10 100 0];
[numCL,denCL]=cloop(num,den);
GCL=tf(numCL,denCL)
step(GCL)
```

Transfer function:

$$100 s^2 + 3000 s + 10000$$

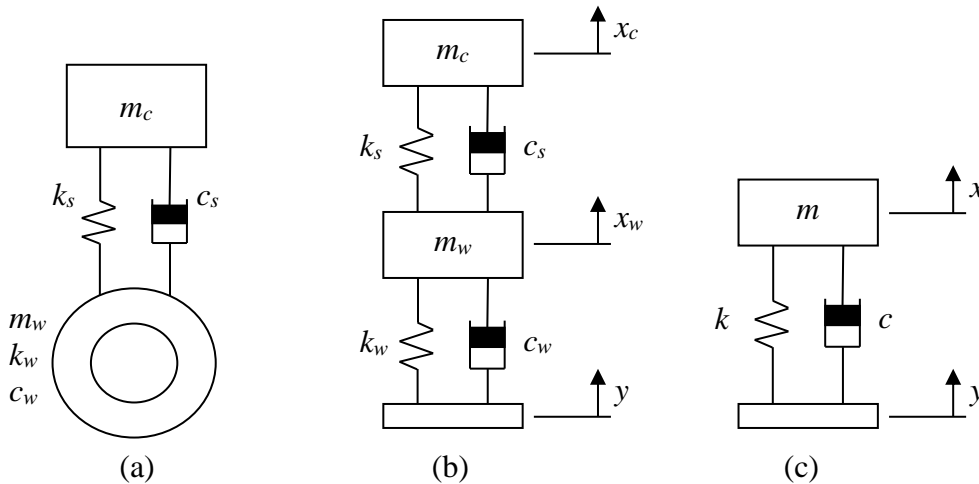
$$s^3 + 110 s^2 + 3100 s + 10000$$



To obtain a better response continue adjusting KD and KP.

7-66) For the sake simplicity, this problem we assume the control force $f(t)$ is applied in parallel to the spring K and damper B . We will not concern the details of what actuator or sensors are used.

Lets look at



Quarter car model realization: (a) quarter car, (b) 2 degree of freedom, and (c) 1 degree of freedom model.

The equation of motion of the system is defined as follows:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = c\dot{y}(t) + ky(t)$$

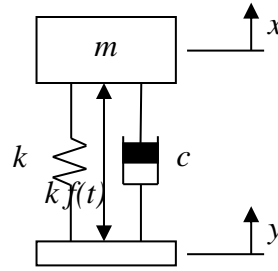
which can be simplified by substituting the relation $z(t) = x(t) - y(t)$ and non-dimensionalizing the coefficients to the form

$$\ddot{z}(t) + 2\zeta\omega_n\dot{z}(t) + \omega_n^2 z(t) = -\ddot{y}(t)$$

The Laplace transform of Eq. (4-323) yields the input output relationship

$$\frac{Z(s)}{\ddot{Y}(s)} = \frac{-1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Now let's apply control.



For simplicity and better presentation, we have scaled the control force as $kf(t)$ we rewrite the above as:

$$\begin{aligned}
 m\ddot{x}(t) + c\dot{x}(t) + kx(t) &= c\dot{y}(t) + ky(t) + kf(t) \\
 \ddot{z}(t) + 2\zeta\omega_n\dot{z}(t) + \omega_n^2 z(t) &= -\ddot{y}(t) + \omega_n^2 f(t) \\
 s^2 + 2\zeta\omega_n s + \omega_n^2 &= -A(s) + \omega_n^2 F(s) \\
 A(s) &= \ddot{Y}(s)
 \end{aligned}$$

Setting the controller structure such that the vehicle bounce $Z(s) = X(s) - Y(s)$ is minimized:

$$F(s) = 0 - \left(K_p + K_D s + \frac{K_I}{s} \right) Z(s)$$

$$\frac{Z(s)}{A(s)} = \frac{-1}{s^2 + 2\zeta\omega_n s + \omega_n^2 \left(1 + K_p + K_D s + \frac{K_I}{s} \right)}$$

$$\frac{Z(s)}{A(s)} = \frac{-s}{s^3 + 2\zeta\omega_n s^2 + \omega_n^2 \left((1 + K_p)s + K_D s^2 + K_I \right)}$$

For proportional control $K_D = K_I = 0$.

Pick $\zeta = 0.707$ and $\omega_n = 1$ for simplicity. This is now an underdamped system.

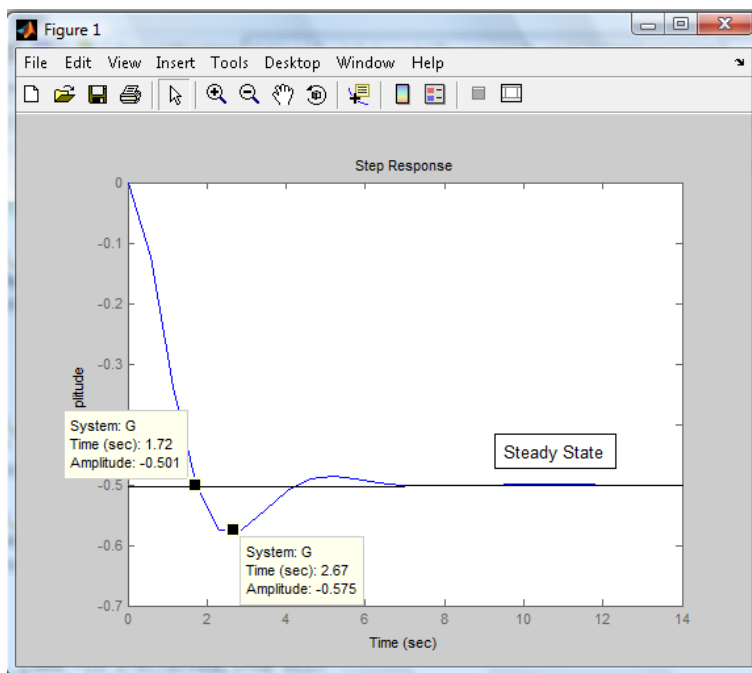
Use MATLAB to obtain response now.

```
clear all
Kp=1;
Kd=0;
Ki=0;
num = [-1 0];
den =[1 2*0.707+Kd 1+Kp Ki];
G=tf(num,den)
step(G)
```

Transfer function:

$-s$

$s^3 + 1.414 s^2 + 2 s$



Adjust parameters to get the desired response if necessary.

The process is the same for parts b, c and d.

7-67) Replace F(s) with

$$F(s) = X_{ref} - \left(K_P + K_D s + \frac{K_I}{s} \right) X(s)$$

$$2\zeta\omega_n = \frac{B}{M}$$

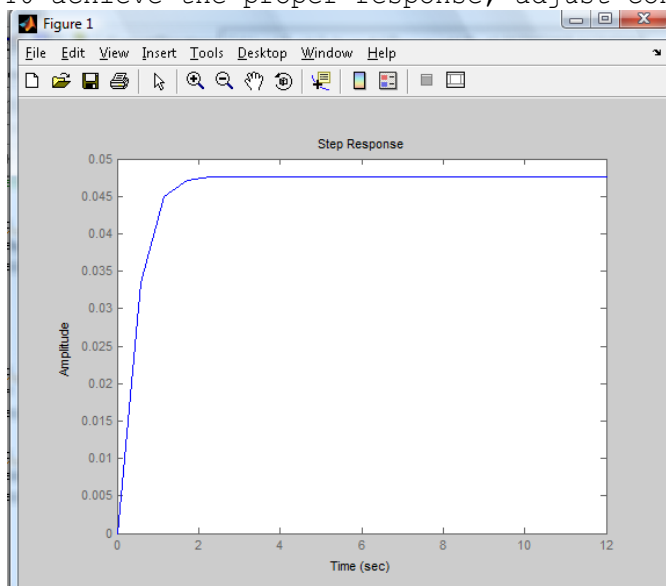
$$\omega_n^2 = \frac{K}{M}$$

$$\frac{X(s)}{X_{ref}(s)} = \frac{1}{s^2 + 2\zeta\omega_n s + \left(\omega_n^2 + K_P + K_D s + \frac{K_I}{s} \right)}$$

Use MATLAB to obtain response now.

```
clear all
Kp=1;
Kd=0;
Ki=0;
B=10;
K=20;
M=1;
omega=sqrt(K/M);
zeta=(B/M)/2/omega;
num = [1 0];
den =[1 2*zeta*omega+Kd omega^2+Kp Ki];
G=tf(num,den)
step(G)
ransfer function:
      s
-----
s^3 + 10 s^2 + 21 s
```

To achieve the proper response, adjust controller gains accordingly.



7-68)

a) Rotational kinetic energy: $T_{rot} = \frac{1}{2} J \dot{\theta}^2$

Translational kinetic energy: $T_T = \frac{1}{2} m \dot{y}^2$

Relation between translational displacement and rotational displacement:

$$y = r\theta$$

$$\dot{y} = r\dot{\theta}$$

$$T_{Rot} = \frac{1}{2} \frac{J}{r^2} \dot{y}^2$$

Potential energy: $U = \frac{1}{2} K y^2$

As we know $T_{Rot} + T_T + U = \text{constant}$, then:

$$\frac{1}{2} \frac{J}{r^2} \dot{y}^2 + \frac{1}{2} m \dot{y}^2 + \frac{1}{2} K y^2 = \text{constant}$$

By differentiating, we have:

$$\begin{aligned} \frac{J}{r^2} \dot{y} \ddot{y} + m \dot{y} \ddot{y} + K y \dot{y} &= 0 \\ \dot{y} \left(\frac{J}{r^2} \ddot{y} + m \ddot{y} + K y \right) &= 0 \end{aligned}$$

Since \dot{y} cannot be zero, then $J \frac{\ddot{y}}{r^2} + m \ddot{y} + K y = 0$

b)

$$\ddot{y} = r \ddot{\theta}$$

$$J \ddot{\theta}^2 + m \ddot{y} + K y = 0$$

$$\frac{Y(s)}{\theta(s)} = - \frac{J}{m s^2 + K}$$

c)

$$T_{max} = \frac{1}{2} m \dot{y}_{max}^2 + \frac{1}{2} \frac{J}{r^2} \dot{y}_{max}^2 = \frac{1}{2} \left(m + \frac{J}{r^2} \right) \dot{y}_{max}^2$$

$$\dot{y}_{max}^2 = \omega_n^2$$

where $\dot{y} = A$ at the maximum energy.

$$U_{max} = \frac{1}{2} K y_{max}^2 = \frac{1}{2} K A^2$$

Then:

$$\frac{1}{2} \left(m + \frac{J}{r^2} \right) \omega_n^2 A^2 = \frac{1}{2} K A^2$$

Or:

$$\omega_n = \sqrt{\frac{K}{m + \frac{J}{r^2}}} = r \sqrt{\frac{K}{r^m + J}}$$

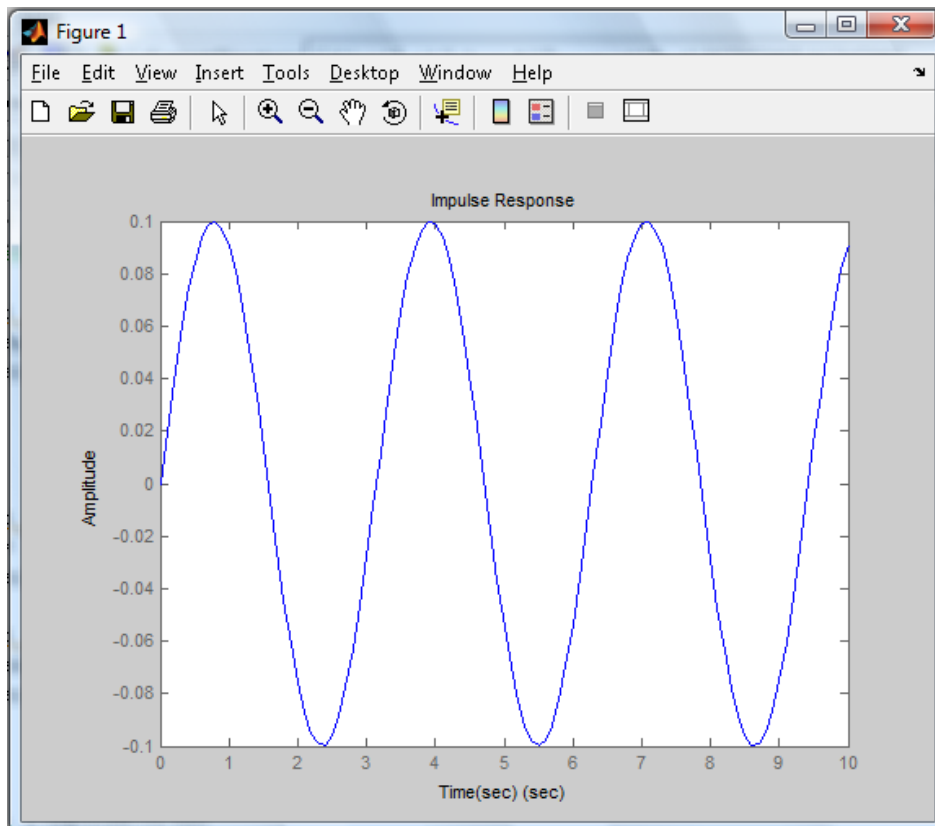
d) $G(s) = \frac{J}{(ms^2 + K)}$

```
% select values of m, J and K
K=100;
J=5;
m=25;
G=tf([J],[m 0 K])
Pole(G)
impz(G,10)
xlabel('Time(sec)');
ylabel('Amplitude');

Transfer function:
      5
-----
25 s^2 + 100

ans =
      0 + 2.0000i
      0 - 2.0000i
```

Uncontrolled

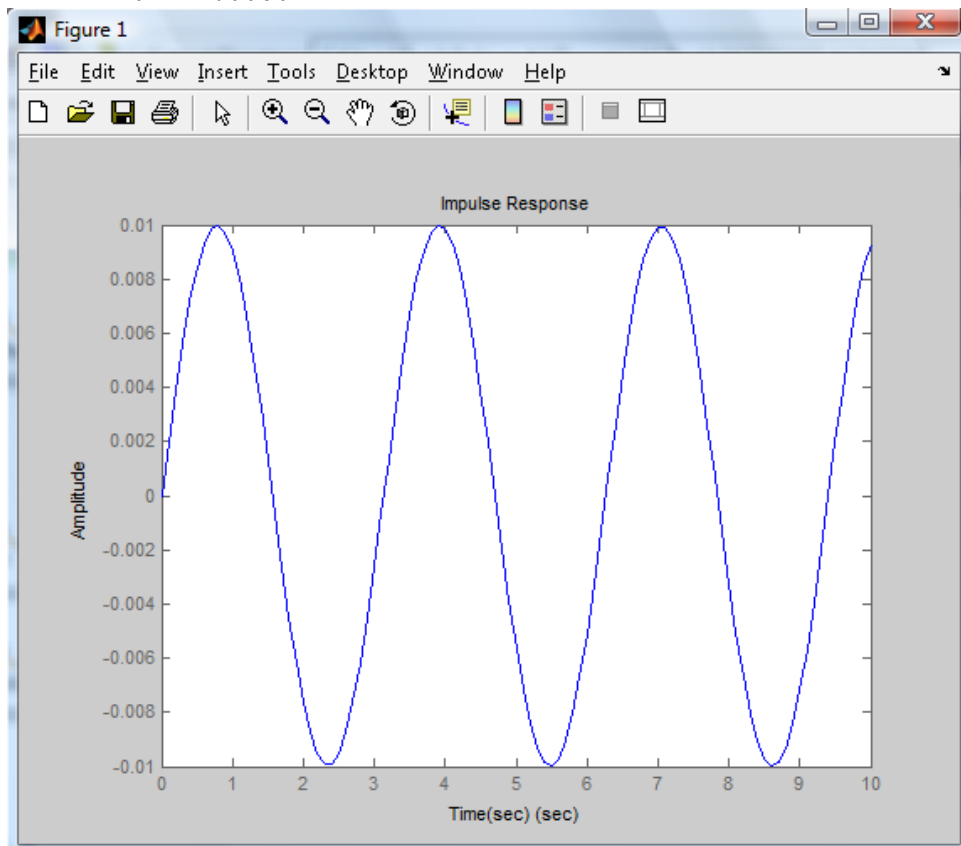


With a proportional controller one can adjust the oscillation amplitude the transfer function is rewritten as:

$$G_{cl}(s) = \frac{JK_p}{ms^2 + K + JK_p}$$

```
% select values of m, J and K
Kp=0.1
K=100;
J=5;
m=25;
G=tf([J*Kp],[m 0 (K+J*Kp)])
Pole(G)
impz(G,10)
xlabel('Time(sec)');
ylabel('Amplitude');
```

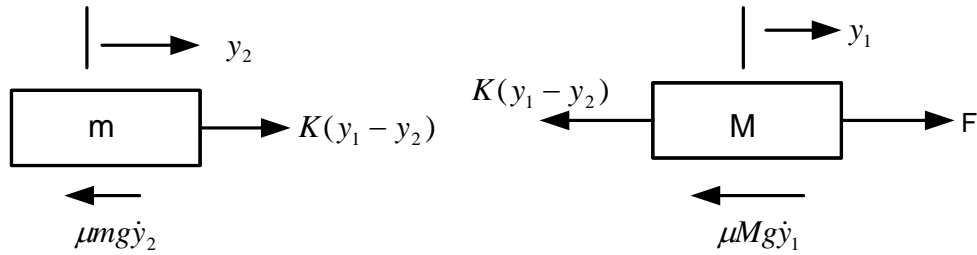
```
Kp =  
    0.1000  
  
Transfer function:  
    0.5  
-----  
25 s^2 + 100.5  
  
ans =  
    0 + 2.0050i  
    0 - 2.0050i
```



A PD controller must be used to damp the oscillation and reduce overshoot. Use Example 7-7-1 as a guide.

7-69) Recall:

a)



b) From Newton's Law:

$$M\ddot{y}_1 = F - K(y_1 - y_2) - \mu M g \dot{y}_1$$

$$m\ddot{y}_2 = K(y_1 - y_2) - \mu m g \dot{y}_2$$

If y_1 and y_2 are considered as a position and v_1 and v_2 as velocity variables

$$\text{Then: } \begin{cases} \dot{y}_1 = v_1 \\ \dot{y}_2 = v_2 \\ M\dot{v}_1 = F - K(y_1 - y_2) - \mu M g v_1 \\ m\dot{v}_2 = K(y_1 - y_2) - \mu m g v_2 \end{cases}$$

The output equation can be the velocity of the engine, which means $z = v_2$

c)

$$\begin{cases} Ms^2 Y_1(s) = F - K(Y_1(s) - Y_2(s)) - \mu M g s Y_1(s) \\ ms^2 Y_2(s) = K(Y_1(s) - Y_2(s)) - \mu m g s Y_2(s) \\ Z(s) = V_2(s) = s Y_2(s) \end{cases}$$

Obtaining $\frac{Z(s)}{F(s)}$ requires solving above equation with respect to $Y_2(s)$

From the first equation:

$$(Ms^2 + K + \mu M g s) Y_1(s) = F + K Y_2(s)$$

$$Y_1(s) = \frac{F + K Y_2(s)}{Ms^2 + \mu M g s + K}$$

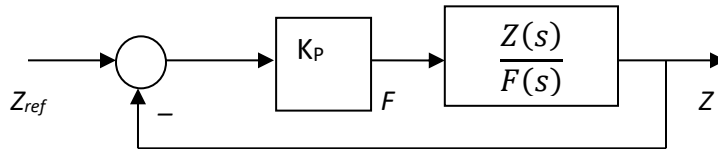
Substituting into the second equation:

$$ms^2 Y_2(s) = \frac{KF + K^2 Y_2(s)}{Ms^2 + \mu M g s + K} - K Y_2(s) - \mu m g s Y_2(s)$$

By solving above equation:

$$\frac{Z(s)}{F(s)} = \frac{sY_2(s)}{F(s)} = \frac{ms^2 + m\mu g s + 1}{Mms^3 + (2Mm\mu g)s^2 + (Mk + Mm(\mu g)^2 + mK)s + K\mu g(M + m)}$$

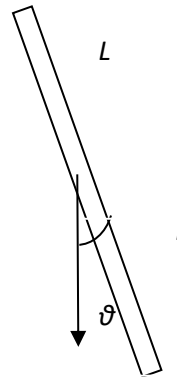
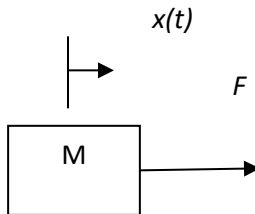
Replace Force F with a proportional controller so that $F=K(Z-Z_{ref})$:



$$\frac{Z(s)}{K(Z_{ref}(s) - Z(s))} = \frac{ms^2 + m\mu g s + 1}{Mms^3 + (2Mm\mu g)s^2 + (Mk + Mm(\mu g)^2 + mK)s + K\mu g(M + m)}$$

$$\begin{aligned} \frac{Z(s)}{Z_{ref}(s)} &= \frac{K_P(ms^2 + m\mu g s + 1)}{Mms^3 + (2Mm\mu g)s^2 + (Mk + Mm(\mu g)^2 + mK)s + K\mu g(M + m) + K_P(ms^2 + m\mu g s + 1)} \end{aligned}$$

7-70)



Here is an alternative representation including friction (damping) μ . In this case the angle θ is measured differently.

Let's find the dynamic model of the system:

$$1) (M + m)\ddot{x} + \mu\dot{x} - ml\ddot{\theta} \cos \theta - ml\dot{\theta}^2 \sin \theta = F$$

$$2) (I + ml^2)\ddot{\theta} + mgl \sin \theta = -ml\ddot{x} \cos \theta$$

Let $\theta = \pi + \Phi$. If Φ is small enough then $\cos \Phi \rightarrow 1$ and $\sin \Phi \rightarrow \Phi$, therefore

$$\begin{cases} (M + m)\ddot{x} + \mu\dot{x} - ml\ddot{\Phi} = F \\ (I + ml^2)\ddot{\Phi} - mgl\Phi = ml\ddot{x} \end{cases}$$

which gives:

$$\frac{\Phi(s)}{F(s)} = \frac{mls^2}{[(M + m)(I + ml^2) - (ml)^2]s^3 + \mu(l + ml^2)s^2 - (M + m)mgl s - \mu mgl}$$

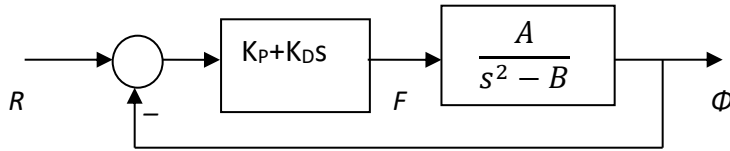
Ignoring friction $\mu = 0$.

$$\frac{\Phi(s)}{F(s)} = \frac{ml}{[(M + m)(I + ml^2) - (ml)^2]s^2 - (M + m)mgl} = \frac{A}{s^2 - B}$$

where

$$A = \frac{ml}{[(M + m)(I + ml^2) - (ml)^2]}; B = \frac{(M + m)mgl}{[(M + m)(I + ml^2) - (ml)^2]}$$

Ignoring actuator dynamics (DC motor equations), we can incorporate feedback control using a series PD compensator and unity feedback. Hence,



$$F(s) = K_p (R(s) - \Phi) - K_D s (R(s) - \Phi)$$

The system transfer function is:

$$\frac{\Phi}{R} = \frac{A(K_p + K_D s)}{(s^2 + K_D s + A(K_p - B))}$$

Control is achieved by ensuring stability ($K_p > B$)

Use Routh Hurwitz to establish stability first. Use Acsys to do that as demonstrated in this chapter problems. Also Chapter 2 has many examples.

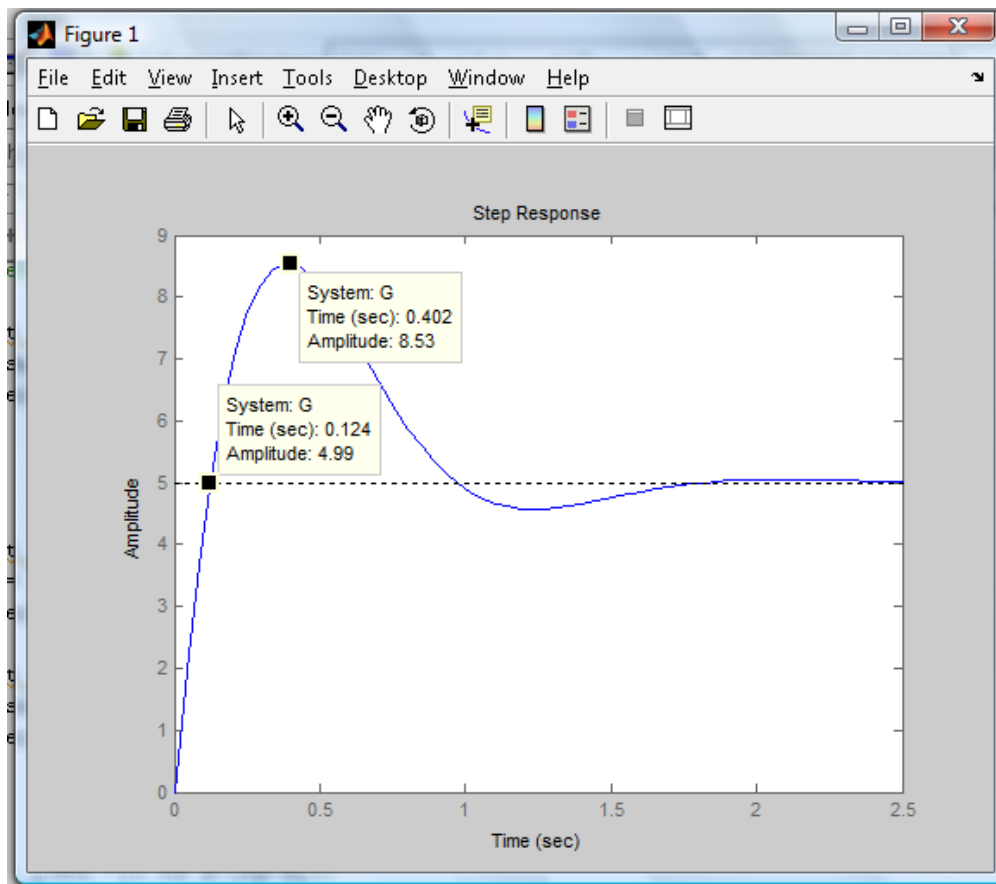
Use MATLAB to simulate response:

```
clear all
Kp=10;
Kd=5;
A=10;
B=8;
num = [A*Kd A*Kp];
den =[1 Kd A*(Kp-B)];
G=tf(num,den)
step(G)
```

Transfer function:

$$50 s + 100$$

$$s^2 + 5 s + 20$$



Adjust parameters to achieve desired response. Use THE PROCEDURE in Example 7-7-1.

You may look at the root locus of the forward path transfer function to get a better perspective.

$$\frac{\Phi}{E} = \frac{A(K_p + K_D s)}{s^2 - AB} = \frac{AK_D(z + s)}{s^2 - AB}$$

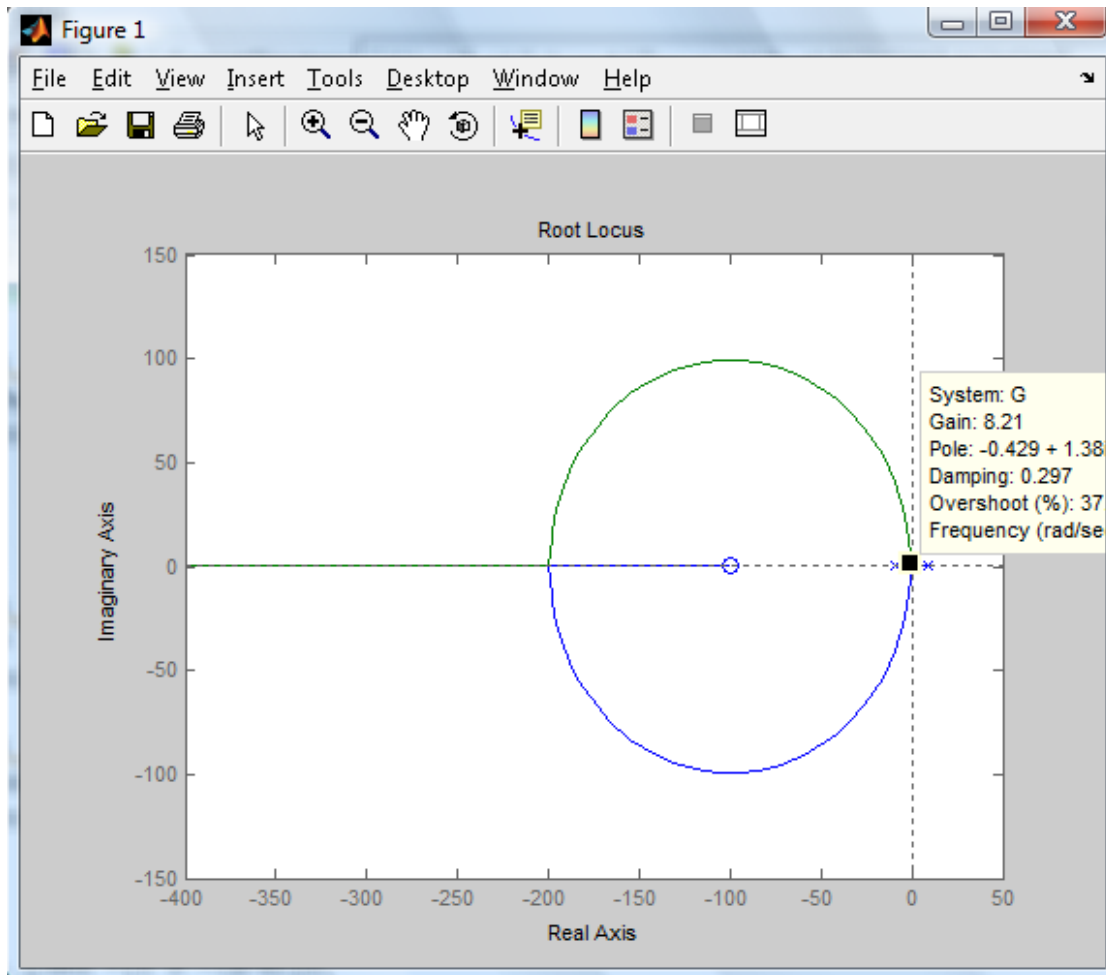
fix z and vary K_D .

```
clear all
z=100;
Kd=0.01;
A=10;
B=8;
num = [A*Kd A*Kd*z];
den = [1 0 -(A*B)];
G=tf(num,den)
rlocus(G)
```

Transfer function:

0.1 s + 10

s^2 - 80



For $z=10$, a large $K_D=0.805$ results in:

```

clear all
Kd=0.805;
Kp=10*Kd;
A=10;
B=8;
num = [A*Kd A*Kp];
den =[1 Kd A*(Kp-B)];
G=tf(num,den)
pole(G)
zero(G)
step(G)

```

Transfer function:

$$\frac{8.05 s + 80.5}{s^2 + 0.805 s + 0.5}$$

```

ans =

-0.4025 + 0.5814i
-0.4025 - 0.5814i

```

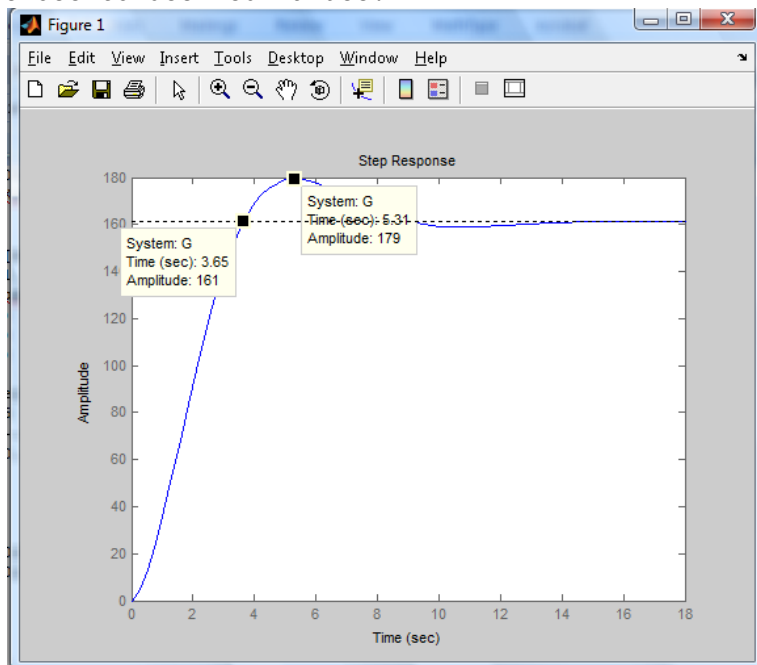
```

ans =

-10

```

Looking at dominant poles we expect to see an oscillatory response with overshoot close to desired values.



For a better design, and to meet rise time criterion, use Example 7-7-1.

Chapter 8

8-1) (a) State variables: $x_1 = y, \quad x_2 = \frac{dy}{dt}$

State equations:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix} r$$

Output equation:

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1$$

(b) State variables: $x_1 = y, \quad x_2 = \frac{dy}{dt}, \quad x_3 = \frac{d^2y}{dt^2}$

State equations:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2.5 & -1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.5 \end{bmatrix} r$$

Output equation:

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1$$

(c) State variables: $x_1 = \int_0^t y(\tau) d\tau, \quad x_2 = \frac{dx_1}{dt}, \quad x_3 = \frac{dy}{dt}, \quad x_4 = \frac{d^2y}{dt^2}$

State equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -3 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r$$

Output equation:

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1$$

(d) State variables: $x_1 = y, \quad x_2 = \frac{dy}{dt}, \quad x_3 = \frac{d^2y}{dt^2}, \quad x_4 = \frac{d^3y}{dt^3}$

State equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -2.5 & 0 & -1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r$$

Output equation:

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1$$

8-2) a) $G(s) = \frac{Y(s)}{U(s)} = \frac{s+3}{s^2+3s+2}$

$$\Rightarrow (s^2 + 3s + 2)Y(s) = (s + 3)U(s)$$

$$\Rightarrow sY(s) + 3Y(s) = -\frac{2}{s}Y(s) + \frac{3}{5}U(s) + V(s)$$

$$\text{Let } X(s) = -\frac{2}{s}Y(s) + \frac{3}{5}U(s)$$

$$\text{Then } \begin{cases} sY(s) = X(s) + U(s) + 3Y(s) \\ sX(s) = -2Y(s) + 3U(s) \end{cases} \Rightarrow \begin{cases} \dot{y} = -3y + x + u \\ \dot{x} = -2y + 3u \end{cases}$$

If $y = x_1$ and $x = x_2$, then

$$\begin{cases} \dot{x}_1 = -3x_1 + x_2 + u \\ \dot{x}_2 = -2x_1 + 3u \end{cases}$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & +1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

b) $G(s) = \frac{Y(s)}{U(s)} = \frac{6}{s^3+6s^2+11s+6}$

$$\Rightarrow Y(s)(s^3 + 6s^2 + 11s + 6) = 6U(s)$$

$$\Rightarrow sY(s) + 6Y(s) = -\frac{6}{s^2}Y(s) - \frac{11}{s}Y(s) + \frac{6}{s^2}U(s)$$

$$\text{Let } X(s) = -\frac{6}{s^2}Y(s) - \frac{11}{s}Y(s) + \frac{6}{s}U(s), \text{ therefore } sX(s) = -\frac{6}{s}Y(s) - 11Y(s) + \frac{6}{s}U(s)$$

$$\text{and Let } Z(s) = -\frac{6}{s}Y(s) + \frac{6}{s}U(s), \text{ then } sZ(s) = -6Y(s) + 6U(s). \text{ As a result:}$$

$$\begin{cases} sY(s) = -6Y(s) + X(s) \\ sX(s) = -11Y(s) + Z(s) \\ sZ(s) = -6Y(s) + 6U(s) \end{cases}$$

or

$$\begin{cases} \dot{y} = -6y + x \\ \dot{x} = -11y + z \\ \dot{z} = -6y + 6u \end{cases}$$

If $y = x_1$, $x = x_2$ and $z = x_3$, then

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -6 & 1 & 0 \\ -11 & 0 & 1 \\ -6 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$c) G(s) = \frac{Y(s)}{U(s)} = \frac{s+2}{s^2+7s+12}$$

$$\Rightarrow Y(s)(s^2 + 7s + 12) = (s + 2)U(s)$$

$$\Rightarrow sY(s) = -7Y(s) - \frac{12}{s}Y(s) + U(s) + \frac{2}{s}U(s)$$

$$\text{Let } sX(s) = -\frac{12}{s}Y(s) + \frac{2}{s}U(s), \text{ then } sX(s) = -12Y(s) + 2U(s). \text{ As a result:}$$

$$\begin{cases} \dot{y} = -7y + x + u \\ \dot{x} = -12y + 2u \end{cases}$$

Let $y = x_1$ and $x = x_2$, then

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -7 & 1 \\ -12 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$d) G(s) = \frac{Y(s)}{U(s)} = \frac{s^3+11s^2+35s+250}{s^2(s^3+4s^2+39s+108)}$$

$$\Rightarrow (s^3 + 4s^2 + 39s + 108)Y(s) = \left[s + 11 + \frac{35}{s} + \frac{250}{s^2} \right] U(s)$$

$$\Rightarrow sY(s) = -4Y(s) + \frac{39}{s}Y(s) + \frac{108}{s^2}Y(s) + \left[\frac{1}{s} + \frac{11}{s^2} + \frac{35}{s^3} + \frac{250}{s^4} \right] U(s)$$

$$\text{Let } X_2(s) = \frac{39}{s}Y(s) + \frac{108}{s^2}Y(s) + \left[\frac{1}{s} + \frac{11}{s^2} + \frac{35}{s^3} + \frac{250}{s^4} \right] U(s), \text{ then}$$

$$sX_2(s) = 39Y(s) + \frac{108}{s}Y(s) + U(s) + \left[\frac{11}{s^2} + \frac{35}{s^3} + \frac{250}{s^4} \right] U(s)$$

$$\text{Now, let } X_3(s) = \frac{108}{s}Y(s) + \frac{11}{s^2}U(s) + \frac{35}{s^3}U(s) + \frac{250}{s^4}U(s), \text{ therefore}$$

$$\begin{cases} sX_2(s) = 39Y(s) + X_3(s) + U(s) \\ sX_3(s) = 108Y(s) + \frac{11}{s}U(s) + \frac{35}{s^2}U(s) + \frac{250}{s^3}U(s) \end{cases}$$

$$\text{Let } X_4(s) = \frac{11}{s}U(s) + \frac{35}{s^2}U(s) + \frac{250}{s^3}U(s), \text{ then } sX_4(s) = 11U(s) + \frac{35}{s}U(s) + \frac{250}{s^2}U(s)$$

$$\text{Let } X_5(s) = \frac{35}{s}U(s) + \frac{250}{s^2}U(s), \text{ or } sX_5(s) = 35U(s) + \frac{250}{s}U(s)$$

$$\text{Let } X_6(s) = \frac{250}{s}U(s), \text{ then } sX_6(s) = 250U(s). \text{ If } Y(s) = X_1(s), \text{ then:}$$

$$\begin{cases} \dot{x}_1 = -4x_1 + x_2 \\ \dot{x}_2 = 39x_1 + x_2 + u \\ \dot{x}_3 = 108x_1 + x_4 \\ \dot{x}_4 = 11u + x_5 \\ \dot{x}_5 = 35u + 36x_6 \\ \dot{x}_6 = 250u \end{cases}$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} -4 & 1 & 0 & 0 & 0 & 0 \\ 39 & 0 & 1 & 0 & 0 & 0 \\ 108 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 11 \\ 35 \\ 250 \end{bmatrix} u$$

8-3) (a) Alternatively use equations (8-225), (8-232) and (8-233)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s + 3}{s^2 + 3s + 2}$$

The state variables are defined as

$$x_1(t) = y(t)$$

$$x_2(t) = \frac{dy(t)}{dt}$$

Then the state equations are represented by the vector-matrix equation

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

where $\mathbf{x}(t)$ is the 2×1 state vector, $u(t)$ the scalar input, and

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad (\text{Also see section 2-3-3 or 8-6})$$

$$\mathbf{C} = [1 \quad 0] \quad \mathbf{D} = 0$$

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} + \mathbf{D}$$

MATLAB

```
>> clear all
```

```
>> syms s
```

```
>> A=[0,1;-2,-3]
```

```
A =
```

```
0    1
```

```
-2   -3
```

```
>> B=[0;1]
```

```
B =
```

```
0
```

```
1
```

```
>> C=[3,1]
```

```
C =
```

```
3    1
```

```
>> s*eye(2)-A
```

```
ans =
```

```
[  s, -1]
```

```
[  2, s+3]
```

```
>> inv(ans)
```

```
ans =
```

```
[ (s+3)/(s^2+3*s+2),    1/(s^2+3*s+2)]
```

```
[ -2/(s^2+3*s+2),    s/(s^2+3*s+2)]
```

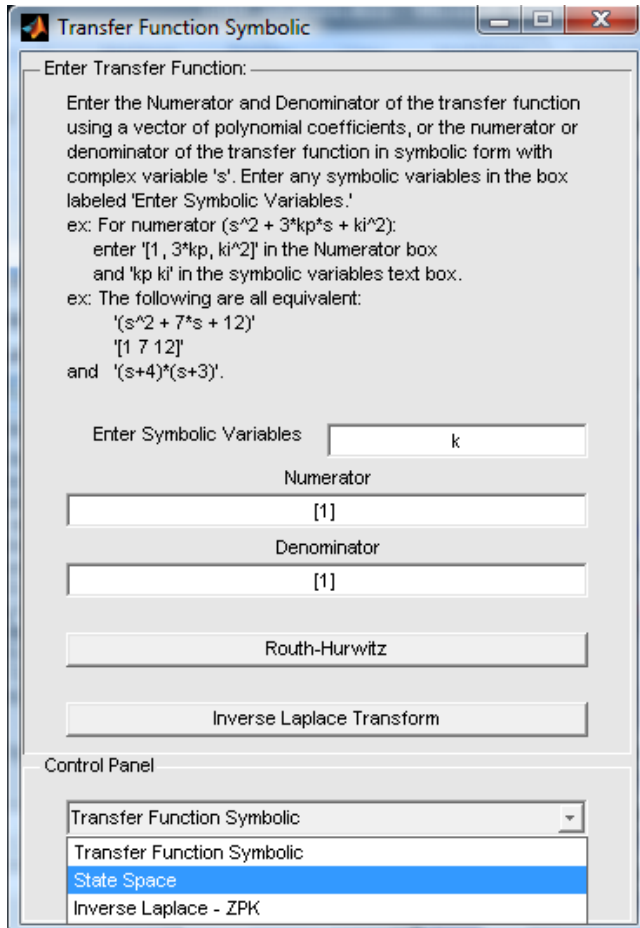
```
>> C*ans*B
```

```
ans =
```

```
3/(s^2+3*s+2)+s/(s^2+3*s+2)
```

Use ACSYS as demonstrate in section 8-19-2

- 1) Activate MATLAB
- 2) Go to the folder containing ACSYS
- 3) Type in Acsys
- 4) Click the “Transfer Function Symbolic” pushbutton
- 5) Enter the transfer function
- 6) Use the “State Space” option as shown below:



You get the next window. Enter the A,B,C, and D values.

Transfer Function Symbolic

Enter Matrix:

Enter the Coefficient Matrices (empty matrices will give error)

E.g. For a 2x2 identity matrix type in: [1 0; 0 1]
 [1 ; 0 ; 1] is a 3x1 column vector & [1 0 1] is a 1x3 row vector

A = [0,1;-2,-3]

B = [0;1]

C = [3,1]

D = [0]

u = [1]

ICs:

State Space

Control Panel

State Space

Close

State Space Analysis

Inputs:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -3 \end{bmatrix} \quad \begin{bmatrix} 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

State Space Representation:

$$\dot{D}x = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\begin{bmatrix} -2 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \end{bmatrix} u$$

$$y = |3 \ 1|x + |0|u$$

Determinant of (s*I-A):

$$\begin{vmatrix} s+3 & 1 \\ 0 & s+2 \end{vmatrix}$$

Characteristic Equation of the Transfer Function:

$$(s+3)(s+2) = 0$$

The eigen values of A and poles of the Transfer Function are:

$$-3$$

$$-2$$

Inverse of (s*I-A) is:

$$\begin{bmatrix} s+3 & 1 \\ 0 & s+2 \end{bmatrix}^{-1} = \frac{1}{(s+3)(s+2)} \begin{bmatrix} s+2 & -1 \\ 0 & s+3 \end{bmatrix}$$

State transition matrix (phi) of A:

$$\begin{bmatrix} 2\exp(-t) - \exp(-2t) & \exp(-t) - \exp(-2t) \\ -2\exp(-t) + 2\exp(-2t) & -\exp(-t) + 2\exp(-2t) \end{bmatrix}$$

Transfer function between u(t) and y(t) is:

$$\frac{s + 3}{s^2 + 3s + 2}$$

No Initial Conditions Specified

States (X) in Laplace Domain:

$$\begin{bmatrix} 1 \\ -\frac{1}{(s+2)(s+1)} \end{bmatrix}$$

Inverse Laplace x(t):

$$\begin{bmatrix} \exp(-t) - \exp(-2t) \\ -\exp(-t) + 2\exp(-2t) \end{bmatrix}$$

Output Y(s):

$$\frac{s + 3}{(s + 2)(s + 1)}$$

Inverse Laplace y(t):

$$2\exp(-t) - \exp(-2t)$$

Use the same procedure for parts b, c and d.

8-4) a) $x_1 = \frac{-x_4 + u}{s+2} \rightarrow (s+2)X_1 = -X_4 + U, \rightarrow \dot{x}_1 = -x_4 - 2x_1 + u$

$$x_4 = \frac{x_3 + x_1}{s} \rightarrow sX_4 = X_3 + X_1 \rightarrow \dot{x}_4 = x_1 + x_3$$

$$x_2 = \frac{0.5}{s}x_1 \rightarrow sX_2 = 0.5X_1 \rightarrow \dot{x}_2 = 0.5x_1$$

$$x_3 = \frac{x_2}{s} \rightarrow sX_3 = X_2 \rightarrow \dot{x}_3 = x_2$$

$$y = x_1 + x_2 + x_3$$

As a result:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 & -1 \\ 0.5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [1 \quad 1 \quad 1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

b) $X_1(s) = \frac{1}{s+2}U(s) \rightarrow sX_1(s) = -2X_1(s) + U(s) \rightarrow \dot{x}_1 = -2x_1 + u$

$$X_2 = \frac{s+4}{s+3}X_1 \rightarrow sX_2(s) = sX_1 + 4X_1 - 3X_2 \rightarrow \dot{x}_2 = \dot{x}_1 + 4x_1 - 3x_2 = 2x_1 - 3x_2 + u$$

$$X_3 = \frac{X_2 + X_1 - 6X_3}{s} \rightarrow sX_3(s) = X_2 + X_1 - 6X_3 \rightarrow \dot{x}_3 = x_2 + x_1 - x_3$$

$$y = x_3 + x_1$$

As a result:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 1 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

c) $X_1(s) = \frac{1}{s+5}U(s) \rightarrow sX_1 = -5X_1 + U \rightarrow \dot{x}_1 = -5x_1 + u$

$$X_2 = \frac{X_1 + U - X_3}{s+2} \rightarrow sX_2 = X_1 - 2X_2 - X_3 + U \rightarrow \dot{x}_2 = x_1 - 2x_2 - x_3 + u$$

$$X_3 = \frac{X_2}{s+4} \rightarrow sX_3 = X_2 - 4X_3 \rightarrow \dot{x}_3 = x_2 - 4x_3$$

$$X_4 = \frac{2X_3}{s} \Rightarrow sX_4 = 2X_3 \Rightarrow \dot{x}_4 = 2x_3$$

$$y = x_2 + x_4$$

As a result:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 0 & 0 \\ 1 & -2 & -1 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [0 \quad 1 \quad 0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

8-5) We shall first show that

$$\Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \frac{\mathbf{I}}{s} + \frac{\mathbf{A}}{s^2} + \frac{1}{2!} \frac{\mathbf{A}^2}{s^2} + \dots$$

We multiply both sides of the equation by $(s\mathbf{I} - \mathbf{A})$, and we get $\mathbf{I} = \mathbf{I}$. Taking the inverse Laplace transform

on both sides of the equation gives the desired relationship for $\phi(t)$.

8-6)

(a) USE MATLAB

```
Amat=[0 1;-2 -1]
[mA,nA]=size(Amat);
rankA=rank(Amat);
disp(' Characteristic Polynomial: ')
chareq=poly(Amat);
[mchareq,nchareq]=size(chareq);
syms s;
poly2sym(chareq,s)
[vecss,eigss]=eig(Amat);
disp(' Eigenvalues of A = Diagonal Canonical Form of A is:');
Abar=eigss,
disp('Eigen Vectors are ')
T=vecss
% state transition matrix
ilaplace(inv([s 0;0 s]-Amat))
```

Results in MATLAB COMMAND LINE

Amat =

0 1

-2 -1

Characteristic Polynomial:

ans =

$s^2 + s + 2$

Eigenvalues of A = Diagonal Canonical Form of A is:

Abar =

-0.5000 + 1.3229i 0

0 -0.5000 - 1.3229i

Eigen Vectors are

T =

-0.2041 - 0.5401i -0.2041 + 0.5401i

0.8165 0.8165

phi=ilaplace(inv([s 0;0 s]-Amat))

phi =

[1/7*exp(-1/2*t)*(7*cos(1/2*7^(1/2)*t)+7^(1/2)*sin(1/2*7^(1/2)*t)), 2/7*7^(1/2)*exp(-1/2*t)*sin(1/2*7^(1/2)*t)]

[-4/7*7^(1/2)*exp(-1/2*t)*sin(1/2*7^(1/2)*t), 1/7*exp(-1/2*t)*(7*cos(1/2*7^(1/2)*t)-7^(1/2)*sin(1/2*7^(1/2)*t))]

% use vpa to convert to digital format. Use digit(#) to adjust level of precision if necessary.

vpa(phi)

ans =

[.1428571*exp(.5000000*t)*(7.*cos(1.322876*t)+2.645751*sin(1.322876*t)),

.7559289*exp(-.5000000*t)*sin(1.322876*t)]

[-1.511858*exp(-.5000000*t)*sin(1.322876*t),

$$.1428571 * \exp(-.5000000 * t) * (7. * \cos(1.322876 * t) - 2.645751 * \sin(1.322876 * t))]$$

ANALYTICAL SOLUTION:

Characteristic equation: $\Delta(s) = |s\mathbf{I} - \mathbf{A}| = s^2 + s + 2 = 0$

Eigenvalues: $s = -0.5 + j1.323, \quad -0.5 - j1.323$

State transition matrix:

$$\phi(t) = \begin{bmatrix} \cos 1.323t + 0.378 \sin 1.323t & 0.756 \sin 1.323t \\ -1.512 \sin 1.323t & -1.069 \sin(1.323t - 69.3^\circ) \end{bmatrix} e^{-0.5t}$$

Alternatively**USE ACSYS as illustrated in section 8-19-1**

- 1) Activate MATLAB
- 2) Go to the folder containing ACSYS
- 3) Type in Acsys
- 4) Click the “State Space” pushbutton
- 5) Enter the A,B,C, and D values. Note C must be entered here and must have the same number of columns as A. We use [1,1] arbitrarily as it will not affect the eigenvalues.
- 6) Use the “Calculate/Display” menu and find the eigenvalues.

State Space Tool

Calculate/Display Time Response Accessories

Block Diagram

$$\frac{dx(t)}{dt} = \begin{bmatrix} A \end{bmatrix} x(t) + \begin{bmatrix} B \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} C \end{bmatrix} x(t) + \begin{bmatrix} D \end{bmatrix} u(t)$$

Input Module

Enter coefficient Matrices.
eg. For a 3x3 identity matrix enter [1 0 0;0 1 0;0 0 1]

[1;0;1] is a 3x1 column [1 0 0] is a 1x3 row.

A
[0,1;-2,-1]

B
[0,1;1,0]

C
[1,1]

D
0

Initial Conditions
0

Buttons

Reset

Close Window

From MATLAB Command Window:

The A matrix is:

Amat =

0 1

-2 -1

Characteristic Polynomial:

ans =

$$s^2 + s + 2$$

Eigenvalues of A = Diagonal Canonical Form of A is:

Abar =

$$\begin{bmatrix} -0.5000 + 1.3229i & 0 \\ 0 & -0.5000 - 1.3229i \end{bmatrix}$$

Eigen Vectors are

T =

$$\begin{bmatrix} -0.2041 - 0.5401i & -0.2041 + 0.5401i \\ 0.8165 & 0.8165 \end{bmatrix}$$

THE REST ARE SAME AS PART A.

(b) Characteristic equation: $\Delta(s) = |s\mathbf{I} - \mathbf{A}| = s^2 + 5s + 4 = 0$ **Eigenvalues:** $s = -4, -1$

State transition matrix:

$$\phi(t) = \begin{bmatrix} 1.333e^{-t} - 0.333e^{-4t} & 0.333e^{-t} - 0.333e^{-4t} \\ -1.333e^{-t} - 1.333e^{-4t} & -0.333e^{-t} + 1.333e^{-4t} \end{bmatrix}$$

(c) Characteristic equation: $\Delta(s) = (s + 3)^2 = 0$ **Eigenvalues:** $s = -3, -3$

State transition matrix:

$$\phi(t) = \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{-3t} \end{bmatrix}$$

(d) Characteristic equation: $\Delta(s) = s^2 - 9 = 0$ **Eigenvalues:** $s = -3, 3$

State transition matrix:

$$\phi(t) = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-3t} \end{bmatrix}$$

(e) Characteristic equation: $\Delta(s) = s^2 + 4 = 0$ **Eigenvalues:** $s = j2, -j2$

State transition matrix:

$$\phi(t) = \begin{bmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{bmatrix}$$

(f) Characteristic equation: $\Delta(s) = s^3 + 5s^2 + 8s + 4 = 0$ **Eigenvalues:** $s = -1, -2, -2$

State transition matrix:

$$\phi(t) = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & te^{-2t} \\ 0 & 0 & e^{-2t} \end{bmatrix}$$

(g) Characteristic equation: $\Delta(s) = s^3 + 15s^2 + 75s + 125 = 0$ **Eigenvalues:** $s = -5, -5, -5$

$$\phi(t) = \begin{bmatrix} e^{-5t} & te^{-5t} & 0 \\ 0 & e^{-5t} & te^{-5t} \\ 0 & 0 & e^{-5t} \end{bmatrix}$$

State transition equation: $\mathbf{x}(t) = \phi(t)\mathbf{x}(t) + \int_0^t \phi(t-\tau)\mathbf{B}\mathbf{r}(\tau)d\tau$ $\phi(t)$ for each part is given in Problem 5-3.

8-7) In MATLAB USE ilaplace to find $\mathbb{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{R}(s)]$ see previous problem for codes.

(a)

$$\begin{aligned} \int_0^t \phi(t-\tau)\mathbf{B}\mathbf{r}(\tau)d\tau &= \mathbb{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{R}(s)] = \mathbb{L}^{-1}\left\{\frac{1}{\Delta(s)}\begin{bmatrix} s+1 & 1 \\ -2 & s \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ s \end{bmatrix}\right\} \\ &= \mathbb{L}^{-1}\begin{bmatrix} \frac{s+2}{s(s^2+s+2)} \\ \frac{s-2}{s(s^2+s+2)} \end{bmatrix} = \begin{bmatrix} 1 + 0.378\sin 1.323t - \cos 1.323t \\ -1 + 1.134\sin 1.323t + \cos 1.323t \end{bmatrix} \quad t \geq 0 \end{aligned}$$

(b)

$$\begin{aligned}
 \int_0^t \phi(t-\tau) \mathbf{B}r(\tau) d\tau &= \mathbf{L}^{-1} \left[(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}R(s) \right] = \mathbf{L}^{-1} \left\{ \frac{1}{\Delta(s)} \begin{bmatrix} s+5 & 1 \\ -4 & s \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{s} \right\} \\
 &= \mathbf{L}^{-1} \left[\frac{s+6}{s(s+1)(s+2)} \right] = \mathbf{L}^{-1} \left[\frac{1.5}{s} - \frac{1.67}{s+1} + \frac{0.167}{s+4} \right] = \begin{bmatrix} 1.5 - 1.67e^{-t} + 0.167e^{-4t} \\ -1 + 1.67e^{-t} - 0.667e^{-4t} \end{bmatrix} \quad t \geq 0
 \end{aligned}$$

(c)

$$\begin{aligned}
 \int_0^t \phi(t-\tau) \mathbf{B}r(\tau) d\tau &= \mathbf{L}^{-1} \left[(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}R(s) \right] = \mathbf{L}^{-1} \left\{ \begin{bmatrix} \frac{1}{s+3} & 0 \\ 0 & \frac{1}{s+3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s} \right\} \\
 &= \mathbf{L}^{-1} \left[\frac{0}{s(s+3)} \right] = \begin{bmatrix} 0 \\ 0.333(1 - e^{-3t}) \end{bmatrix} \quad t \geq 0
 \end{aligned}$$

(d)

$$\begin{aligned}
 \int_0^t \phi(t-\tau) \mathbf{B}r(\tau) d\tau &= \mathbf{L}^{-1} \left[(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}R(s) \right] = \mathbf{L}^{-1} \left\{ \begin{bmatrix} \frac{1}{s-3} & 0 \\ 0 & \frac{1}{s+3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s} \right\} \\
 &= \mathbf{L}^{-1} \left[\frac{0}{s(s+3)} \right] = \begin{bmatrix} 0 \\ 0.333(1 - e^{-3t}) \end{bmatrix} \quad t \geq 0
 \end{aligned}$$

(e)

$$\begin{aligned}
 \int_0^t \phi(t-\tau) \mathbf{B}r(\tau) d\tau &= \mathbf{L}^{-1} \left[(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}R(s) \right] = \mathbf{L}^{-1} \left\{ \begin{bmatrix} \frac{1}{s^2+4} & 2 \\ \frac{-2}{s^2+4} & \frac{s}{s^2+4} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s} \right\} \\
 &= \mathbf{L}^{-1} \left[\frac{\frac{2}{s}}{s^2+4} \right] = \begin{bmatrix} 2 \\ 0.5 \sin 2t \end{bmatrix} \quad t \geq 0
 \end{aligned}$$

(f)

$$\int_0^t \phi(t-\tau) \mathbf{B}r(\tau) d\tau = \mathbf{L}^{-1} \left[(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}R(s) \right] = \mathbf{L}^{-1} \left\{ \begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ 0 & \frac{1}{s+2} & \frac{1}{(s+2)^2} \\ 0 & 0 & \frac{1}{s+2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \frac{1}{s} \right\}$$

$$= \mathbf{L}^{-1} \begin{bmatrix} 0 \\ \frac{1}{s(s+2)} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5(1 - e^{-2t}) \\ 0 \end{bmatrix} \quad t \geq 0$$

(g)

$$\int_0^t \phi(t-\tau) \mathbf{B}r(\tau) d\tau = \mathbf{L}^{-1} \left[(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}R(s) \right] = \mathbf{L}^{-1} \left\{ \begin{bmatrix} \frac{1}{s+5} & \frac{1}{(s+5)^2} & 0 \\ 0 & \frac{1}{s+5} & \frac{1}{(s+5)^2} \\ 0 & 0 & \frac{1}{s+5} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \frac{1}{s} \right\}$$

$$= \mathbf{L}^{-1} \begin{bmatrix} 0 \\ \frac{1}{s(s+5)^2} \\ \frac{1}{s(s+5)} \end{bmatrix} = \mathbf{L}^{-1} \begin{bmatrix} 0 \\ \frac{0.04}{s} - \frac{0.04}{s+5} - \frac{0.2}{(s+5)^2} \\ \frac{0.2}{s} - \frac{0.2}{s+5} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.04 - 0.04e^{-5t} - 0.2te^{-5t} \\ 0.2 - 0.2e^{-5t} \end{bmatrix} u_s(t)$$

8-8) State transition equation: $\mathbf{x}(t) = \phi(t)\mathbf{x}(t) + \int_0^t \phi(t-\tau)\mathbf{B}r(\tau) d\tau$ $\phi(t)$ for each part is given in Problem 5-3.

(a)

$$\begin{aligned}\int_0^t \phi(t-\tau) \mathbf{B}r(\tau) d\tau &= \mathbf{L}^{-1} \left[(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}R(s) \right] = \mathbf{L}^{-1} \left\{ \frac{1}{\Delta(s)} \begin{bmatrix} s+1 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{s} \right\} \\ &= \mathbf{L}^{-1} \begin{bmatrix} \frac{s+2}{s(s^2+s+2)} \\ \frac{s-2}{s(s^2+s+2)} \end{bmatrix} = \begin{bmatrix} 1 + 0.378 \sin 1.323t - \cos 1.323t \\ -1 + 1.134 \sin 1.323t + \cos 1.323t \end{bmatrix} \quad t \geq 0\end{aligned}$$

(b)

$$\begin{aligned}\int_0^t \phi(t-\tau) \mathbf{B}r(\tau) d\tau &= \mathbf{L}^{-1} \left[(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}R(s) \right] = \mathbf{L}^{-1} \left\{ \frac{1}{\Delta(s)} \begin{bmatrix} s+5 & 1 \\ -4 & s \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{s} \right\} \\ &= \mathbf{L}^{-1} \begin{bmatrix} \frac{s+6}{s(s+1)(s+2)} \\ \frac{s-4}{s(s+1)(s+4)} \end{bmatrix} = \mathbf{L}^{-1} \begin{bmatrix} \frac{1.5}{s} - \frac{1.67}{s+1} + \frac{0.167}{s+4} \\ \frac{-1}{s} + \frac{1.67}{s+1} - \frac{0.667}{s+4} \end{bmatrix} = \begin{bmatrix} 1.5 - 1.67e^{-t} + 0.167e^{-4t} \\ -1 + 1.67e^{-t} - 0.667e^{-4t} \end{bmatrix} \quad t \geq 0\end{aligned}$$

(c)

$$\begin{aligned}\int_0^t \phi(t-\tau) \mathbf{B}r(\tau) d\tau &= \mathbf{L}^{-1} \left[(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}R(s) \right] = \mathbf{L}^{-1} \left\{ \begin{bmatrix} \frac{1}{s+3} & 0 \\ 0 & \frac{1}{s+3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s} \right\} \\ &= \mathbf{L}^{-1} \begin{bmatrix} 0 \\ \frac{1}{s(s+3)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.333(1 - e^{-3t}) \end{bmatrix} \quad t \geq 0\end{aligned}$$

(d)

$$\begin{aligned}\int_0^t \phi(t-\tau) \mathbf{B}r(\tau) d\tau &= \mathbf{L}^{-1} \left[(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}R(s) \right] = \mathbf{L}^{-1} \left\{ \begin{bmatrix} \frac{1}{s-3} & 0 \\ 0 & \frac{1}{s+3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s} \right\} \\ &= \mathbf{L}^{-1} \begin{bmatrix} 0 \\ \frac{1}{s(s+3)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.333(1 - e^{-3t}) \end{bmatrix} \quad t \geq 0\end{aligned}$$

(e)

$$\int_0^t \phi(t-\tau) \mathbf{B}r(\tau) d\tau = \mathbf{L}^{-1} \left[(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}R(s) \right] = \mathbf{L}^{-1} \left\{ \begin{bmatrix} \frac{1}{s^2+4} & 2 \\ \frac{-2}{s^2+4} & \frac{s}{s^2+4} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s} \right\}$$

$$= \mathbf{L}^{-1} \begin{bmatrix} \frac{2}{s} \\ 1 \\ \frac{1}{(s^2+4)} \end{bmatrix} = \begin{bmatrix} 2 \\ 0.5 \sin 2t \end{bmatrix} \quad t \geq 0$$

(f)

$$\int_0^t \phi(t-\tau) \mathbf{B}r(\tau) d\tau = \mathbf{L}^{-1} \left[(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}R(s) \right] = \mathbf{L}^{-1} \left\{ \begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ 0 & \frac{1}{s+2} & \frac{1}{(s+2)^2} \\ 0 & 0 & \frac{1}{s+2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \frac{1}{s} \right\}$$

$$= \mathbf{L}^{-1} \begin{bmatrix} 0 \\ \frac{1}{s(s+2)} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5(1 - e^{-2t}) \\ 0 \end{bmatrix} \quad t \geq 0$$

(g)

$$\int_0^t \phi(t-\tau) \mathbf{B}r(\tau) d\tau = \mathbf{L}^{-1} \left[(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}R(s) \right] = \mathbf{L}^{-1} \left\{ \begin{bmatrix} \frac{1}{s+5} & \frac{1}{(s+5)^2} & 0 \\ 0 & \frac{1}{s+5} & \frac{1}{(s+5)^2} \\ 0 & 0 & \frac{1}{s+5} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \frac{1}{s} \right\}$$

$$= \mathbf{L}^{-1} \begin{bmatrix} 0 \\ \frac{1}{s(s+5)^2} \\ \frac{1}{s(s+5)} \end{bmatrix} = \mathbf{L}^{-1} \begin{bmatrix} 0 \\ \frac{0.04}{s} - \frac{0.04}{s+5} - \frac{0.2}{(s+5)^2} \\ \frac{0.2}{s} - \frac{0.2}{s+5} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.04 - 0.04e^{-5t} - 0.2te^{-5t} \\ 0.2 - 0.2e^{-5t} \end{bmatrix} u_s(t)$$

8-9) (a) Not a state transition matrix, since $\phi(0) \neq \mathbf{I}$ (identity matrix).

(b) Not a state transition matrix, since $\phi(0) \neq \mathbf{I}$ (identity matrix).

(c) $\phi(t)$ is a state transition matrix, since $\phi(0) = \mathbf{I}$ and

$$[\phi(t)]^{-1} = \begin{bmatrix} 1 & 0 \\ 1 - e^{-t} & e^{-t} \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 1 - e^t & e^t \end{bmatrix} = \phi(-t)$$

(d) $\phi(t)$ is a state transition matrix, since $\phi(0) = \mathbf{I}$, and

$$[\phi(t)]^{-1} = \begin{bmatrix} e^{2t} & -te^{2t} & t^2 e^{2t} / 2 \\ 0 & e^{2t} & -te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix} = \phi(-t)$$

8-10) a) $\dot{x} = Ax + Bu \rightarrow sI - A = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$ and $(sI - A)^{-1} = \frac{1}{s^2+3s+2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$

Therefore:

$$\Phi(t) = L^{-1}\{(sI - A)^{-1}\} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

If $x(0) = 0$, then $x(t) = \int_0^t \Phi(t - \tau)Bu(\tau)d\tau = \begin{bmatrix} 0.5 - e^{-t} + 0.5e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$

b) $\Phi(t) = L^{-1}\{(sI - A)^{-1}\}$

$$\begin{aligned} &= L^{-1}\left\{\frac{1}{s^2+s+0.5} \begin{bmatrix} s & -0.5 \\ 1 & s+1 \end{bmatrix}\right\} \\ &= \begin{bmatrix} e^{-0.5t}(\cos 0.5t - \sin 0.5t) & e^{-0.5t} \sin 0.5t \\ 2e^{-0.5t} \sin 0.5t & e^{-0.5t}(\cos 5t + \sin 0.5t) \end{bmatrix} \end{aligned}$$

If $x(0) = 0$, then

$$\begin{aligned} x(t) &= A^{-1}(e^{At} - I)B = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} 0.5e^{-0.5t}(\cos 0.5t - \sin 0.5t) - 0.5 \\ e^{-0.5t} - \sin 0.5t \end{bmatrix} \\ &= \begin{bmatrix} e^{-0.5t} \sin 5t \\ -e^{-0.5t}(\cos 0.5t + \sin 0.5t) + 1 \end{bmatrix} \end{aligned}$$

and

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 = e^{-0.5t} \sin 0.5t$$

8-11) (a) (1) Eigenvalues of A: 2.325, $-0.3376 + j0.5623$, $-0.3376 - j0.5623$

(2) Transfer function relation:

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}U(s) = \frac{1}{\Delta(s)} \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} U(s) = \frac{1}{\Delta(s)} \begin{bmatrix} s^2 + 3s + 2 & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -2s-1 & s^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} U(s) = \frac{1}{\Delta(s)} \begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix} U(s)$$

$$\Delta(s) = s^3 + 3s^2 + 2s + 1$$

(3) Output transfer function:

$$\frac{Y(s)}{U(s)} = \mathbf{C}(s)(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \frac{1}{\Delta(s)} \begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix} = \frac{1}{s^3 + 3s^2 + 2s + 1}$$

USE ACSYS as illustrated in section 8-19-1

- 7) Activate MATLAB
- 8) Go to the folder containing ACSYS
- 9) Type in Acsys
- 10) Click the “State Space” pushbutton
- 11) Enter the A,B,C, and D values. Note C must be entered here and must have the same number of columns as A. We use [1,1] arbitrarily as it will not affect the eigenvalues.
- 12) Use the “Calculate/Display” menu and find the eigenvalues and other State space calculations.

State Space Tool

Calculate/Display Time Response Accessories

Block Diagram

$$\frac{dx(t)}{dt} = \begin{bmatrix} A \end{bmatrix} x(t) + \begin{bmatrix} B \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} C \end{bmatrix} x(t) + \begin{bmatrix} D \end{bmatrix} u(t)$$

Input Module

Enter coefficient Matrices.
eg. For a 3x3 identity matrix enter [1 0 0; 0 1 0; 0 0 1]
[1;0;1] is a 3x1 column [1 0 0] is a 1x3 row.

A
[0,1,0;0,0,1;-1,-2,-3]

B
[0;0;1]

C
[1,0,0]

D
0

Initial Conditions
0

Buttons

Reset

Close Window

The A matrix is:

Amat =

0 1 0

0 0 1

-1 -2 -3

Characteristic Polynomial:

ans =

$$s^3 + 3s^2 + 2s + 1$$

Eigenvalues of A = Diagonal Canonical Form of A is:

Abar =

$$\begin{bmatrix} -2.3247 & 0 & 0 \\ 0 & -0.3376 + 0.5623i & 0 \\ 0 & 0 & -0.3376 - 0.5623i \end{bmatrix}$$

Eigen Vectors are

T =

$$\begin{bmatrix} 0.1676 & 0.7868 & 0.7868 \\ -0.3896 & -0.2657 + 0.4424i & -0.2657 - 0.4424i \\ 0.9056 & -0.1591 - 0.2988i & -0.1591 + 0.2988i \end{bmatrix}$$

State-Space Model is:

a =

$$\begin{bmatrix} x1 & x2 & x3 \\ x1 & 0 & 1 & 0 \\ x2 & 0 & 0 & 1 \\ x3 & -1 & -2 & -3 \end{bmatrix}$$

b =

$$\begin{bmatrix} u1 \\ x1 & 0 \\ x2 & 0 \\ x3 & 1 \end{bmatrix}$$

c =

x1 x2 x3

y1 1 0 0

d =

u1

y1 0

Continuous-time model.

Characteristic Polynomial:

ans =

s^3+3*s^2+2*s+1

Equivalent Transfer Function Model is:

Transfer function:

1.776e-015 s^2 + 6.661e-016 s + 1

s^3 + 3 s^2 + 2 s + 1

Pole, Zero Form:

Zero/pole/gain:

1.7764e-015 (s^2 + 0.375s + 5.629e014)

(s+2.325) (s^2 + 0.6753s + 0.4302)

The numerator is basically equal to 1

Use the same procedure for other parts.

(b) (1) Eigenvalues of A: $-1, -1$.

(2) Transfer function relation:

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}U(s) = \frac{1}{\Delta(s)} \begin{bmatrix} s+1 & 1 \\ 0 & s+1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} U(s) = \begin{bmatrix} \frac{1}{(s+1)^2} \\ \frac{1}{(s+1)} \end{bmatrix} U(s) \quad \Delta(s) = s^2 + 2s + 1$$

(3) Output transfer function:

$$\frac{Y(s)}{U(s)} = \mathbf{C}(s)(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} = [1 \quad 1] \begin{bmatrix} \frac{1}{(s+1)^2} \\ \frac{1}{s+1} \end{bmatrix} = \frac{1}{(s+1)^2} + \frac{1}{s+1} = \frac{s+2}{(s+1)^2}$$

(c) (1) Eigenvalues of A: $0, -1, -1$.

(2) Transfer function relation:

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}U(s) = \frac{1}{\Delta(s)} \begin{bmatrix} s^2 + 2s + 1 & s+2 & 1 \\ 0 & s(s+2) & s \\ 0 & -s & s^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} U(s) = \frac{1}{\Delta(s)} \begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix} U(s) \quad \Delta(s) = s(s^2 + 2s + 1)$$

(3) Output transfer function:

$$\frac{Y(s)}{U(s)} = \mathbf{C}(s)(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} = [1 \quad 1 \quad 0] \begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix} = \frac{s+1}{s(s+1)^2} = \frac{1}{s(s+1)}$$

8-12) We write $\frac{dy}{dt} = \frac{dx_1}{dt} + \frac{dx_2}{dt} = x_2 + x_3$ $\frac{d^2y}{dt^2} = \frac{dx_2}{dt} + \frac{dx_3}{dt} = -x_1 - 2x_2 - 2x_3 + u$

$$\frac{d\bar{\mathbf{x}}}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dy}{dt} \\ \frac{d^2y}{dt^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \quad (1)$$

$$\bar{\mathbf{x}} = \begin{bmatrix} x_1 \\ y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \mathbf{x} \quad \mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \bar{\mathbf{x}} \quad (2)$$

Substitute Eq. (2) into Eq. (1), we have

$$\frac{d\bar{\mathbf{x}}}{dt} = \mathbf{A}_1 \bar{\mathbf{x}} + \mathbf{B}_1 u = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & -2 \end{bmatrix} \bar{\mathbf{x}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

8-13) For MATLAB Codes see 8-15

(a)

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s & -2 & 0 \\ -1 & s-2 & 0 \\ 1 & 0 & s-1 \end{vmatrix} = s^3 - 3s^2 + 2 = s^3 + a_2s^2 + a_1s + a_0 \quad a_0 = 2, \quad a_1 = 0, \quad a_2 = -3$$

$$\mathbf{M} = \begin{bmatrix} a_1 & a_2 & 1 \\ a_2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 1 \\ -3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \mathbf{S} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B}] = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 2 & 6 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\mathbf{P} = \mathbf{SM} = \begin{bmatrix} -2 & 2 & 0 \\ 0 & -1 & 1 \\ -4 & -2 & 1 \end{bmatrix}$$

(b)

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s & -2 & 0 \\ -1 & s-1 & 0 \\ 1 & -1 & s-1 \end{vmatrix} = s^3 - 3s^2 + 2 = s^3 + a_2s^2 + a_1s + a_0 \quad a_0 = 2, \quad a_1 = 0, \quad a_2 = -3$$

$$\mathbf{M} = \begin{bmatrix} a_1 & a_2 & 1 \\ a_2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 1 \\ -3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \mathbf{S} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B}] = \begin{bmatrix} 1 & 2 & 6 \\ 1 & 3 & 8 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{P} = \mathbf{SM} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(c)

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s+2 & -1 & 0 \\ 0 & s+2 & 0 \\ 1 & 2 & s+3 \end{vmatrix} = s^3 + 7s^2 + 16s + 12 = s^3 + a_2s^2 + a_1s + a_0 \quad a_0 = 12, \quad a_1 = 16, \quad a_2 = 7$$

$$\mathbf{M} = \begin{bmatrix} a_1 & a_2 & 1 \\ a_2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 7 & 1 \\ 7 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \mathbf{S} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B}] = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -2 & 4 \\ 1 & -6 & 23 \end{bmatrix}$$

$$\mathbf{P} = \mathbf{SM} = \begin{bmatrix} 9 & 6 & 1 \\ 6 & 5 & 1 \\ -3 & 1 & 1 \end{bmatrix}$$

(d)

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s+1 & -1 & 0 \\ 0 & s-1 & -1 \\ 0 & 0 & s+1 \end{vmatrix} = s^3 + 3s^2 + 3s + 1 = s^3 + a_2s^2 + a_1s + a_0 \quad a_0 = 1, \quad a_1 = 3, \quad a_2 = 3$$

$$\mathbf{M} = \begin{bmatrix} a_1 & a_2 & 1 \\ a_2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \mathbf{S} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B}] = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\mathbf{P} = \mathbf{SM} = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

(e)

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s-1 & -1 \\ 2 & s+3 \end{vmatrix} = s^2 + 2s - 1 = s^2 + a_1s + a_0 \quad a_0 = -1, \quad a_1 = 2$$

$$\mathbf{M} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \quad \mathbf{S} = [\mathbf{B} \quad \mathbf{AB}] = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}$$

$$\mathbf{P} = \mathbf{SM} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

8-14) For MATLAB codes see 8-15**(a)** From Problem 8-13(a),

$$\mathbf{M} = \begin{bmatrix} 0 & -3 & 1 \\ -3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Then,

$$\mathbf{V} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \quad \mathbf{Q} = (\mathbf{MV})^{-1} = \begin{bmatrix} 0.5 & 1 & 3 \\ 0.5 & 1.5 & 4 \\ -0.5 & -1 & -2 \end{bmatrix}$$

(b) From Problem 8-13(b),

$$\mathbf{M} = \begin{bmatrix} 16 & 7 & 1 \\ 7 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 3 & 1 \\ 2 & 5 & 1 \end{bmatrix} \quad \mathbf{Q} = (\mathbf{MV})^{-1} = \begin{bmatrix} 0.2308 & 0.3077 & 1.0769 \\ 0.1538 & 0.5385 & 1.3846 \\ -0.2308 & -0.3077 & -0.0769 \end{bmatrix}$$

(c) From Problem 8-13(c),

$$\mathbf{V} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -4 & 0 \end{bmatrix}$$

Since \mathbf{V} is singular, the OCF transformation cannot be conducted.

(d) From Problem 8-13(d),

$$\mathbf{M} = \begin{bmatrix} 3 & 3 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Then,

$$\mathbf{V} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & -2 & 2 \end{bmatrix} \quad \mathbf{Q} = (\mathbf{MV})^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

(e) From Problem 8-13(e),

$$\mathbf{M} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{Then,} \quad \mathbf{V} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \mathbf{Q} = (\mathbf{MV})^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}$$

8-15) (a) Eigenvalues of \mathbf{A} : 1, 2.7321, -0.7321

$$\mathbf{T} = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3] = \begin{bmatrix} 0 & 0.5591 & 0.8255 \\ 0 & 0.7637 & -0.3022 \\ 1 & -0.3228 & 0.4766 \end{bmatrix}$$

where \mathbf{p}_1 , \mathbf{p}_2 , and \mathbf{p}_3 are the eigenvectors.

(b) Eigenvalues of \mathbf{A} : 1, 2.7321, -0.7321

$$\mathbf{T} = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3] = \begin{bmatrix} 0 & 0.5861 & 0.7546 \\ 0 & 0.8007 & -0.2762 \\ 1 & 0.1239 & 0.5952 \end{bmatrix}$$

where \mathbf{p}_1 , \mathbf{p}_2 , and \mathbf{p}_3 are the eigenvectors.

(c) Eigenvalues of \mathbf{A} : -3, -2, -2. A nonsingular DF transformation matrix \mathbf{T} cannot be found.

(d) Eigenvalues of \mathbf{A} : -1, -1, -1

The matrix \mathbf{A} is already in Jordan canonical form. Thus, the DF transformation matrix \mathbf{T} is the identity matrix \mathbf{I} .

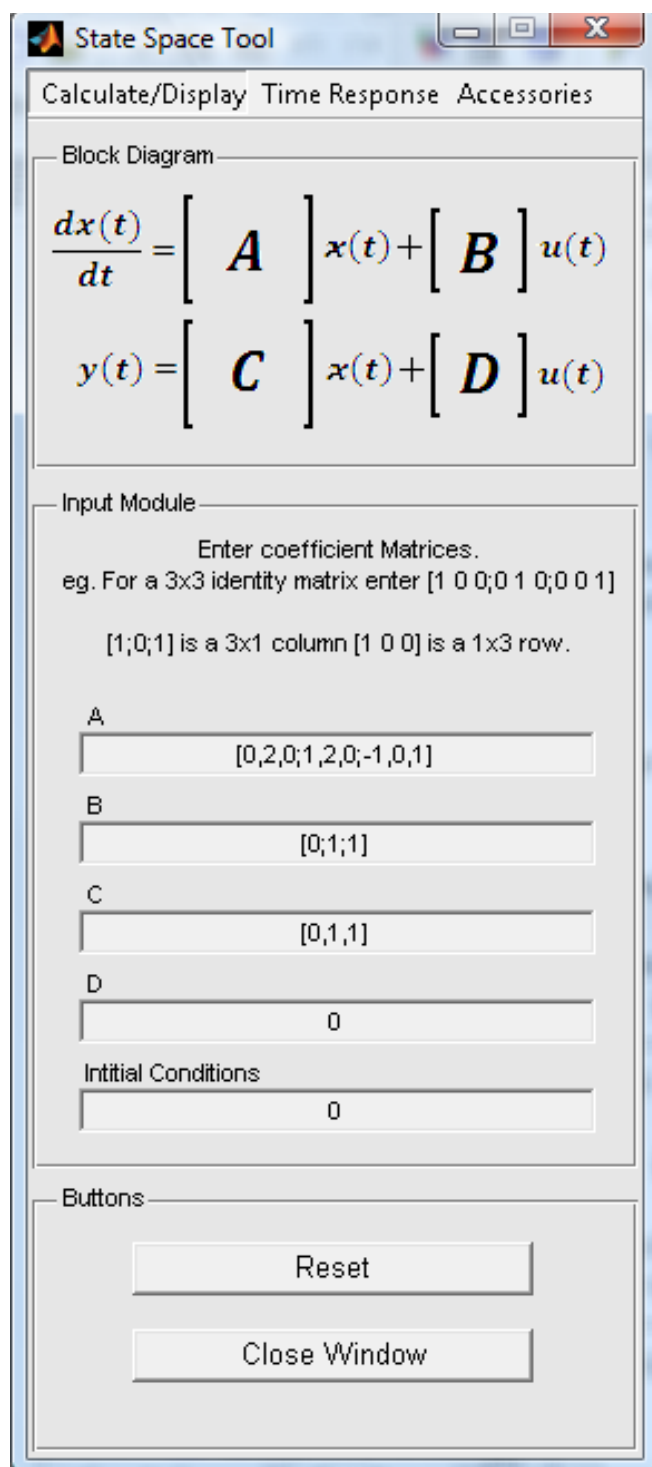
(e) Eigenvalues of \mathbf{A} : 0.4142, -2.4142

$$\mathbf{T} = [\mathbf{p}_1 \quad \mathbf{p}_2] = \begin{bmatrix} 0.8629 & -0.2811 \\ -0.5054 & 0.9597 \end{bmatrix}$$

USE ACSYS as illustrated in section 8-19-1

- 1) Activate MATLAB
- 2) Go to the folder containing ACSYS
- 3) Type in Acsys
- 4) Click the “State Space” pushbutton
- 5) Enter the A,B,C, and D values. Note C must be entered here and must have the same number of columns as A. We use [1,1] arbitrarily as it will not affect the eigenvalues.
- 6) Use the “Calculate/Display” menu and find the eigenvalues.
- 7) Next use the “Calculate/Display” menu and conduct State space calculations.
- 8) Next use the “Calculate/Display” menu and conduct Controllability calculations.

NOTE: the above order of calculations MUST BE followed in the order stated, otherwise you will get an error.

SOLVE PART (a)

State Space Tool

Calculate/Display Time Response Accessories

Block Diagram

$$\frac{dx(t)}{dt} = \begin{bmatrix} A \end{bmatrix} x(t) + \begin{bmatrix} B \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} C \end{bmatrix} x(t) + \begin{bmatrix} D \end{bmatrix} u(t)$$

Input Module

Enter coefficient Matrices.
eg. For a 3x3 identity matrix enter [1 0 0;0 1 0;0 0 1]

[1;0;1] is a 3x1 column [1 0 0] is a 1x3 row.

A

[0,2,0;1,2,0;-1,0,1]

B

[0;1;1]

C

[0,1,1]

D

0

Initial Conditions

0

Buttons

Reset

Close Window

The A matrix is:

A_{mat} =

$$\begin{bmatrix} 0 & 2 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 1 \end{bmatrix}$$

Characteristic Polynomial:

ans =

$$s^3 - 3s^2 + 2$$

Eigenvalues of A = Diagonal Canonical Form of A is:

A_{bar} =

$$\begin{bmatrix} 1.0000 & 0 & 0 \\ 0 & 2.7321 & 0 \\ 0 & 0 & -0.7321 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2.7321 & 0 \\ 0 & 0 & -0.7321 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & -0.7321 \end{bmatrix}$$

Eigen Vectors are

T =

$$\begin{bmatrix} 0 & 0.5591 & 0.8255 \\ 0 & 0.7637 & -0.3022 \\ 1.0000 & -0.3228 & 0.4766 \end{bmatrix}$$

State-Space Model is:

a =

$$\begin{array}{c} \mathbf{x1} \quad \mathbf{x2} \quad \mathbf{x3} \\ \mathbf{x1} \quad 0 \quad 2 \quad 0 \\ \mathbf{x2} \quad 1 \quad 2 \quad 0 \\ \mathbf{x3} \quad -1 \quad 0 \quad 1 \end{array}$$

b =

$$\begin{array}{c} \mathbf{u1} \\ \mathbf{x1} \quad 0 \\ \mathbf{x2} \quad 1 \\ \mathbf{x3} \quad 1 \end{array}$$

c =

$$\begin{array}{c} \mathbf{x1} \quad \mathbf{x2} \quad \mathbf{x3} \\ \mathbf{y1} \quad 0 \quad 1 \quad 1 \end{array}$$

d =

u1

y1 0

Continuous-time model.

Characteristic Polynomial:

ans =

s^3-3*s^2+2

Equivalent Transfer Function Model is:

Transfer function:

2 s^2 - 3 s - 4

s^3 - 3 s^2 + 8.882e-016 s + 2

Pole, Zero Form:

Zero/pole/gain:

2 (s-2.351) (s+0.8508)

$$(s-2.732)(s-1)(s+0.7321)$$

The Controllability Matrix $[B \ AB \ A^2B \ \dots]$ is =

$$S_{mat} =$$

$$\begin{bmatrix} 0 & 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$$

The system is therefore Controllable, rank of S Matrix is =

$$\text{rank}S =$$

$$3$$

$$M_{mat} =$$

$$\begin{bmatrix} 0 & -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

The Controllability Canonical Form (CCF) Transformation matrix is:

P_{tran} =

$$\begin{bmatrix} -2 & 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -4 & -2 & 1 \end{bmatrix}$$

The transformed matrices using CCF are:

A_{bar} =

$$\begin{bmatrix} 0 & 1.0000 & 0.0000 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -0.0000 & 1.0000 \end{bmatrix}$$

$$\begin{bmatrix} -2.0000 & 0.0000 & 3.0000 \end{bmatrix}$$

B_{bar} =

$$\begin{bmatrix} 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \end{bmatrix}$$

C_{bar} =

$$\begin{bmatrix} -4 & -3 & 2 \end{bmatrix}$$

D_{bar} =

$$\begin{bmatrix} 0 \end{bmatrix}$$

8-16) a) $\frac{Y(s)}{U(s)} = \frac{s^2-1}{s^2(s^2-2)}$

Consider:

$$Y(s) = (s^{-2} - s^{-4})X(s)$$

$$X(s) = U(s) - 2s^{-2}X(s) = U(s) + 2s^{-2}X(s)$$

Therefore:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

As $\frac{Y(s)}{U(s)} = \frac{s^2-1}{s^2(s^2-2)}$, therefore $sY(s) = \frac{2}{s}Y(s) + \frac{U(s)}{s} - \frac{U(s)}{s^3}$

Let $X_2(s) = \frac{2}{s}Y(s) + \frac{U(s)}{s} - \frac{U(s)}{s^3}$. If $y = x_1$, then $sY(s) = sX_1(s) = X_2$, or $\dot{x}_1 = x_2$. As a result:

$$sX_2 = 2X_1 + U(s) - \frac{U(s)}{s^2}$$

Now consider $X_3 = -\frac{U(s)}{s^2}$, and $sX_3 = \frac{U}{s} = X_4$, then

$$\dot{x}_2 = 2x_1 - x_3 + u$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = u$$

Therefore:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} u$$

b) $\frac{Y(s)}{U(s)} = \frac{2s+1}{s^2+4s+4}$

Consider:

$$Y(s) = (2s^{-1} + s^{-2})X(s)$$

$$X(s) = U(s) - (4s^{-1} + 4s^{-2})X(s)$$

Therefore:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

As $\frac{Y(s)}{U(s)} = \frac{2s+1}{s^2+4s+4}$, therefore $sY(s) = -4Y(s) + \frac{4}{s}Y(s) + 2U(s) + \frac{U(s)}{s}$. As a result:

$$\begin{cases} y = x_1 \rightarrow \dot{x}_1 = -4x_1 + 2u + x_2 \\ X_2 = \frac{21}{s}Y(s) + \frac{U(s)}{s} \rightarrow sX_2 = 4Y(s) + U(s) \rightarrow \dot{x}_2 = 4x_1 + u \end{cases}$$

8-17) For MATLAB codes see 8-15

(a)

$$\mathbf{S} = [\mathbf{B} \quad \mathbf{AB}] = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \quad \mathbf{S} \text{ is singular.}$$

(b)

$$\mathbf{S} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B}] = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 3 & -3 & 3 \end{bmatrix} \quad \mathbf{S} \text{ is singular.}$$

(c)

$$\mathbf{S} = [\mathbf{B} \quad \mathbf{AB}] = \begin{bmatrix} 2 & 2+2\sqrt{2} \\ \sqrt{2} & 2+\sqrt{2} \end{bmatrix} \quad \mathbf{S} \text{ is singular.}$$

(d)

$$\mathbf{S} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B}] = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 0 & 0 \\ 1 & -4 & 14 \end{bmatrix} \quad \mathbf{S} \text{ is singular.}$$

8-18) a, d and e are controllable

b, c, and f are not controllable

USE ACSYS as illustrated in section 8-19-1

- 9) Activate MATLAB
- 10) Go to the folder containing ACSYS
- 11) Type in Acsys
- 12) Click the “State Space” pushbutton
- 13) Enter the A,B,C, and D values. Note C must be entered here and must have the same number of columns as A. We use [1,1] arbitrarily as it will not affect the eigenvalues.
- 14) Use the “Calculate/Display” menu and find the eigenvalues.
- 15) Next use the “Calculate/Display” menu and conduct State space calculations.
- 16) Next use the “Calculate/Display” menu and conduct Controllability calculations.

NOTE: the above order of calculations MUST BE followed in the order stated, otherwise you will get an error.

State Space Tool

Calculate/Display Time Response Accessories

Block Diagram

$$\frac{dx(t)}{dt} = \begin{bmatrix} A \end{bmatrix} x(t) + \begin{bmatrix} B \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} C \end{bmatrix} x(t) + \begin{bmatrix} D \end{bmatrix} u(t)$$

Input Module

Enter coefficient Matrices.
eg. For a 3x3 identity matrix enter [1 0 0; 0 1 0; 0 0 1]

[1;0;1] is a 3x1 column [1 0 0] is a 1x3 row.

A
[-1,0;0,-2]

B
[2;0]

C
[1,1]

D
0

Initial Conditions
0

Buttons

Reset

Close Window

For part b, the system is not Controllable because $[B \ AB]$ is singular (rank is less than 2):

The A matrix is:

A_{mat} =

$$\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2 \end{bmatrix}$$

Characteristic Polynomial:

ans =

$$s^2 + 3s + 2$$

Eigenvalues of A = Diagonal Canonical Form of A is:

A_{bar} =

$$\begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \end{bmatrix}$$

Eigen Vectors are

T =

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix}$$

Characteristic Polynomial:

ans =

$$s^2 + 3s + 2$$

Equivalent Transfer Function Model is:

Transfer function:

$$\frac{2}{s + 1}$$

$$s + 1$$

Pole, Zero Form:

Zero/pole/gain:

2

(s+1)

The Controllability Matrix $[B \ AB \ A^2B \ \dots]$ is =

Smat =

2 -2

0 0

← Rank is 1, and this is a singular matrix

The system is therefore Not Controllable, rank of S Matrix is =

rankS =

1

Mmat =

3 1

1 0

The Controllability Canonical Form (CCF) Transformation matrix is:

Ptran =

4 2

0 0

8-19) a, d, and e are observable

b, c, and f are not observable

Using ACSYS (also see the previous problem for more details):

State Space Tool

Calculate/Display Time Response Accessories

Block Diagram

$$\frac{dx(t)}{dt} = \begin{bmatrix} A \end{bmatrix} x(t) + \begin{bmatrix} B \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} C \end{bmatrix} x(t) + \begin{bmatrix} D \end{bmatrix} u(t)$$

Input Module

Enter coefficient Matrices.
eg. For a 3x3 identity matrix enter [1 0 0; 0 1 0; 0 0 1]

[1;0;1] is a 3x1 column [1 0 0] is a 1x3 row.

A
[-1,0;0,-2]

B
[2;0]

C
[0,1]

D
0

Initial Conditions
0

Buttons

Reset

Close Window

For part b, the system is not observable. Note: you must choose a B matrix arbitrarily.

The A matrix is:

Amat =

-1 0

0 -2

Characteristic Polynomial:

ans =

$s^2 + 3s + 2$

Eigenvalues of A = Diagonal Canonical Form of A is:

$A_{bar} =$

$$\begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Eigen Vectors are

$T =$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Characteristic Polynomial:

ans =

$$s^2 + 3s + 2$$

Equivalent Transfer Function Model is:

Transfer function:

$$0$$

Pole, Zero Form:

Zero/pole/gain:

$$0$$

The Observability Matrix (transpose:[C CA CA² ...]) is =

Vmat =

$$\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$$

The System is therefore Not Observable, rank of V Matrix is =

rankV =

$$1$$

Mmat =

$$\begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$$

8-20) (a) Rewrite the differential equations as:

$$\frac{d^2\theta_m}{dt^2} = -\frac{B}{J} \frac{d\theta_m}{dt} - \frac{K}{J} \theta_m + \frac{K_i}{J} i_a \quad \frac{di_a}{dt} = -\frac{K_b}{L_a} \frac{d\theta_m}{dt} - \frac{R_a}{L_a} i_a + \frac{K_a K_s}{L_a} (\theta_r - \theta_m)$$

State variables: $x_1 = \theta_m, \quad x_2 = \frac{d\theta_m}{dt}, \quad x_3 = i_a$

State equations:

Output equation:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{K}{J} & -\frac{B}{J} & \frac{K_i}{J} \\ -\frac{K_a K_s}{L_a} & -\frac{K_b}{L_a} & -\frac{R_a}{L_a} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{K_a K_s}{L_a} \end{bmatrix} \theta_r \quad y = [1 \quad 0 \quad 0] \mathbf{x} = x_1$$

(b) Forward-path transfer function:

$$G(s) = \frac{\Theta_m(s)}{E(s)} = [1 \quad 0 \quad 0] \begin{bmatrix} s & -1 & 0 \\ \frac{K}{J} & s + \frac{B}{J} & -\frac{K_i}{J} \\ 0 & \frac{K_b}{L_a} & s + \frac{R_a}{L_a} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \frac{K_a}{L_a} \end{bmatrix} = \frac{K_i K_a}{\Delta_o(s)}$$

$$\Delta_o(s) = JL_a s^3 + (BL_a + R_a J) s^2 + (KL_a + K_i K_b + R_a B) s + KR_a = 0$$

Closed-loop transfer function:

$$M(s) = \frac{\Theta_m(s)}{\Theta_r(s)} = [1 \quad 0 \quad 0] \begin{bmatrix} s & -1 & 0 \\ \frac{K}{J} & s + \frac{B}{J} & -\frac{K_i}{J} \\ \frac{K_a K_s}{L_a} & \frac{K_b}{L_a} & s + \frac{R_a}{L_a} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \frac{K_a K_s}{L_a} \end{bmatrix} = \frac{K_s G(s)}{1 + K_s(s)}$$

$$= \frac{K_i K_a K_s}{J L_a s^3 + (B L_a + R_a J) s^2 + (K L_a + K_i K_b + R_a B) s + K_i K_a K_s + K R_a}$$

8-21) (a)

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \mathbf{A}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \mathbf{A}^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \mathbf{A}^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(1) Infinite series expansion:

$$\phi(t) = \mathbf{I} + \mathbf{A}t + \frac{1}{2!} \mathbf{A}^2 t^2 + \dots = \begin{bmatrix} 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots & t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots \\ -t + \frac{t^3}{3!} - \frac{t^5}{5!} + \dots & 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

(2) Inverse Laplace transform:

$$\Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}^{-1} = \frac{1}{s^2 + 1} \begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix} \quad \phi(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

(b)

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \quad \mathbf{A}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \quad \mathbf{A}^3 = \begin{bmatrix} -1 & 0 \\ 0 & -8 \end{bmatrix} \quad \mathbf{A}^4 = \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix}$$

(1) Infinite series expansion:

$$\phi(t) = \mathbf{I} + \mathbf{A}t + \frac{1}{2!} \mathbf{A}^2 t^2 + \dots = \begin{bmatrix} 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \dots & 0 \\ 0 & 1 - 2t + \frac{4t^2}{2!} - \frac{8t^3}{3!} + \dots \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$$

(2) Inverse Laplace transform:

$$\Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s+1 & 0 \\ 0 & s+2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+2} \end{bmatrix} \quad \phi(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$$

(c)

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \mathbf{A}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{A}^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \mathbf{A}^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(1) Infinite series expansion:

$$\phi(t) = \mathbf{I} + \mathbf{A}t + \frac{1}{2!} \mathbf{A}^2 t^2 + \dots = \begin{bmatrix} 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \dots & t + \frac{t^3}{3!} + \frac{t^5}{5!} + \dots \\ t + \frac{t^3}{3!} + \frac{t^5}{5!} + \dots & 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \dots \end{bmatrix} = \begin{bmatrix} e^{-t} + e^t & -e^{-t} + e^t \\ -e^{-t} + e^t & e^{-t} + e^t \end{bmatrix}$$

(2) Inverse Laplace transform:

$$\Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s & -1 \\ -1 & s \end{bmatrix}^{-1} = \frac{1}{s^2 - 1} \begin{bmatrix} s & 1 \\ 1 & s \end{bmatrix} = \begin{bmatrix} \frac{0.5}{s+1} - \frac{0.5}{s-1} & \frac{-0.5}{s+1} + \frac{0.5}{s-1} \\ \frac{-0.5}{s+1} + \frac{0.5}{s-1} & \frac{0.5}{s+1} + \frac{0.5}{s-1} \end{bmatrix}$$

$$\phi(t) = 0.5 \begin{bmatrix} e^{-t} + e^t & -e^{-t} + e^t \\ -e^{-t} + e^t & e^{-t} + e^t \end{bmatrix}$$

$$\mathbf{8-22) (a)} \quad e = K_s (\theta_r - \theta_y) \quad e_a = e - e_s \quad e_s = R_s i_a \quad e_u = K e_a$$

$$i_a = \frac{e_u - e_b}{R_a + R_s} \quad e_b = K_b \frac{d\theta_y}{dt} \quad T_m = K i_a = (J_m + J_L) \frac{d^2 \theta_y}{dt^2}$$

Solve for i_a in terms of θ_y and $\frac{d\theta_y}{dt}$, we have

$$i_a = \frac{KK_s (\theta_r - \theta_y) - K_b \frac{d\theta_y}{dt}}{R_s + R_a + KK_s}$$

Differential equation:

$$\frac{d^2\theta_y}{dt^2} = \frac{K_i i_a}{J_m + J_L} = \frac{K_i}{(J_m + J_L)(R_a + R_s + KR_s)} \left(-K_b \frac{d\theta_y}{dt} - KK_s \theta_y + KK_s \theta_r \right)$$

State variables: $x_1 = \theta_y, \quad x_2 = \frac{d\theta_y}{dt}$

State equations:

$$\begin{aligned} \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ \frac{-KK_s K_i}{(J_m + J_L)(R_a + R_s + KR_s)} & \frac{-K_b K_i}{(J_m + J_L)(R_a + R_s + KR_s)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-KK_s K_i}{(J_m + J_L)(R_a + R_s + KR_s)} \end{bmatrix} \theta_r \\ &= \begin{bmatrix} 0 & 1 \\ -322.58 & -80.65 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 322.58 \end{bmatrix} \theta_r \end{aligned}$$

We can let $v(t) = 322.58\theta_r$, then the state equations are in the form of CCF.

(b)

$$\begin{aligned} (s\mathbf{I} - \mathbf{A})^{-1} &= \begin{bmatrix} s & -1 \\ 322.58 & s + 80.65 \end{bmatrix}^{-1} = \frac{1}{s^2 + 80.65s + 322.58} \begin{bmatrix} s + 80.65 & 1 \\ -322.58 & s \end{bmatrix} \\ &= \begin{bmatrix} \frac{-0.06}{s + 76.42} - \frac{1.059}{s + 4.22} & \frac{-0.014}{s + 76.42} + \frac{0.014}{s + 4.22} \\ \frac{4.468}{s + 76.42} - \frac{4.468}{s + 4.22} & \frac{1.0622}{s + 76.42} - \frac{0.0587}{s + 4.22} \end{bmatrix} \end{aligned}$$

For a unit-step function input, $u_s(t) = 1/s$.

$$\begin{aligned} (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} \frac{1}{s} &= \begin{bmatrix} \frac{322.2}{s(s + 76.42)(s + 4.22)} \\ \frac{322.2}{s(s + 76.42)(s + 4.22)} \end{bmatrix} = \begin{bmatrix} \frac{1}{s} + \frac{0.0584}{s + 76.42} - \frac{1.058}{s + 4.22} \\ \frac{-4.479}{s + 76.42} + \frac{4.479}{s + 4.22} \end{bmatrix} \\ \mathbf{x}(t) &= \begin{bmatrix} -0.06e^{-76.42t} - 1.059e^{-4.22t} & -0.014e^{-76.42t} + 0.01e^{-4.22t} \\ 4.468e^{-76.42t} - 4.468e^{-4.22t} & 1.0622e^{-76.42t} - 0.0587e^{-4.22t} \end{bmatrix} \mathbf{x}(0) \\ &= \begin{bmatrix} 1 + 0.0584e^{-76.42t} - 1.058e^{-4.22t} \\ -4.479e^{-76.42t} + 4.479e^{-4.22t} \end{bmatrix} \quad t \geq 0 \end{aligned}$$

(c) Characteristic equation: $\Delta(s) = s^2 + 80.65s + 322.58 = 0$

(d) From the state equations we see that whenever there is R_a there is $(1+K)R_s$. Thus, the purpose of R_s is to increase the effective value of R_a by $(1+K)R_s$. This improves the time constant of the system.

8-23) (a) State equations:

$$\begin{aligned} \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ \frac{-KK_s K_i}{J(R+R_s+KR_s)} & \frac{-K_b K_i}{J(R+R_s+KR_s)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{KK_s K_i}{J(R+R_s+KR_s)} \end{bmatrix} \theta_r \\ &= \begin{bmatrix} 0 & 1 \\ -818.18 & -90.91 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 818.18 \end{bmatrix} \theta_r \end{aligned}$$

Let $v = 818.18\theta_r$. The equations are in the form of CCF with v as the input.

$$\text{(b)} \quad (s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s & -1 \\ 818.18 & s + 90.91 \end{bmatrix}^{-1} = \frac{1}{(s + 10.128)(s + 80.782)} \begin{bmatrix} s + 90.91 & 1 \\ -818.18 & s \end{bmatrix}$$

$$\begin{aligned} \mathbf{x}(t) &= \begin{bmatrix} 1.143e^{-10.128t} - 0.142e^{-80.78t} & 0.01415e^{-10.128t} - 0.0141e^{-80.78t} \\ -11.58e^{-10.128t} + 0.1433e^{-80.78t} & -0.1433e^{-10.128t} + 1.143e^{-80.78t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \\ &+ \begin{bmatrix} 11.58e^{-10.128t} - 11.58e^{-80.78t} \\ 1 - 1.1434e^{-10.128t} + 0.1433e^{-80.78t} \end{bmatrix} v \quad t \geq 0 \end{aligned}$$

(c) Characteristic equation: $\Delta(s) = s^2 + 90.91s + 818.18 = 0$

Eigenvalues: $-10.128, -80.782$

(d) Same remark as in part (d) of Problem 5-14.

8-24) If $\dot{x} = Ax$ and P diagonalizing A , let consider $x = P\hat{x}$, therefore $\dot{x} = P\dot{\hat{x}}$ or $\dot{\hat{x}} = P^{-1}AP\hat{x} = D\hat{x}$

The solution for \hat{x} is $\hat{x} = e^{Dt}\hat{x}(0)$, therefore

$$x(t) = P\hat{x}(t) = Pe^{Dt}P^{-1}x(0) \quad (1)$$

on the other hand

$$x(t) = e^{At}x(0) \quad (2)$$

From equation (1) and (2):

$$e^{At} = P e^{Dt} P^{-1}$$

8-25) Consider $\dot{x} = Ax$ and $s^{-1}As = J$. If $x = S\hat{x}$, then $\dot{x} = S\dot{\hat{x}}$ or $\dot{\hat{x}} = s^{-1}As\hat{x} = J\hat{x}$

The solution for \hat{x} is $\hat{x}(t) = e^{Jt}\hat{x}(0)$, therefore:

$$x(t) = s\hat{x}(t) = se^{Jt}s^{-1}x(0) \quad (1)$$

On the other hand:

$$x(t) = e^{At}x(0) \quad (2)$$

From equation (1) and (2):

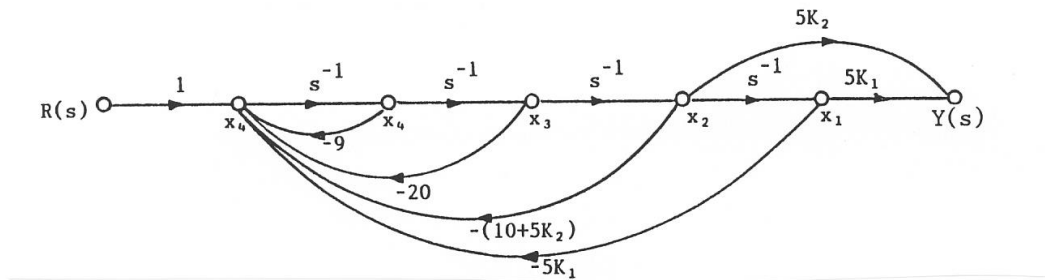
$$e^{At} = se^{Jt}s^{-1}$$

8-26 (a) Forward-path transfer function:

Closed-loop transfer function:

$$G(s) = \frac{Y(s)}{E(s)} = \frac{5(K_1 + K_2s)}{s[s(s+4)(s+5)+10]} \quad M(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{5(K_1 + K_2s)}{s^4 + 9s^3 + 20s^2 + (10 + 5K_2)s + 5K_1}$$

(b) State diagram by direct decomposition:



State equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5K_1 & -(10+5K_2) & -20 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r$$

Output equation:

$$y = [5K_1 \quad 5K_2 \quad 0] \mathbf{x}$$

(c) Final value: $r(t) = u_s(t)$, $R(s) = \frac{1}{s}$.

$$\lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} sY(s) = \lim_{s \rightarrow 0} \frac{5(K_1 + K_2 s)}{s^4 + 9s^3 + 20s^2 + (10 + 5K_2)s + 5K_1} = 1$$

8-27 In CCF form,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & -1 & 0 & 0 & \cdots & 0 \\ 0 & s & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & -1 \\ a_0 & a_1 & a_2 & a_3 & \cdots & s + a_n \end{bmatrix}$$

$$|s\mathbf{I} - \mathbf{A}| = s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1s + a_0$$

Since \mathbf{B} has only one nonzero element which is in the last row, only the last column of $\text{adj}(s\mathbf{I} - \mathbf{A})$ is

going to contribute to $\text{adj}(s\mathbf{I} - \mathbf{A})\mathbf{B}$. The last column of $\text{adj}(s\mathbf{I} - \mathbf{A})$ is obtained from the cofactors of

the last row of $(s\mathbf{I} - \mathbf{A})$. Thus, the last column of $\text{adj}(s\mathbf{I} - \mathbf{A})\mathbf{B}$ is $\begin{bmatrix} 1 & s & s^2 & \cdots & s^{n-1} \end{bmatrix}^T$.

8-28 (a) State variables: $x_1 = y$, $x_2 = \frac{dy}{dt}$, $x_3 = \frac{d^2y}{dt^2}$

State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}r(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(b) State transition matrix:

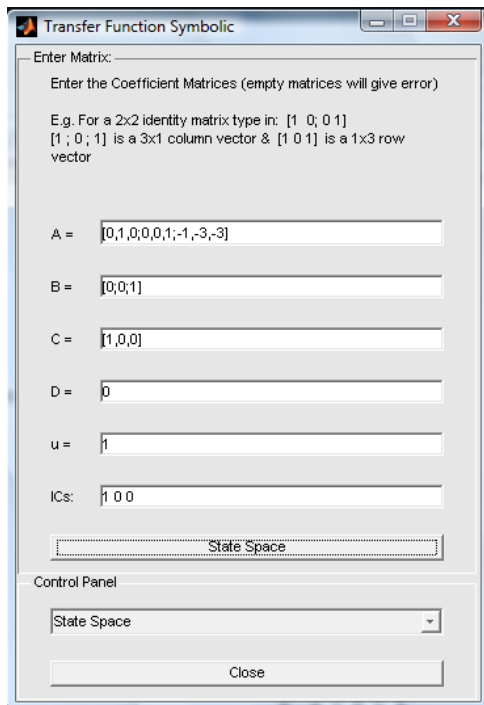
$$\Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 3 & s+3 \end{bmatrix}^{-1} = \frac{1}{\Delta(s)} \begin{bmatrix} s^2 + 3s + 3 & s + 3 & 1 \\ -1 & s^2 + 3s & s \\ -s & -3s - 1 & s^2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{1}{(s+1)^3} & \frac{1}{(s+1)^2} + \frac{2}{(s+1)^3} & \frac{1}{(s+1)^3} \\ \frac{-1}{(s+1)^3} & \frac{1}{s+1} + \frac{1}{(s+1)^2} - \frac{2}{(s+1)^3} & \frac{s}{(s+1)^3} \\ \frac{-s}{(s+1)^3} & \frac{-3}{(s+1)^2} + \frac{2}{(s+1)^3} & \frac{s^2}{(s+1)^3} \end{bmatrix}$$

$$\Delta(s) = s^3 + 3s^2 + 3s + 1 = (s+1)^3$$

$$\phi(t) = \begin{bmatrix} (1+t+t^2/2)e^{-t} & (t+t^2)e^{-t} & t^2e^{-t}/2 \\ -t^2e^{-t}/2 & (1+t-t^2)e^{-t} & (t-t^2/2)e^{-t} \\ (-t+t^2/2)e^{-t} & t^2e^{-t} & (1-2t+t^2/2)e^{-t} \end{bmatrix}$$

(c) Use ACSYS or MATLAB and follow the procedure shown in solution to 8-3.



State Space Analysis

Inputs:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

State Space Representation:

$$\dot{D}x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \end{bmatrix} u$$

Determinant of $(sI - A)$:

$$\begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 3 & s+3 \end{vmatrix} = s^3 + 3s^2 + 3s + 1$$

Characteristic Equation of the Transfer Function:

$$s^3 + 3s^2 + 3s + 1 = 0$$

$$s^3 + 3s^2 + 3s + 1$$

The eigen values of A and poles of the Transfer Function are:

$$\begin{matrix} -1 \\ -1 \\ -1 \end{matrix}$$

Inverse of (s*I-A) is:

$$\begin{bmatrix} 2 & & \\ s^2 + 3s + 3 & s + 3 & 1 \\ \hline \%1 & \%1 & \%1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & s(s+3) & s \\ \hline \%1 & \%1 & \%1 \end{bmatrix}$$

$$\begin{bmatrix} s & 3s+1 & s \\ \hline \%1 & \%1 & \%1 \end{bmatrix}$$

$$\%1 := s^3 + 3s^2 + 3s + 1$$

State transition matrix (phi) of A:

$$\begin{bmatrix} 1/2 \exp(-t) (2 + 2t + t^2), (t + t^2) \exp(-t), 1/2 t^2 \exp(-t) \\ -1/2 t^2 \exp(-t), -(t - 1 + t^2) \exp(-t), -1/2 \exp(-t) (-2t + t^2) \\ 1/2 \exp(-t) (-2t + t^2), \exp(-t) (-3t + t^2), 1/2 \exp(-t) (2 - 4t + t^2) \end{bmatrix}$$

Transfer function between u(t) and y(t) is:

$$\frac{1}{s^3 + 3s^2 + 3s + 1}$$

No Initial Conditions Specified

States (X) in Laplace Domain:

$$\begin{bmatrix} 1 \\ \hline 3 \\ (s+1) \end{bmatrix}$$

$$\begin{bmatrix} s \\ \hline 3 \\ (s+1) \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ s \\ \hline 3 \\ (s+1) \end{bmatrix}$$

Inverse Laplace x(t):

$$\begin{bmatrix} 2 \\ 1/2 t \exp(-t) \\ \hline 2 \\ -1/2 \exp(-t) (-2t + t) \\ \hline 2 \\ 1/2 \exp(-t) (2 - 4t + t) \end{bmatrix}$$

Output Y(s):

$$\frac{1}{(s+1)^3}$$

Inverse Laplace y(t):

$$\frac{1}{2} t^2 \exp(-t)$$

State Space Analysis

Inputs:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

State Space Representation:

$$Dx = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \end{bmatrix} u$$

Determinant of $(sI-A)$:

$$s^3 + 3s^2 + 3s + 1$$

Characteristic Equation of the Transfer Function:

$$s^3 + 3s^2 + 3s + 1$$

The eigen values of A and poles of the Transfer Function are:

$$\begin{matrix} -1 \\ -1 \\ -1 \end{matrix}$$

Inverse of $(sI-A)$ is:

$$\begin{bmatrix} 2 & & \\ s^2 + 3s + 3 & s + 3 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & s(s+3) & s \\ \hline 1 & 1 & 1 \\ \hline 2 & & \\ s & 3s+1 & s \\ \hline 1 & 1 & 1 \end{bmatrix}$$

$$\%1 := s^3 + 3s^2 + 3s + 1$$

State transition matrix (phi) of A:

$$\begin{bmatrix} 1/2 \exp(-t) (2 + 2t + t^2), (t + t^2) \exp(-t), 1/2 t^2 \exp(-t) \end{bmatrix}$$

$$\begin{bmatrix} -1/2 t^2 \exp(-t), -(t - 1 + t^2) \exp(-t), -1/2 \exp(-t) (-2t + t^2) \end{bmatrix}$$

$$\begin{bmatrix} \\ \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 1/2 \exp(-t) (-2t + t), \exp(-t) (-3t + t), \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 1/2 \exp(-t) (2 - 4t + t^2) \end{bmatrix}$$

Transfer function between $u(t)$ and $y(t)$ is:

$$\frac{1}{s^3 + 3s^2 + 3s + 1}$$

Initial Conditions:

$$\begin{bmatrix} x(0) = 1 \\ 0 \\ 0 \end{bmatrix}$$

States (X) in Laplace Domain:

$$\begin{bmatrix} 2 \\ s^2 + 3s + 4 \end{bmatrix} \quad \begin{bmatrix} 3 \\ (s+1) \end{bmatrix} \quad \begin{bmatrix} 3 \\ (s+1) \end{bmatrix}$$

Inverse Laplace $x(t)$:

$$\begin{bmatrix} 2 \\ (t+1+t^2) \exp(-t) \end{bmatrix} \quad \begin{bmatrix} 3 \\ -(t+1+t^2) \exp(-t) \end{bmatrix} \quad \begin{bmatrix} 3 \\ (-3t+1+t^2) \exp(-t) \end{bmatrix}$$

Output $Y(s)$ (with initial conditions):

$$\frac{2}{s^2 + 3s + 4}$$

$$\frac{3}{(s+1)}$$

Inverse Laplace $y(t)$:

$$\frac{2}{(t+1+t)} \exp(-t)$$

(d) Characteristic equation: $\Delta(s) = s^3 + 3s^2 + 3s + 1 = 0$

Eigenvalues: $-1, -1, -1$

8-29 (a) State variables: $x_1 = y, \quad x_2 = \frac{dy}{dt}$

State equations:

$$\begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t)$$

State transition matrix:

$$\Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s & -1 \\ 1 & s+2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{s+2}{(s+1)^2} & \frac{1}{(s+1)^2} \\ \frac{-1}{(s+1)^2} & \frac{s}{(s+1)^2} \end{bmatrix} \quad \phi(t) = \begin{bmatrix} (1+t)e^{-t} & te^{-t} \\ -te^{-t} & (1-t)e^{-t} \end{bmatrix}$$

Characteristic equation: $\Delta(s) = (s+1)^2 = 0$

(b) State variables: $x_1 = y, \quad x_2 = y + \frac{dy}{dt}$

State equations:

$$\frac{dx_1}{dt} = \frac{dy}{dt} = x_2 - y = x_2 - x_1 \quad \frac{dx_2}{dt} = \frac{d^2y}{dt^2} + \frac{dy}{dt} = -y - \frac{dy}{dt} + r = -x_2 + r$$

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

State transition matrix:

$$\Phi(s) = \begin{bmatrix} s+1 & -2 \\ 0 & s+1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s+1} & \frac{-2}{(s+1)^2} \\ 0 & \frac{1}{s+1} \end{bmatrix} \quad \phi(t) = \begin{bmatrix} e^{-t} & -te^{-t} \\ 0 & e^{-t} \end{bmatrix}$$

(c) Characteristic equation: $\Delta(s) = (s+1)^2 = 0$ which is the same as in part (a).

8-30 (a) State transition matrix:

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s - \sigma & \omega \\ -\omega & s - \sigma \end{bmatrix} \quad (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{\Delta(s)} \begin{bmatrix} s - \sigma & -\omega \\ \omega & s - \sigma \end{bmatrix} \quad \Delta(s) = s^2 - 2\sigma s + (\sigma^2 + \omega^2)$$

$$\phi(t) = \mathcal{L}^{-1} \left[(s\mathbf{I} - \mathbf{A})^{-1} \right] = \begin{bmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{bmatrix} e^{\sigma t}$$

(b) Eigenvalues of A: $\sigma + j\omega, \sigma - j\omega$

8-31 (a)

$$\frac{Y_1(s)}{U_1(s)} = \frac{s^{-3}}{1 + s^{-1} + 2s^{-2} + 3s^{-3}} = \frac{1}{s^3 + s^2 + 2s + 3}$$

$$\frac{Y_2(s)}{U_2(s)} = \frac{s^{-3}}{1 + s^{-1} + 2s^{-2} + 3s^{-3}} = \frac{1}{s^3 + s^2 + 2s + 3} = \frac{Y_1(s)}{U_1(s)}$$

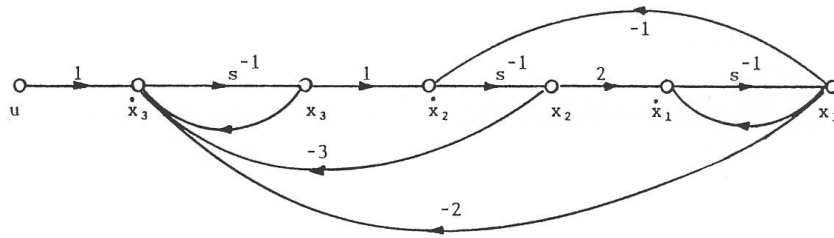
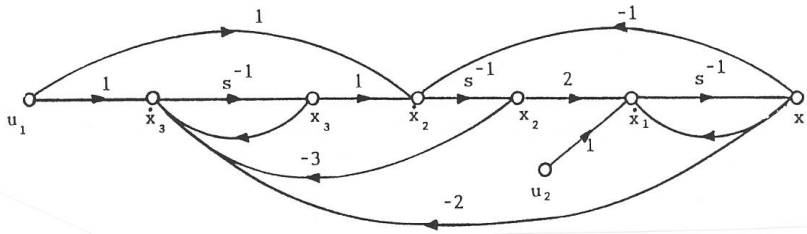
(b) State equations [Fig. 5-21(a)]: $\dot{\mathbf{x}} = \mathbf{A}_1 \mathbf{x} + \mathbf{B}_1 u_1$ **Output equation:** $y_1 = \mathbf{C}_1 \mathbf{x}$

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -2 & -1 \end{bmatrix} \quad \mathbf{B}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{C}_1 = [1 \quad 0 \quad 0]$$

State equations [Fig. 5-21(b)]: $\dot{\mathbf{x}} = \mathbf{A}_2 \mathbf{x} + \mathbf{B}_2 u_2$ **Output equation:** $y_2 = \mathbf{C}_2 \mathbf{x}$

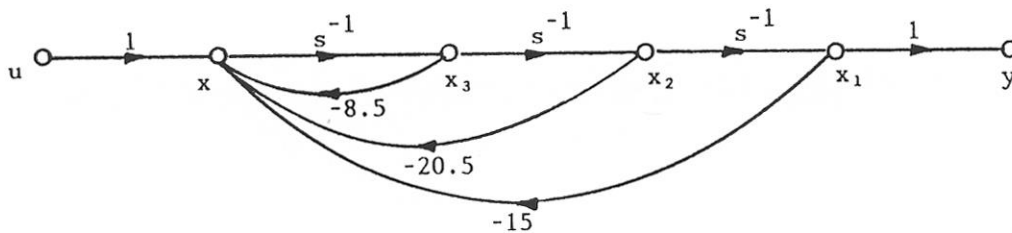
$$\mathbf{A}_2 = \begin{bmatrix} 0 & 0 & -3 \\ 1 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix} \quad \mathbf{B}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{C}_2 = [0 \quad 0 \quad 1]$$

Thus, $\mathbf{A}_2 = \mathbf{A}_1'$

8-32 (a) State diagram:**(b) State diagram:****8-33 (a)**

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10s^{-3}}{1 + 8.5s^{-1} + 20.5s^{-2} + 15s^{-3}} \frac{X(s)}{X(s)} \quad Y(s) = 10X(s)$$

$$X(s) = U(s) - 8.5s^{-1}X(s) - 20.5s^{-2}X(s) - 15s^{-3}X(s)$$

State diagram:

State equation: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

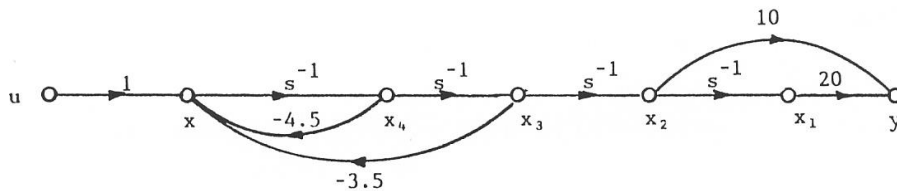
$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -15 & -20.5 & -8.5 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{A} \text{ and } \mathbf{B} \text{ are in CCF}$$

(b)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10s^{-3} + 20s^{-4}}{1 + 4.5s^{-1} + 3.5s^{-2}} \frac{X(s)}{X(s)}$$

$$Y(s) = 10s^{-3}X(s) + 20s^{-4}X(s) \quad X(s) = -4.5s^{-1}X(s) - 3.5s^{-2}X(s) + U(s)$$

State diagram:



State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

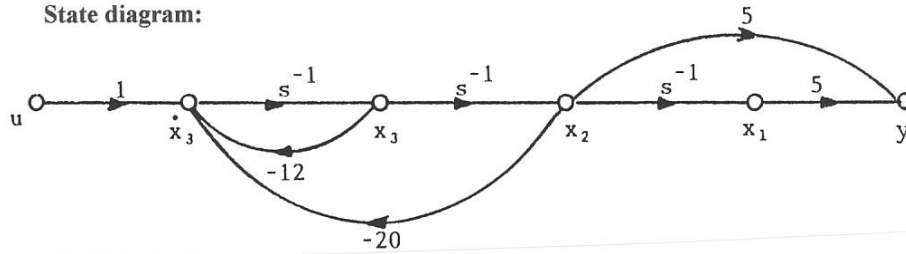
$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -3.5 & -4.5 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{A} \text{ and } \mathbf{B} \text{ are in CCF}$$

(c)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{5(s+1)}{s(s+2)(s+10)} = \frac{5s^{-2} + 5s^{-3}}{1 + 12s^{-1} + 20s^{-2}} \frac{X(s)}{X(s)}$$

$$Y(s) = 5s^{-2}X(s) + 5s^{-3}X(s) \quad X(s) = U(s) - 12s^{-1}X(s) - 20s^{-2}X(s)$$

State diagram:



State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -20 & -12 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

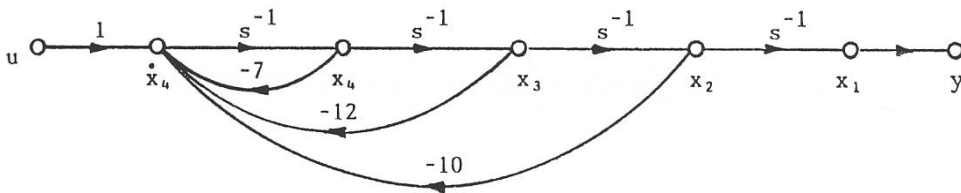
\mathbf{A} and \mathbf{B} are in CCF

(d)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s(s+5)(s^2+2s+2)} = \frac{s^{-4}}{1+7s^{-1}+12s^{-2}+10s^{-3}} \frac{X(s)}{X(s)}$$

$$Y(s) = s^{-4}X(s) \quad X(s) = U(s) - 7s^{-1}X(s) - 12s^{-2}X(s) - 10s^{-3}X(s)$$

State diagram:



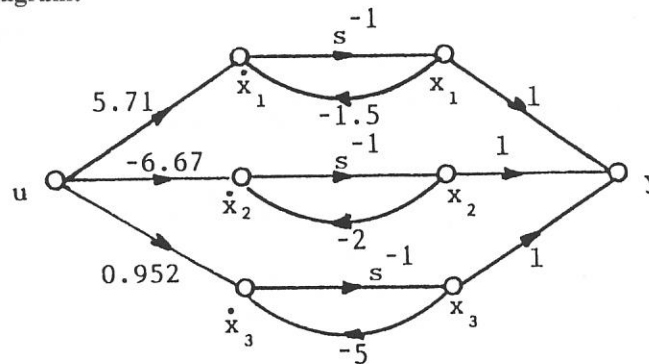
State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -10 & -12 & -7 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{A} \text{ and } \mathbf{B} \text{ are in CCF}$$

8-34 (a)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10}{s^3 + 8.5s^2 + 20.5s + 15} = \frac{5.71}{s+15} - \frac{6.67}{s+2} + \frac{0.952}{s+5}$$

State diagram:

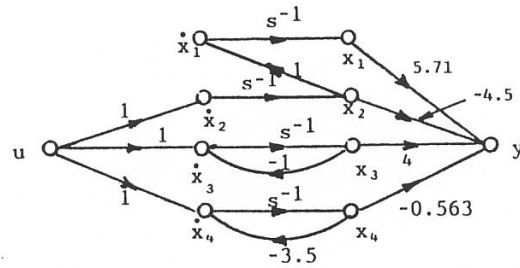
State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} -1.5 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -5 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 5.71 \\ -6.67 \\ 0.952 \end{bmatrix}$$

The matrix \mathbf{B} is not unique. It depends on how the input and the output branches are allocated.**(b)**

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10(s+2)}{s^2(s+1)(s+3.5)} = \frac{-4.5}{s} + \frac{0.49}{s+3.5} + \frac{4}{s+1} + \frac{5.71}{s^2}$$

State diagram:

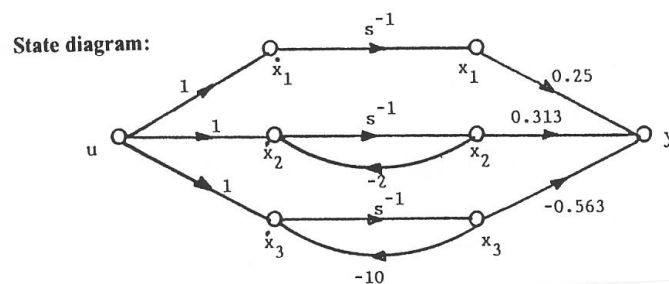


State equation: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3.5 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(b)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{5(s+1)}{s(s+2)(s+10)} = \frac{2.5}{s} + \frac{0.313}{s+2} - \frac{0.563}{s+10}$$



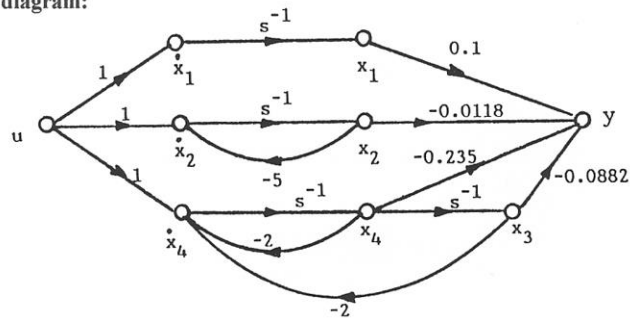
State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -10 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(d)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s(s+5)(s^2+2s+2)} = \frac{0.1}{s} - \frac{0.0118}{s+5} - \frac{0.0882s+0.235}{s^2+2s+2}$$

State diagram:

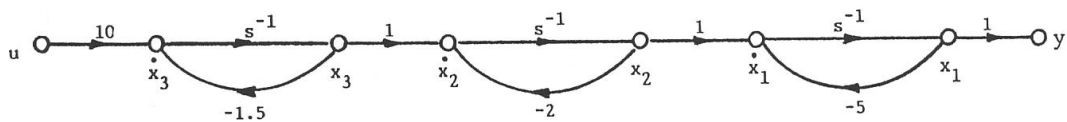
State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

8-35 (a)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10}{(s+1.5)(s+2)(s+5)}$$

State diagram:

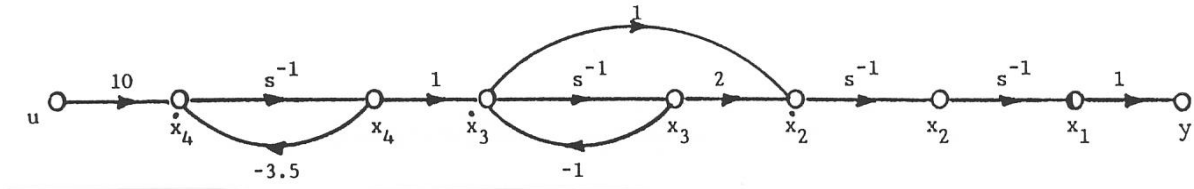
State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} -5 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1.5 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$$

(b)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10(s+2)}{s^2(s+1)(s+3.5)} = \left(\frac{10}{s^2}\right) \left(\frac{s+2}{s+1}\right) \left(\frac{1}{s+3.5}\right)$$

State diagram:



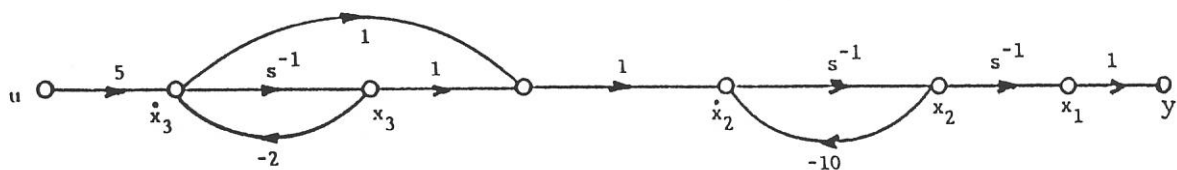
State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -3.5 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 10 \end{bmatrix}$$

(c)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{59s+1}{s(s+2)(s+10)} = \left(\frac{5}{s}\right) \left(\frac{s+1}{s+2}\right) \left(\frac{1}{s+10}\right)$$

State diagram:



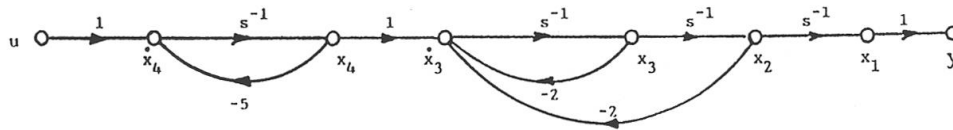
State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -10 & -1 \\ 0 & 0 & -2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$$

(d)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s(s+5)(s^2+2s+2)}$$

State diagram:



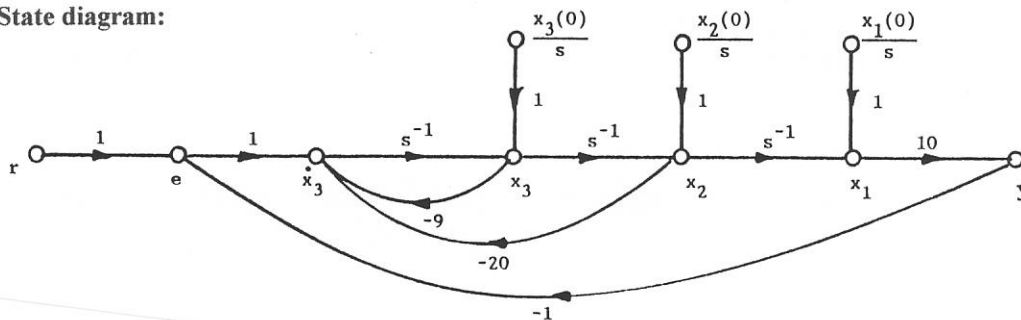
State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & -2 & 1 \\ 0 & 0 & 0 & -5 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

8-36 (a)

$$G(s) = \frac{Y(s)}{E(s)} = \frac{10}{s(s+4)(s+5)} = \frac{10s^{-3}}{1+9s^{-1}+20s^{-2}} \frac{X(s)}{X(s)}$$

State diagram:



(b) Dynamic equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -20 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r \quad y = [10 \quad 0 \quad 0] \mathbf{x}$$

(c) State transition equation:

$$\begin{aligned}
 \begin{bmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{bmatrix} &= \frac{1}{\Delta(s)} \begin{bmatrix} s^{-1}(1+9s^{-1}+20s^{-2}) & s^{-2}(1+9s^{-1}) & s^{-3} \\ -10s^{-3} & s^{-1}(1+9s^{-1}) & s^{-2} \\ -10s^{-2} & -20s^{-2} & s^{-1} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} + \frac{1}{\Delta(s)} \begin{bmatrix} s^{-3} \\ s^{-2} \\ s^{-1} \end{bmatrix} \frac{1}{s} \\
 &= \frac{1}{\Delta_c(s)} \begin{bmatrix} s^2+9s+20 & s+9 & 1 \\ -10 & s(s+9) & s \\ -10s & -10(2s+1) & s^2 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} + \frac{1}{\Delta_c(s)} \begin{bmatrix} \frac{1}{s} \\ 1 \\ s \end{bmatrix} \\
 \Delta(s) &= 1+9s^{-1}+20s^{-2}+10s^{-3} \quad \Delta_c(s) = s^3+9s^2+20s+10 \\
 \mathbf{x}(t) &= \left\{ \begin{bmatrix} 1.612 & 0.946 & 0.114 \\ -1.14 & -0.669 & -0.081 \\ 0.807 & 0.474 & 0.057 \end{bmatrix} e^{-0.708t} + \begin{bmatrix} -0.706 & -1.117 & -0.169 \\ 1.692 & 2.678 & 4.056 \\ -4.056 & -6.420 & -0.972 \end{bmatrix} e^{-2.397t} + \begin{bmatrix} 0.0935 & 0.171 & 0.055 \\ -0.551 & -1.009 & -0.325 \\ 3.249 & 5.947 & 1.915 \end{bmatrix} e^{-5.895t} \right\} \mathbf{x}(0) \\
 &+ \begin{bmatrix} 0.1 - 0.161e^{-0.708t} + 0.0706e^{-2.397t} - 0.00935e^{-5.895t} \\ 0.114e^{-0.708t} - 0.169e^{-2.397t} + 0.055e^{-5.895t} \\ -0.087e^{-0.708t} + 0.406e^{-2.397t} - 0.325e^{-5.895t} \end{bmatrix} \quad t \geq 0
 \end{aligned}$$

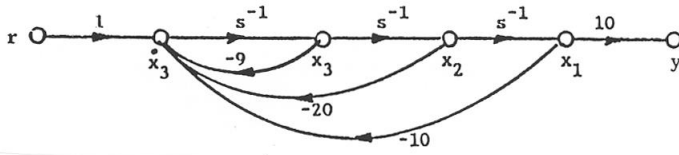
(d) Output:

$$\begin{aligned}
 y(t) = 10x_1(t) &= 10 \left(1.612e^{-0.708t} - 0.706e^{-2.397t} + 0.0935e^{-5.895t} \right) x_1(0) + 10 \left(0.946e^{-0.708t} - 1.117e^{-2.397t} + 0.171e^{-5.895t} \right) x_2(0) \\
 &+ 10 \left(1.141e^{-0.708t} - 0.169e^{-2.397t} + 0.055e^{-5.895t} \right) x_3(0) + 1 - 1.61e^{-0.708t} + 0.706e^{-2.397t} - 0.0935e^{-5.895t} \quad t \geq 0
 \end{aligned}$$

8-37(a) Closed-loop transfer function:

$$\frac{Y(s)}{R(s)} = \frac{10}{s^3 + 9s^2 + 20s + 10}$$

(b) State diagram:



(c) State equations:

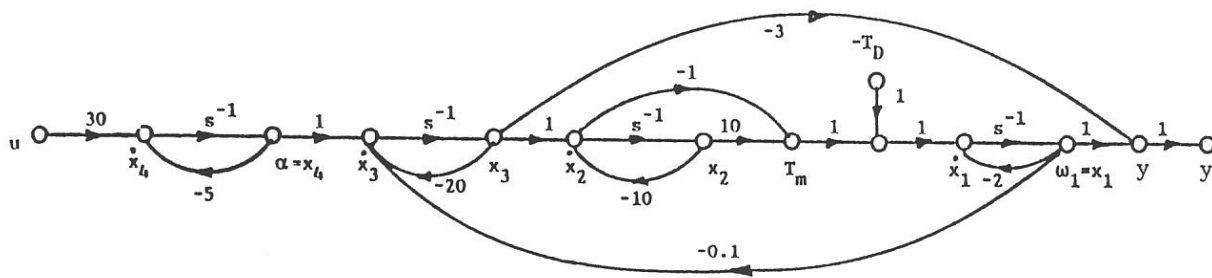
(d) State transition equations:

[Same answers as Problem 5-26(d)]

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -20 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

(e) Output: [Same answer as Problem 5-26(e)]

8-38 (a) State diagram:



(b) State equations:

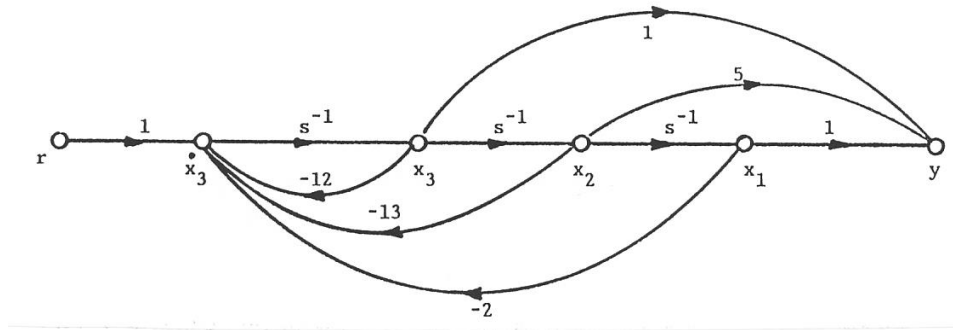
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -2 & 20 & -1 & 0 \\ 0 & -10 & 1 & 0 \\ -0.1 & 0 & -20 & 1 \\ 0 & 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ 30 & 0 \end{bmatrix} \begin{bmatrix} u \\ T_d \end{bmatrix}$$

(c) Transfer function relations:

From the system block diagram,

$$Y(s) = \frac{1}{\Delta(s)} \left(\frac{-1}{s+2} T_d(s) + \frac{0.3}{(s+2)(s+20)} T_d(s) + \frac{30e^{-0.2s} U(s)}{(s+2)(s+5)(s+20)} + \frac{90U(s)}{(s+5)(s+20)} \right)$$

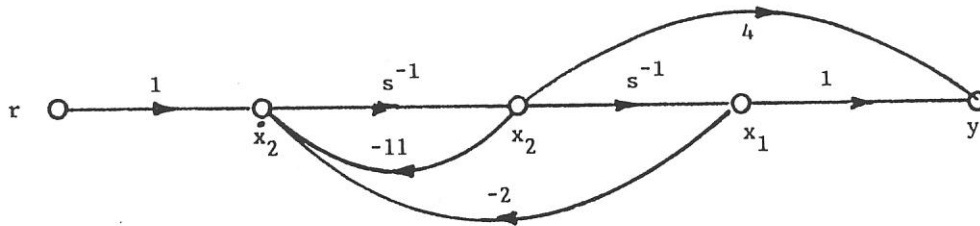
$$\Delta(s) = 1 + \frac{0.1e^{-0.2s}}{(s+2)(s+20)} = \frac{(s+2)(s+20) + 0.1e^{-0.2s}}{(s+2)(s+20)}$$



(d) When $K = 4$:

$$\frac{Y(s)}{R(s)} = \frac{4s^2 + 5s + 1}{(s+1)(s^2 + 11s + 2)} = \frac{(s+1)(4s+1)}{(s+1)(s^2 + 11s + 2)} = \frac{4s+1}{s^2 + 11s + 2}$$

State diagram:



(e)

$$\frac{Y(s)}{R(s)} = \frac{Ks^2 + 5s + 1}{(s+1)(s^2 + 11s + 2)} \quad (s+1)(s^2 + 11s + 2) = 0$$

MATLAB

```

solve(s^2+11*s+2)
ans = -11/2+1/2*113^(1/2)
-11/2-1/2*113^(1/2)
>> vpa(ans)
ans =
-10.8
-0.20

```

$$\frac{Y(s)}{R(s)} = \frac{Ks^2 + 5s + 1}{(s+1)(s+0.2)(s+10.82)}$$

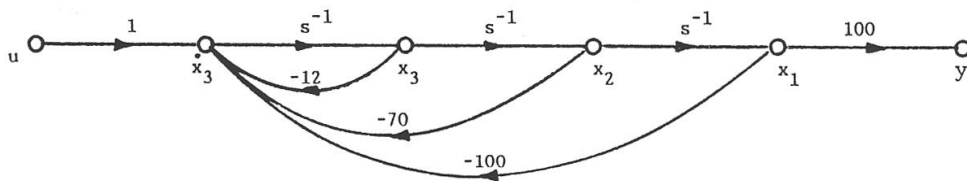
$K = 4, 2.1914, 0.4536$

Pole zero cancelation occurs for the given values of K .

8-41 (a)

$$G_p(s) = \frac{Y(s)}{U(s)} = \frac{1}{(1+0.5s)(1+0.2s+0.02s^2)} = \frac{100}{s^3 + 12s^2 + 70s + 100}$$

State diagram by direct decomposition:



State equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -100 & -70 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

(b) Characteristic equation of closed-loop system:

$$s^3 + 12s^2 + 70s + 200 = 0$$

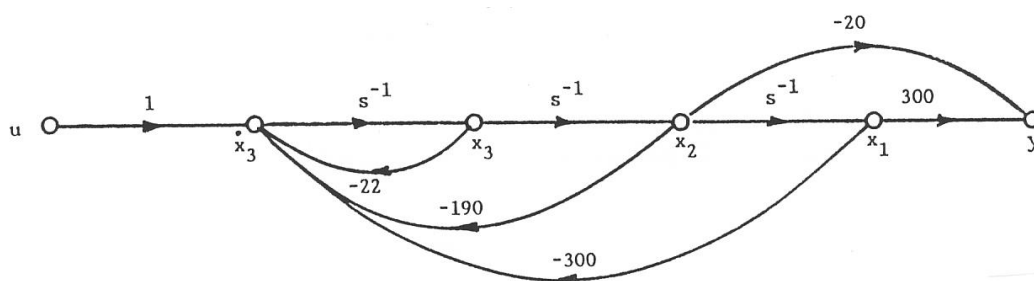
Roots of characteristic equation:

$$-5.88, \quad -3.06 + j4.965, \quad -3.06 - j4.965$$

8-42 (a)

$$G_p(s) = \frac{Y(s)}{U(s)} \cong \frac{1 - 0.066s}{(1+0.5s)(1+0.133s+0.0067s^2)} = \frac{-20(s-15)}{s^3 + 22s^2 + 190s + 300}$$

State diagram by direct decomposition:



State equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -300 & -190 & -22 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Characteristic equation of closed-loop system:

$$s^3 + 22s^2 + 170s + 600 = 0$$

Roots of characteristic equation:

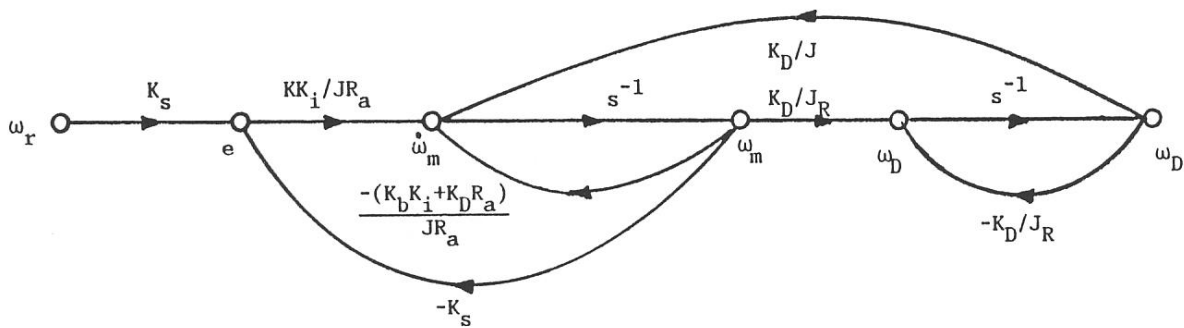
$$-12, -5 + j5, -5 - j5$$

8-43 (a) State variables: $x_1 = \omega_m$ and $x_2 = \omega_D$

State equations:

$$\frac{d\omega_m}{dt} = -\frac{K_b K_i + K_b R_a}{J R_a} \omega_m + \frac{K_D}{J} \omega_D + \frac{K K_i}{J R_a} e \quad \frac{d\omega_D}{dt} = \frac{K_D}{J_R} \omega_m - \frac{K_D}{J_R} \omega_D$$

(b) State diagram:



(c) Open-loop transfer function:

$$\frac{\Omega_m(s)}{E(s)} = \frac{KK_i(J_R s + K_D)}{JJ_R R_a s^2 + (K_b J_R K_i + K_D R_a J_R + K_D J R_a)s + K_D K_b K_i}$$

Closed-loop transfer function:

$$\frac{\Omega_m(s)}{\Omega_r(s)} = \frac{K_s K K_i (J_R s + K_D)}{J J_R R_a s^2 + (K_b J_R K_i + K_D R_a J_R + K_D J R_a + K_s K K_i J_R) s + K_D K_b K_i + K_s K K_i K_D}$$

(d) Characteristic equation of closed-loop system:

$$\Delta(s) = J J_R R_a s^2 + (K_b J_R K_i + K_D R_a J_R + K_D J R_a + K_s K K_i J_R) s + K_D K_b K_i + K_s K K_i K_D = 0$$

$$\Delta(s) = s^2 + 1037s + 20131.2 = 0$$

Characteristic equation roots: -19.8, -1017.2

8-44 (a) State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}r(t)$

$$\mathbf{A} = \begin{bmatrix} -b & d \\ c & -a \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \mathbf{S} = [\mathbf{B} \quad \mathbf{AB}] = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

Since \mathbf{S} is nonsingular, the system is controllable.

(b)

$$\mathbf{S} = [\mathbf{B} \quad \mathbf{AB}] = \begin{bmatrix} 0 & d \\ 1 & -a \end{bmatrix} \quad \text{The system is controllable for } d \neq 0.$$

8-45 (a)

$$\mathbf{S} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B}] = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \quad \mathbf{S} \text{ is singular. The system is uncontrollable.}$$

(b)

$$\mathbf{S} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B}] = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ 1 & -3 & 9 \end{bmatrix} \quad \mathbf{S} \text{ is nonsingular. The system is controllable.}$$

8-46 (a) State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} -2 & 3 \\ 1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \mathbf{S} = [\mathbf{B} \quad \mathbf{AB}] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \mathbf{S} \text{ is singular. The system is uncontrollable.}$$

Output equation: $y = [1 \quad 0]\mathbf{x} = \mathbf{C}\mathbf{x} \quad \mathbf{C} = [1 \quad 0]$

$$\mathbf{V} = [\mathbf{C}^T \quad \mathbf{A}^T \mathbf{C}^T] = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \quad \mathbf{V} \text{ is nonsingular. The system is observable.}$$

(b) Transfer function:

$$\frac{Y(s)}{U(s)} = \frac{s+3}{s^2+2s-3} = \frac{1}{s-1}$$

Since there is pole-zero cancellation in the input-output transfer function, the system is either uncontrollable or unobservable or both. In this case, the state variables are already defined, and the system is uncontrollable as found out in part (a).

8-47 (a) $\alpha = 1, 2, \text{ or } 4$. These values of α will cause pole-zero cancellation in the transfer function.

(b) The transfer function is expanded by partial fraction expansion,

$$\frac{Y(s)}{R(s)} = \frac{\alpha-1}{3(s+1)} - \frac{\alpha-2}{2(s+2)} + \frac{\alpha-4}{6(s+4)}$$

By parallel decomposition, the state equations are: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}r(t)$, output equation: $y(t) = \mathbf{C}\mathbf{x}(t)$.

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} \alpha-1 \\ \alpha-2 \\ \alpha-4 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \end{bmatrix}$$

The system is uncontrollable for $\alpha = 1$, or $\alpha = 2$, or $\alpha = 4$.

(c) Define the state variables so that

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{2} \\ -\frac{1}{6} \end{bmatrix} \quad \mathbf{D} = [\alpha - 1 \quad \alpha - 2 \quad \alpha - 4]$$

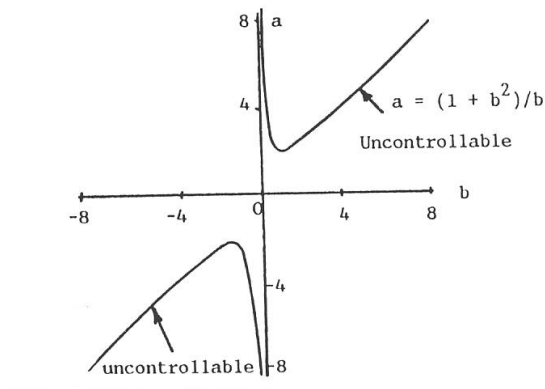
The system is unobservable for $\alpha = 1$, or $\alpha = 2$, or $\alpha = 4$.

8-48

$$\mathbf{S} = [\mathbf{B} \quad \mathbf{AB}] = \begin{bmatrix} 1 & b \\ b & ab - 1 \end{bmatrix} \quad |\mathbf{S}| = ab - 1 - b^2 \neq 0$$

The boundary of the region of controllability is described by $ab - 1 - b^2 = 0$.

Regions of controllability:



8-49

$$\mathbf{S} = [\mathbf{B} \quad \mathbf{AB}] = \begin{bmatrix} b_1 & b_1 + b_2 \\ b_2 & b_2 \end{bmatrix} \quad |\mathbf{S}| = 0 \text{ when } b_1 b_2 - b_1 b_2 - b_2^2 = 0, \text{ or } b_2 = 0$$

The system is completely controllable when $b_2 \neq 0$.

$$\mathbf{V} = [\mathbf{C}^T \quad \mathbf{A}^T \mathbf{C}^T] = \begin{bmatrix} d_1 & d_2 \\ d_2 & d_1 + d_2 \end{bmatrix} \quad |\mathbf{V}| = 0 \text{ when } d_1 \neq 0.$$

The system is completely observable when $d_2 \neq 0$.

8-50 (a) State equations:

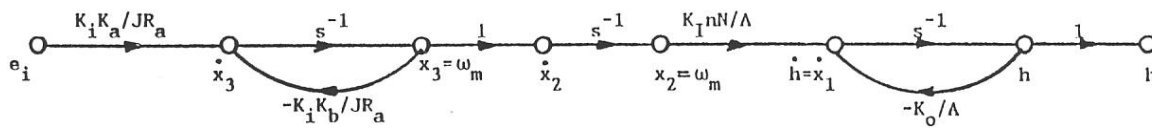
$$\frac{dh}{dt} = \frac{1}{A}(q_i - q_o) = \frac{K_I n N}{A} \theta_m - \frac{K_o}{A} h \quad \frac{d\theta_m}{dt} = \omega_m \quad \frac{d\omega_m}{dt} = -\frac{K_i K_b}{J R_a} \omega_m + \frac{K_i K_a}{J R_a} e_i \quad J = J_m + n^2 J_L$$

State variable: $x_1 = h, \quad x_2 = \theta_m, \quad x_3 = \frac{d\theta_m}{dt} = \omega_m$

State equations: $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}e_i$

$$\mathbf{A} = \begin{bmatrix} \frac{-K_o}{A} & \frac{K_I n N}{A} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{K_i K_b}{J R_a} \end{bmatrix} = \begin{bmatrix} -1 & 0.016 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -11.767 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \frac{K_i K_a}{J R_a} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 8333.33 \end{bmatrix}$$

State diagram:

**(b) Characteristic equation of A:**

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s + \frac{K_o}{A} & \frac{-K_I n N}{A} & 0 \\ 0 & s & -1 \\ 0 & 0 & s + \frac{K_i K_b}{J R_a} \end{vmatrix} = s \left(s + \frac{K_o}{A} \right) \left(s + \frac{K_i K_b}{J R_a} \right) = s(s+1)(s+11.767)$$

Eigenvalues of A: $0, -1, -11.767.$

(c) Controllability:

$$\mathbf{S} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B}] = \begin{bmatrix} 0 & 0 & 133.33 \\ 0 & 8333.33 & -98058 \\ 8333.33 & -98058 & 1153848 \end{bmatrix} \quad |\mathbf{S}| \neq 0. \text{ The system is controllable.}$$

(d) Observability:

(1) $\mathbf{C} = [1 \ 0 \ 0]$:

$$\mathbf{V} = [\mathbf{C}^T \quad \mathbf{A}^T\mathbf{C}^T \quad (\mathbf{A}^T)^2\mathbf{C}^T] = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0.016 & -0.016 \\ 0 & 0 & 0.016 \end{bmatrix} \quad \mathbf{V} \text{ is nonsingular. The system is observable.}$$

(2) $\mathbf{C} = [0 \ 1 \ 0]$:

$$\mathbf{V} = [\mathbf{C}^T \quad \mathbf{A}^T\mathbf{C}^T \quad (\mathbf{A}^T)^2\mathbf{C}^T] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -11.767 \end{bmatrix} \quad \mathbf{V} \text{ is singular. The system is unobservable.}$$

(3) $\mathbf{C} = [0 \ 0 \ 1]$:

$$\mathbf{V} = [\mathbf{C}^T \quad \mathbf{A}^T\mathbf{C}^T \quad (\mathbf{A}^T)^2\mathbf{C}^T] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -11.767 & 138.46 \end{bmatrix} \quad \mathbf{V} \text{ is singular. The system is unobservable.}$$

8-51 (a) Characteristic equation: $\Delta(s) = |s\mathbf{I} - \mathbf{A}^*| = s^4 - 25.92s^2 = 0$

Roots of characteristic equation: $-5.0912, 5.0912, 0, 0$

(b) Controllability:

$$\mathbf{S} = [\mathbf{B}^* \quad \mathbf{A}^*\mathbf{B}^* \quad \mathbf{A}^{*2}\mathbf{B}^* \quad \mathbf{A}^{*3}\mathbf{B}^*] = \begin{bmatrix} 0 & -0.0732 & 0 & -1.8973 \\ -0.0732 & 0 & -1.8973 & 0 \\ 0 & 0.0976 & 0 & 0.1728 \\ 0.0976 & 0 & 0.1728 & 0 \end{bmatrix}$$

\mathbf{S} is nonsingular. Thus, $[\mathbf{A}^*, \mathbf{B}^*]$ is controllable.

(c) Observability:

$$(1) \quad \mathbf{C}^* = [1 \quad 0 \quad 0 \quad 0]$$

$$\mathbf{V} = [\mathbf{C}^{*'} \quad \mathbf{A}^{*'} \mathbf{C}^{*'} \quad (\mathbf{A}^{*'})^2 \mathbf{C}^{*'} \quad (\mathbf{A}^{*'})^3 \mathbf{C}^{*'}] = \begin{bmatrix} 1 & 0 & 25.92 & 0 \\ 0 & 1 & 0 & 25.92 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

\mathbf{S} is singular. The system is unobservable.

$$(2) \quad \mathbf{C}^* = [0 \quad 1 \quad 0 \quad 0]$$

$$\mathbf{V} = [\mathbf{C}^{*'} \quad \mathbf{A}^{*'} \mathbf{C}^{*'} \quad (\mathbf{A}^{*'})^2 \mathbf{C}^{*'} \quad (\mathbf{A}^{*'})^3 \mathbf{C}^{*'}] = \begin{bmatrix} 0 & 25.92 & 0 & 671.85 \\ 1 & 0 & 25.92 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

\mathbf{S} is singular. The system is unobservable.

$$(3) \quad \mathbf{C}^* = [0 \quad 0 \quad 1 \quad 0]$$

$$\mathbf{V} = [\mathbf{C}^{*'} \quad \mathbf{A}^{*'} \mathbf{C}^{*'} \quad (\mathbf{A}^{*'})^2 \mathbf{C}^{*'} \quad (\mathbf{A}^{*'})^3 \mathbf{C}^{*'}] = \begin{bmatrix} 0 & 0 & -2.36 & 0 \\ 0 & 0 & 0 & -2.36 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

\mathbf{S} is nonsingular. The system is observable.

$$(4) \quad \mathbf{C}^* = [0 \quad 0 \quad 0 \quad 1]$$

$$\mathbf{V} = [\mathbf{C}^{*'} \quad \mathbf{A}^{*'} \mathbf{C}^{*'} \quad (\mathbf{A}^{*'})^2 \mathbf{C}^{*'} \quad (\mathbf{A}^{*'})^3 \mathbf{C}^{*'}] = \begin{bmatrix} 0 & -2.36 & 0 & -61.17 \\ 0 & 0 & -2.36 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

\mathbf{S} is singular. The system is unobservable.

8-52 The controllability matrix is

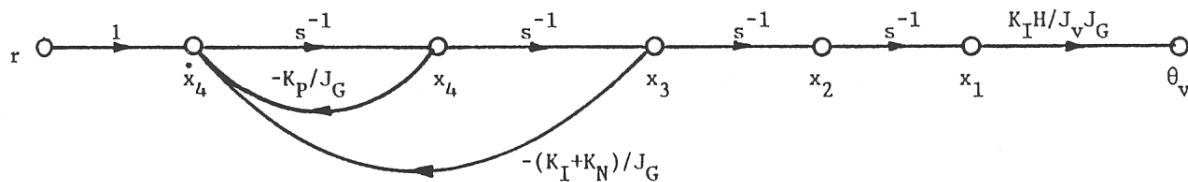
$$\mathbf{S} = \begin{bmatrix} 0 & -1 & 0 & -16 & 0 & -384 \\ -1 & 0 & -16 & 0 & -384 & 0 \\ 0 & 0 & 0 & 16 & 0 & 512 \\ 0 & 0 & 16 & 0 & 512 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of \mathbf{S} is 6. The system is controllable.

8-53 (a) Transfer function:

$$\frac{\Theta_v(s)}{R(s)} = \frac{K_I H}{J_v s^2 (J_G s^2 + K_p s + K_I + K_N)}$$

State diagram by direct decomposition:



State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}r(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-(K_I + K_N)}{J_G} & \frac{-K_p}{J_G} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(b) Characteristic equation: $J_v s^2 (J_G s^2 + K_p s + K_I + K_N) = 0$

8-54 (a) State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u_1(t)$

$$\mathbf{A} = \begin{bmatrix} -3 & 1 \\ 0 & -2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \mathbf{S} = [\mathbf{B} \quad \mathbf{AB}] = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

\mathbf{S} is nonsingular. $[\mathbf{A}, \mathbf{B}]$ is controllable.

Output equation: $y_2 = \mathbf{C}\mathbf{x} \quad \mathbf{C} = [-1 \quad 1]$

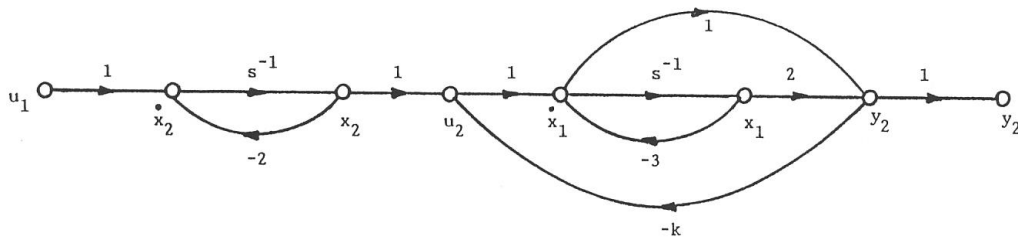
$$\mathbf{V} = [\mathbf{C}' \quad \mathbf{A}'\mathbf{C}'] = \begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix} \quad \mathbf{V} \text{ is singular. The system is unobservable.}$$

(b) With feedback, $u_2 = -kc_2$, the state equation is: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u_1(t)$.

$$\mathbf{A} = \begin{bmatrix} \frac{-3-2k}{1+g} & \frac{1}{1+k} \\ 0 & -2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \mathbf{S} = \begin{bmatrix} 0 & \frac{1}{1+k} \\ 1 & -2 \end{bmatrix}$$

\mathbf{S} is nonsingular for all finite values of k . The system is controllable.

State diagram:



Output equation: $y_2 = \mathbf{C}\mathbf{x} \quad \mathbf{C} = \begin{bmatrix} \frac{-1}{1+k} & \frac{1}{1+k} \end{bmatrix}$

$$\mathbf{V} = [\mathbf{D}' \quad \mathbf{A}'\mathbf{D}'] = \begin{bmatrix} \frac{-1}{1+K} & \frac{3+2k}{(1+k)^2} \\ \frac{1}{1+k} & -\frac{3+2k}{(1+k)^2} \end{bmatrix}$$

\mathbf{V} is singular for any k . The system with feedback is unobservable.

8-55 (a)

$$\mathbf{S} = [\mathbf{B} \quad \mathbf{AB}] = \begin{bmatrix} 1 & 2 \\ 2 & -7 \end{bmatrix} \quad \mathbf{S} \text{ is nonsingular. System is controllable.}$$

$$\mathbf{V} = [\mathbf{C}^T \quad \mathbf{A}^T \mathbf{C}^T] = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} \quad \mathbf{V} \text{ is nonsingular. System is observable.}$$

$$\text{(b)} \quad u = -[k_1 \quad k_2]\mathbf{x}$$

$$\mathbf{A}_c = \mathbf{A} - \mathbf{BK} = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ 2k_1 & 2k_2 \end{bmatrix} = \begin{bmatrix} -k_1 & 1-k_2 \\ -1-2k_1 & -3-2k_2 \end{bmatrix}$$

$$\mathbf{S} = [\mathbf{B} \quad \mathbf{A}_c \mathbf{B}] = \begin{bmatrix} 1 & -k_1 - 2k_2 + 2 \\ 2 & -7 - 2k_1 - 4k_2 \end{bmatrix} \quad |\mathbf{S}| = -11 - 2k_2 \neq 0$$

$$\text{For controllability, } k_2 \neq -\frac{11}{2}$$

$$\mathbf{V} = [\mathbf{C}^T \quad \mathbf{A}_c^T \mathbf{C}^T] = \begin{bmatrix} -1 & -1-3k_1 \\ 1 & -2-3k_2 \end{bmatrix}$$

$$\text{For observability, } |\mathbf{V}| = -1 + 3k_1 - 3k_2 \neq 0$$

8-56

Same as 8-21 (a)

8-57

$$= \begin{bmatrix} 0 & 1 \\ -322.58 & -80.65 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 322.58 \end{bmatrix} \theta_r$$

From 8-22
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -322.58 & -80.65 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 322.58 \end{bmatrix} \theta_r$$

$$A = \begin{bmatrix} 0 & 1 \\ -322.58 & -80.65 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 322.58 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$D = 0$$

Use the state space tool of ACSYS

State Space Tool

Calculate/Display Time Response Accessories

Block Diagram

$$\frac{dx(t)}{dt} = \begin{bmatrix} A \end{bmatrix} x(t) + \begin{bmatrix} B \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} C \end{bmatrix} x(t) + \begin{bmatrix} D \end{bmatrix} u(t)$$

Input Module

Enter coefficient Matrices.
eg. For a 3x3 identity matrix enter [1 0 0; 0 1 0; 0 0 1]

[1;0;1] is a 3x1 column [1 0 0] is a 1x3 row.

A
[0,1,-322.58,-80.65]

B
[0;322.58]

C
[1,0]

D
0

Initial Conditions
0

Buttons

Reset

Close Window

The A matrix is:

Amat =

0 1.0000
-322.5800 -80.6500

Characteristic Polynomial:

ans =

$s^2 + 1613/20*s + 16129/50$

Eigenvalues of A = Diagonal Canonical Form of A is:

Abar =

-4.2206 0
0 -76.4294

Eigen Vectors are

T =

0.2305 -0.0131
-0.9731 0.9999

State-Space Model is:

a =

$$\begin{array}{cc} & x_1 & x_2 \\ x_1 & 0 & 1 \\ x_2 & -322.6 & -80.65 \end{array}$$

b =

$$\begin{array}{c} u_1 \\ x_1 & 0 \\ x_2 & 322.6 \end{array}$$

c =

$$\begin{array}{cc} x_1 & x_2 \\ y_1 & 1 & 0 \end{array}$$

d =

$$\begin{array}{c} u_1 \\ y_1 & 0 \end{array}$$

Continuous-time model.

Characteristic Polynomial:

ans =

$$s^2 + 1613/20 s + 16129/50$$

Equivalent Transfer Function Model is:

Transfer function:

$$322.6$$

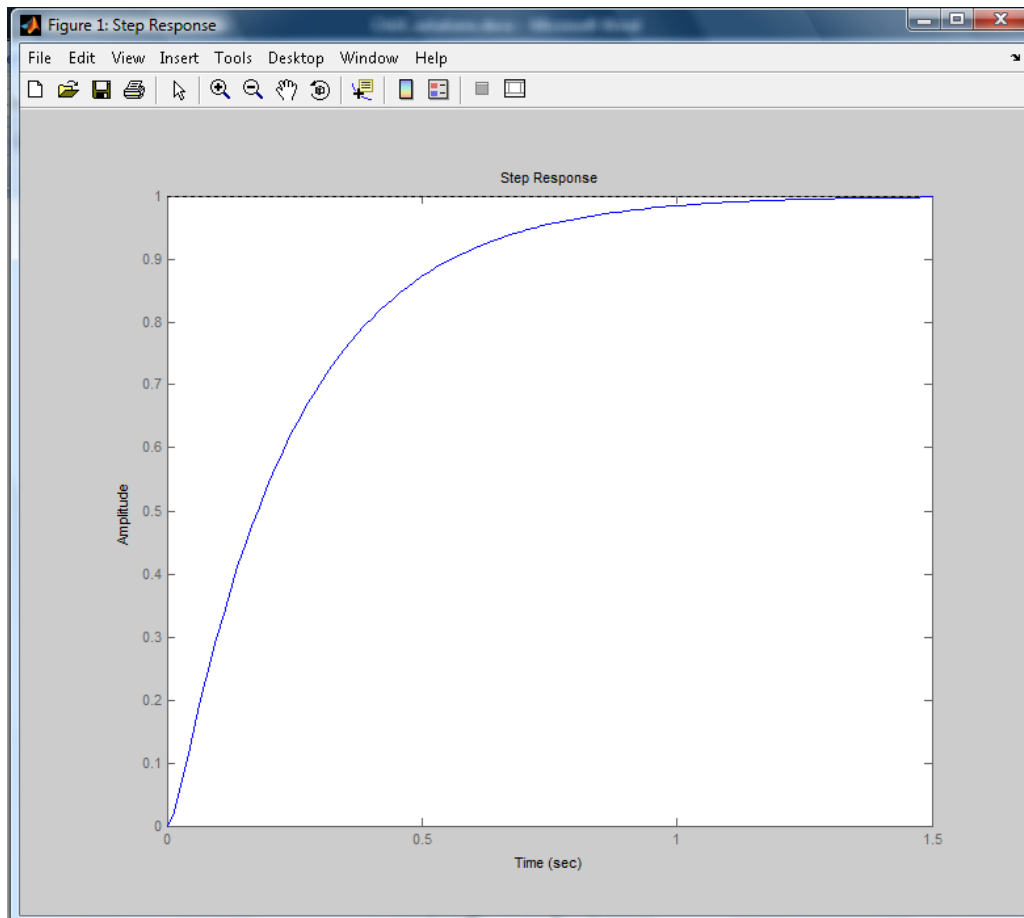
$$s^2 + 80.65 s + 322.6$$

Pole, Zero Form:

Zero/pole/gain:

$$322.58$$

$$(s+76.43) (s+4.221)$$



8-58

Closed-loop System Transfer Function.

$$\frac{Y(s)}{R(s)} = \frac{1}{s^3 + (4 + k_3)s^2 + (3 + k_2 + k_3)s + k_1}$$

For zero steady-state error to a step input, $k_1 = 1$. For the complex roots to be located at $-1 + j$ and $-1 - j$,

we divide the characteristic polynomial by $s^2 + 2s + 2$ and solve for zero remainder.

$$\begin{array}{r}
 s + (2 + k_2) \\
 s^2 + 2s + 2 \overline{) s^3 + (4 + k_3)s^2 + (3 + k_2 + k_3)s + 1} \\
 \underline{s^3 + \quad 2s^2 \quad \quad + 2s} \\
 (2 + k_3)s^2 + (1 + k_2 + k_3)s + 1
 \end{array}$$

$$\frac{(2+k_3)s^2 + (4+2k_3)s + 4+2k_3}{(-3+k_2-k_3)s - 3-2k_3}$$

For zero remainder, $-3-2k_3 = 0$ Thus $k_3 = -1.5$

$-3+k_2-k_3 = 0$ Thus $k_2 = 1.5$

The third root is at -0.5 . Not all the roots can be arbitrarily assigned, due to the requirement on the steady-state error.

8-59 (a) Open-loop Transfer Function.

$$G(s) = \frac{X_1(s)}{E(s)} = \frac{k_3}{s[s^2 + (4+k_2)s + 3+k_1+k_2]}$$

Since the system is type 1, the steady-state error due to a step input is zero for all values of k_1 , k_2 , and k_3 that correspond to a stable system. The characteristic equation of the closed-loop system is

$$s^3 + (4+k_2)s^2 + (3+k_1+k_2)s + k_3 = 0$$

For the roots to be at $-1+j$, $-1-j$, and -10 , the equation should be:

$$s^3 + 12s^2 + 22s + 20 = 0$$

Equating like coefficients of the last two equations, we have

$$4+k_2 = 12 \quad \text{Thus} \quad k_2 = 8$$

$$3+k_1+k_2 = 22 \quad \text{Thus} \quad k_1 = 11$$

$$k_3 = 20 \quad \text{Thus} \quad k_3 = 20$$

(b) Open-loop Transfer Function.

$$\frac{Y(s)}{E(s)} = \frac{G_c(s)}{(s+1)(s+3)} = \frac{20}{s(s^2 + 12s + 22)} \quad \text{Thus} \quad G_c(s) = \frac{20(s+1)(s+3)}{s(s^2 + 12s + 22)}$$

8-60 (a)

$$\mathbf{A}^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 25.92 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -2.36 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{B}^* = \begin{bmatrix} 0 \\ -0.0732 \\ 0 \\ 0.0976 \end{bmatrix}$$

The feedback gains, from k_1 to k_4 :

−2.4071E+03 −4.3631E+02 −8.4852E+01 −1.0182E+02

The $\mathbf{A}^* - \mathbf{B}^* \mathbf{K}$ matrix of the closed-loop system

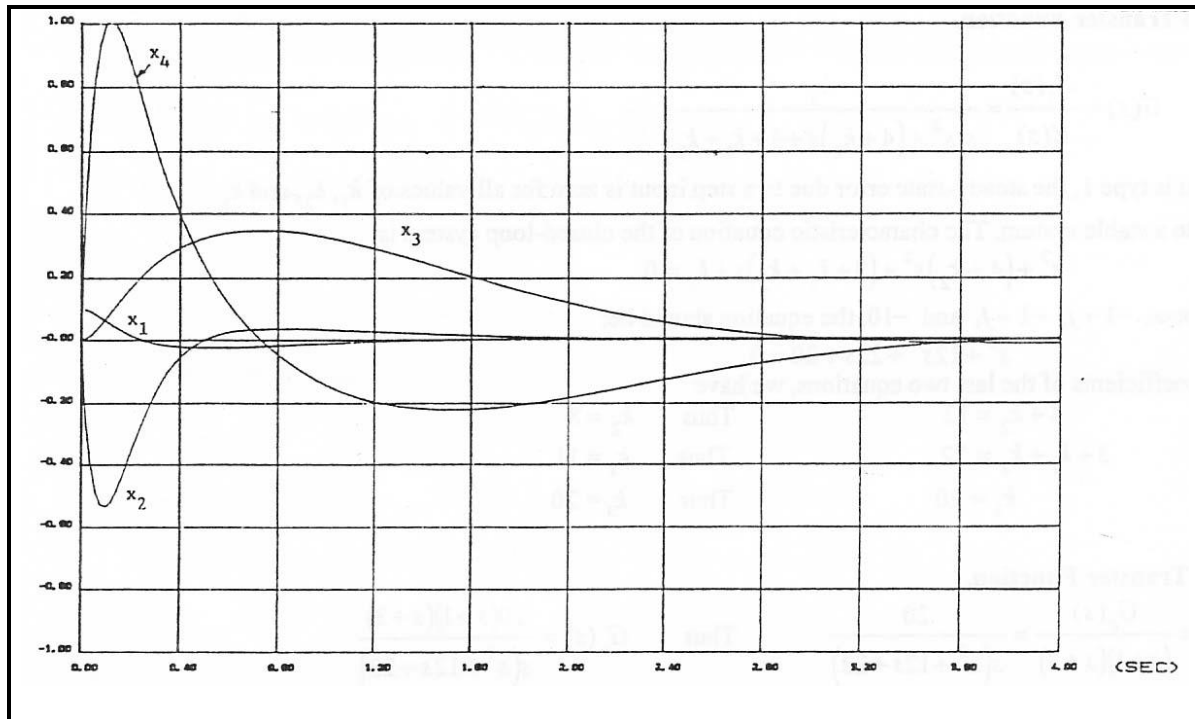
0.0000E+00	1.0000E+00	0.0000E+00	0.0000E+00
−1.5028E+02	−3.1938E+01	−6.2112E+00	−7.4534E+00
0.0000E+00	0.0000E+00	0.0000E+00	1.0000E+00
2/3258E+02	4.2584E+01	8.2816E+00	9.9379E+00

The \mathbf{B} vector

0.0000E+00
−7.3200E−02
0.0000E+00

9.7600E-02

Time Responses:

8-60
(b)The feedback gains, from k_1 to k_2 :

$-9.9238\text{E}+03$ $-1.6872\text{E}+03$ $-1.3576\text{E}+03$ $-8.1458\text{E}+02$

The $\mathbf{A}^* - \mathbf{B}^* \mathbf{K}$ matrix of the closed-loop system

0.0000E+00	1.0000E+00	0.0000E+00	0.0000E+00
$-7.0051\text{E}+02$	$-1.2350\text{E}+02$	$-9.9379\text{E}+01$	$-5.9627\text{E}+01$
0.0000E+00	0.0000E+00	0.0000E+00	1.0000E+00
$9.6621\text{E}+02$	$1.6467\text{E}+02$	$1.3251\text{E}+02$	$7.9503\text{E}+01$

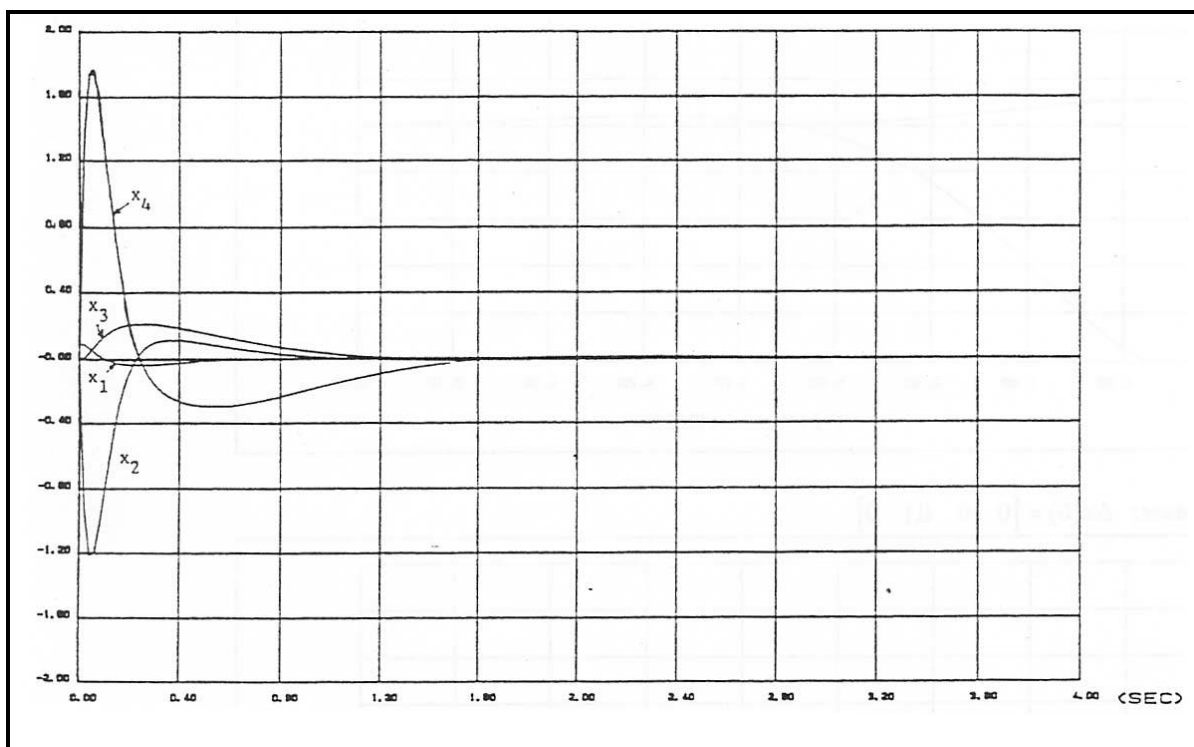
The \mathbf{B} vector

0.0000E+00

-7.3200E-02

0.0000E+00

9.7600E-02

Time Responses:**8-61**
The

solutions using MATLAB

(a) The feedback gains, from k_1 to k_2 :

-6.4840E+01

-5.6067E+00

2.0341E+01

2.2708E+00

The $\mathbf{A}^* - \mathbf{B}^* \mathbf{K}$ matrix of the closed-loop system

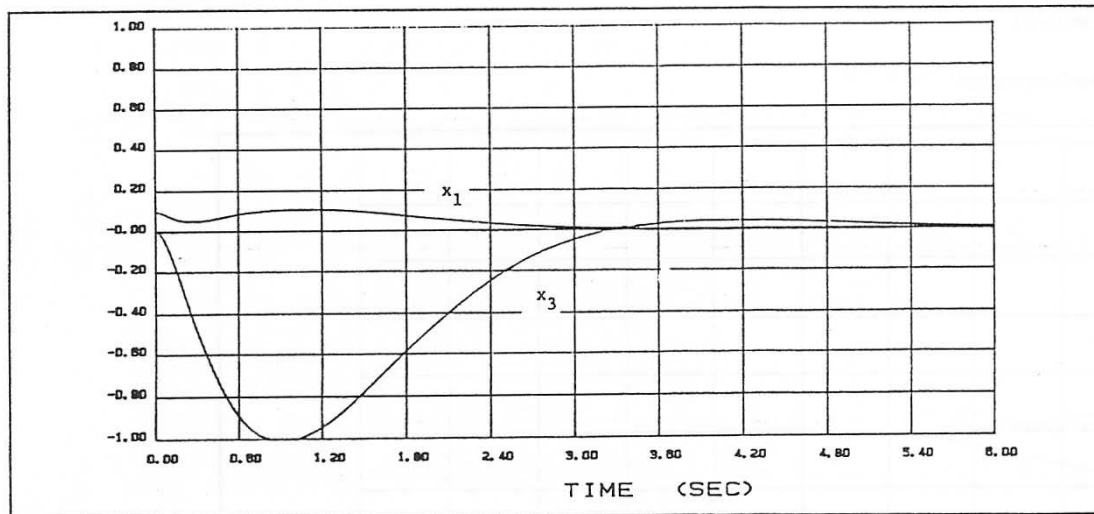
0.0000E+00	1.0000E+00	0.0000E+00	0.0000E+00
-3.0950E+02	-3.6774E+01	1.1463E+02	1.4874E+01
0.0000E+00	0.0000E+00	0.0000E+00	1.0000E+00
-4.6190E+02	-3.6724E+01	1.7043E+02	1.477eE+01

The \mathbf{B} vector

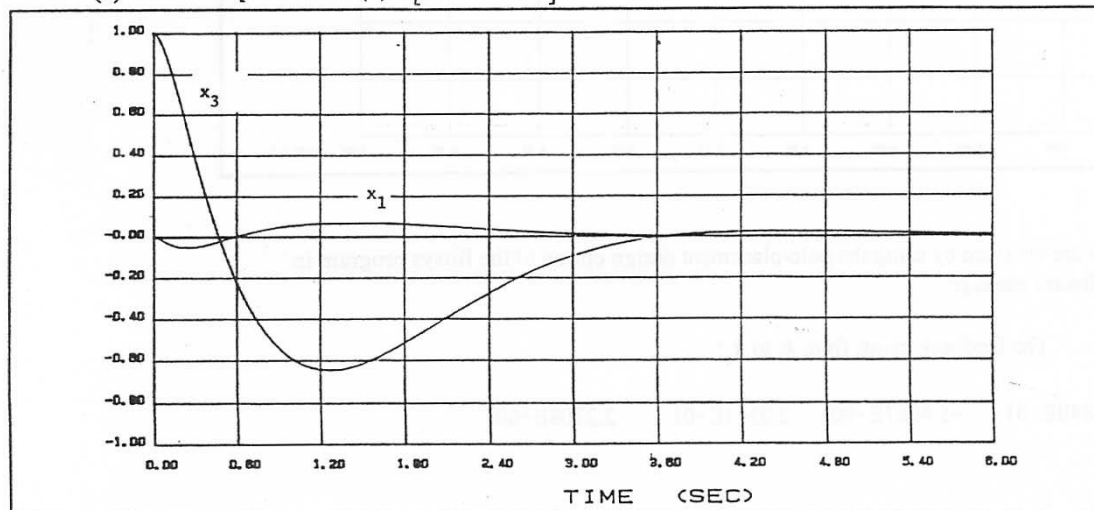
0.0000E+00
-6.5500E+00
0.0000E+00
-6.5500E+00

(b) Time Responses: $\Delta \mathbf{x}(0) = [0.1 \ 0 \ 0 \ 0]^T$

With the
initial
states



(c) Time Responses: $\Delta \mathbf{x}(0) = [0 \ 0 \ 0.1 \ 0]^T$



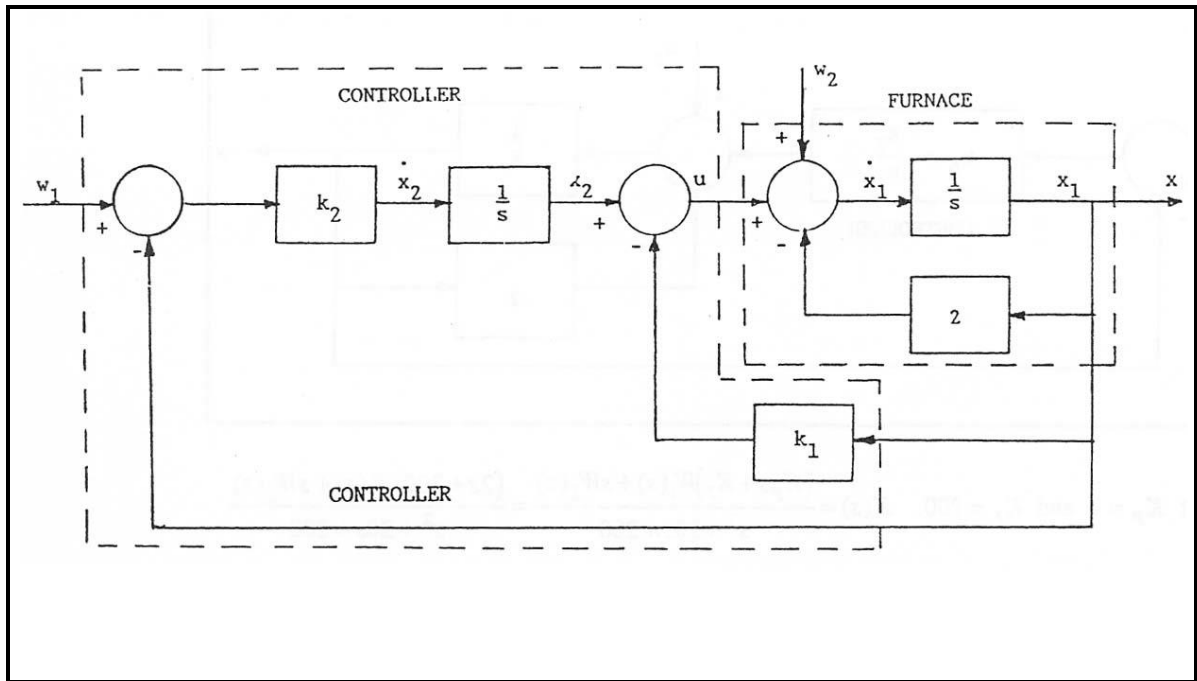
$\Delta \mathbf{x}(0) = [0.1 \ 0 \ 0 \ 0]^T$, the initial position of Δx_1 or Δy_1 is perturbed downward

from its stable equilibrium position. The steel ball is initially pulled toward the magnet, so $\Delta x_3 = \Delta y_2$ is negative at first. Finally, the feedback control pulls both bodies back to the equilibrium position.

With the initial states $\Delta \mathbf{x}(0) = [0 \ 0 \ 0.1 \ 0]^T$, the initial position of Δx_3 or Δy_2 is perturbed downward from its stable equilibrium. For $t > 0$, the ball is going to be attracted up by the magnet toward the equilibrium position. The magnet will initially be attracted toward the fixed iron plate, and then settles to the stable equilibrium position. Since the steel ball has a small mass, it will move more

actively.

8-62 (a) Block Diagram of System.



$$u = -k_1 x_1 + k_2 \int (-x_1 + w_1) dt$$

State Equations of Closed-loop System:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} -2 - k_1 & 1 \\ -k_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ k_2 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

Characteristic Equation:

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s + 2 + k_1 & -1 \\ k_2 & s \end{vmatrix} = s^2 + (2 + k_1)s + k_2 = 0$$

For $s = -10, -10$, $|s\mathbf{I} - \mathbf{A}| = s^2 + 20s + 200 = 0$ Thus $k_1 = 18$ and $k_2 = 200$

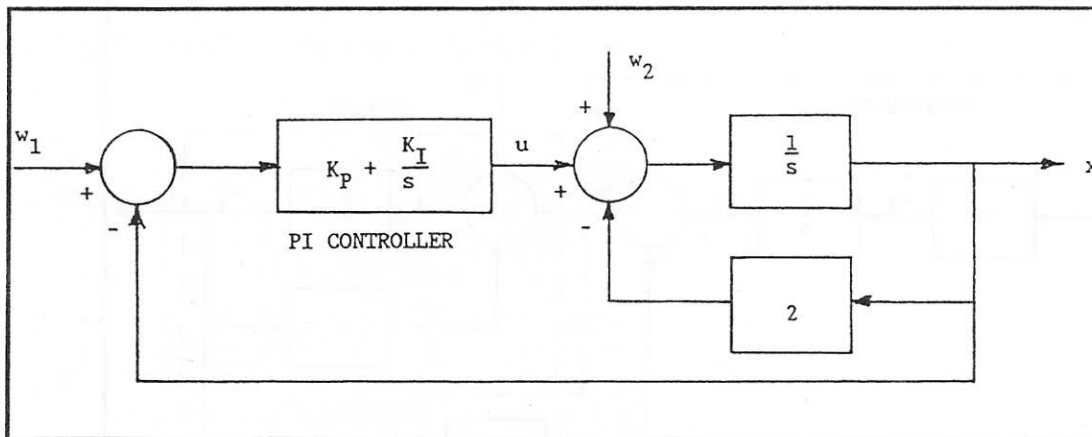
$$X(s) = X_1(s) = \frac{200W_1(s)s^{-2} + s^{-1}W_2(s)}{1 + 2s^{-1} + 18s^{-1} + 200s^{-2}} = \frac{200W_1(s) + sW_2(s)}{s^2 + 20s + 200}$$

$$W_1(s) = \frac{1}{s} \quad W_2(s) = \frac{W_2}{s} \quad W_2 = \text{constant}$$

$$X(s) = \frac{200 + W_2 s}{s(s^2 + 20s + 200)} \quad \lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} sX(s) = 1$$

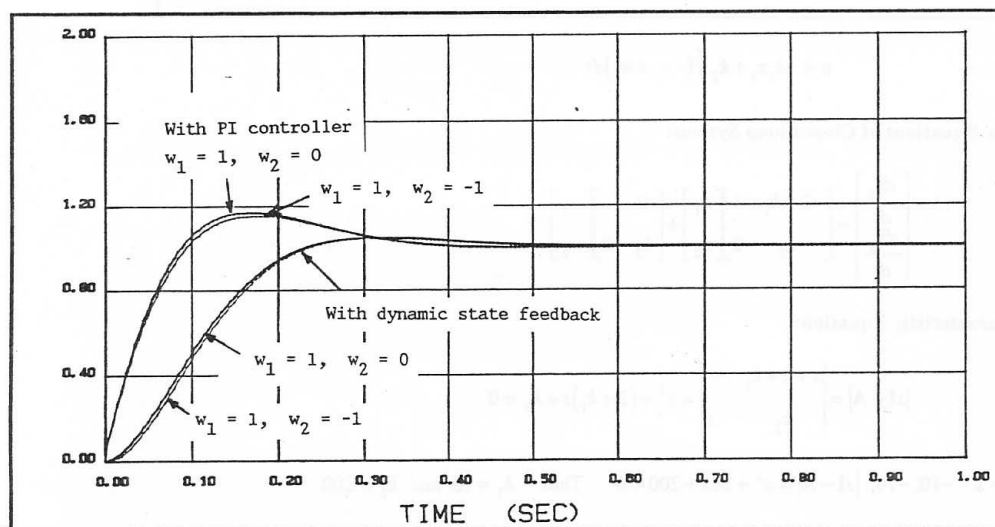
8-62 (b) With PI Controller:

Block Diagram of System:



Set $K_p = 2$ and $K_I = 200$.
$$X(s) = \frac{(K_p s + K_I)W_1(s) + sW_2(s)}{s^2 + 20s + 200} = \frac{(2s + 200)W_1(s) + sW_2(s)}{s^2 + 20s + 200}$$

Time Responses:



8-63)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10}{(s+1)(s+2)(s+3)} = \frac{10}{s^3 + 6s^2 + 11s + 6}$$

Consider:

$$\begin{cases} Y(s) = s^{-3}X(s) \\ X(s) = 10U(s) - (6s^{-1} + 11s^{-2} + 6s^{-3})X(s) \end{cases}$$

Therefore:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

As a result:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} \quad C = [1 \quad 0 \quad 0] \quad D = [0]$$

Using MATLAB, we'll find:

$$K = [15.4 \quad 4.5 \quad 0.8]$$

8-64)**Inverted Pendulum on a cart**

The equations of motion from Problem 4-21 are obtained (by ignoring all the pendulum inertia term):

$$(M+m)\ddot{x} - ml\ddot{\theta} \cos \theta + ml\dot{\theta}^2 \sin \theta = f$$

$$ml(-g \sin \theta - \ddot{x} \cos \theta + l\ddot{\theta}) = 0$$

These equations are nonlinear, but they can be linearized. Hence

$$\theta \approx 0$$

$$\cos \theta \approx 1$$

$$\sin \theta \approx \theta$$

$$(M+m)\ddot{x} + ml\ddot{\theta} = f$$

$$ml(-g\theta - \ddot{x} + l\ddot{\theta}) = 0$$

Or

$$\begin{bmatrix} (M+m) & ml \\ -ml & ml^2 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} f \\ mlg\theta \end{bmatrix}$$

Pre-multiply by inverse of the coefficient matrix

$$\text{inv}([(M+m), m*l; -m*l, m*l^2])$$

ans =

$$\begin{bmatrix} 1/(M+2*m), & -1/l/(M+2*m) \end{bmatrix}$$

$$\begin{bmatrix} 1/l/(M+2*m), & (M+m)/m/l^2/(M+2*m) \end{bmatrix}$$

For values of M=2, m=0.5, l=1, g=9.8

ans =

$$0.3333 \quad -0.3333$$

$$0.3333 \quad 1.6667$$

Hence

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 1/3 & -1/3 \\ 1/3 & 5/3 \end{bmatrix} \begin{bmatrix} f \\ 49/10\theta \end{bmatrix}$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 1/3*f-49/30\theta \\ 1/3*f+49/6\theta \end{bmatrix}$$

The state space model is:

$$\begin{bmatrix} \dot{x}_4 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1/3*f-49/30x_1 \\ 1/3*f+49/6x_1 \end{bmatrix}$$

Or:

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 1/3*f + 49/6 x_1 \\ 1/3*f - 49/30 x_1 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 49/6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -49/30 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/3 \\ 0 \\ 1/3 \end{bmatrix} f$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 49/6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -49/30 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1/3 \\ 0 \\ 1/3 \end{bmatrix}$$

$$C = [1 \quad 0 \quad 1 \quad 0]$$

$$D = 0$$

Use ACSYS State tool and follow the design process stated in Example 8-17-1:

The screenshot shows the 'State Space Tool' window with the following sections:

- Block Diagram:** Displays the state-space equations:
$$\frac{dx(t)}{dt} = \begin{bmatrix} A \end{bmatrix} x(t) + \begin{bmatrix} B \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} C \end{bmatrix} x(t) + \begin{bmatrix} D \end{bmatrix} u(t)$$
- Input Module:** Contains instructions and input fields for matrices A, B, C, D, and Initial Conditions.
 - Enter coefficient Matrices.
eg. For a 3x3 identity matrix enter [1 0 0;0 1 0;0 0 1]
 - [1;0;1] is a 3x1 column [1 0 0] is a 1x3 row.
 - A:** [0,1,0,0;49/6,0,0,0;0,0,0,1;-49/30,0,0,0]
 - B:** [0;1/3;0;1/3]
 - C:** [1,0,1,0]
 - D:** 0
 - Initial Conditions:** 0
- Buttons:** Contains 'Reset' and 'Close Window' buttons.

The A matrix is:

A_{mat} =

$$\begin{bmatrix} 0 & 1.0000 & 0 & 0 \\ 8.1667 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 \\ -1.6333 & 0 & 0 & 0 \end{bmatrix}$$

Characteristic Polynomial:

ans =

$$s^4 - 49/6 s^2$$

Eigenvalues of A = Diagonal Canonical Form of A is:

A_{bar} =

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2.8577 & 0 \\ 0 & 0 & 0 & -2.8577 \end{bmatrix}$$

Eigen Vectors are

T =

$$\begin{bmatrix} 0 & 0 & 0.3239 & -0.3239 \\ 0 & 0 & 0.9256 & 0.9256 \\ 1.0000 & -1.0000 & -0.0648 & 0.0648 \\ 0 & 0.0000 & -0.1851 & -0.1851 \end{bmatrix}$$

State-Space Model is:

a =

	x1	x2	x3	x4
x1	0	1	0	0
x2	8.167	0	0	0
x3	0	0	0	1
x4	-1.633	0	0	0

b =

	u1
x1	0
x2	0.3333

$$\begin{aligned} x_3 &= 0 \\ x_4 &= 0.3333 \end{aligned}$$

$$c = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$d = \begin{bmatrix} u_1 \\ y_1 & 0 \end{bmatrix}$$

Continuous-time model.
Characteristic Polynomial:

ans =

$$s^4 - 49/6 s^2$$

Equivalent Transfer Function Model is:

Transfer function:

$$4.441e-016 s^3 + 0.6667 s^2 - 2.22e-016 s - 3.267$$

$$s^4 - 8.167 s^2$$

Pole, Zero Form:

Zero/pole/gain:

$$4.4409e-016 (s+1.501e015) (s+2.214) (s-2.214)$$

$$s^2 (s-2.858) (s+2.858)$$

The Controllability Matrix $[B \ AB \ A^2B \ \dots]$ is =

Smat =

$$\begin{bmatrix} 0 & 0.3333 & 0 & 2.7222 \\ 0.3333 & 0 & 2.7222 & 0 \\ 0 & 0.3333 & 0 & -0.5444 \\ 0.3333 & 0 & -0.5444 & 0 \end{bmatrix}$$

The system is therefore Not Controllable, rank of S Matrix is =

rankS =

$$4$$

Mmat =

$$\begin{bmatrix} 0 & -8.1667 & 0 & 1.0000 \\ -8.1667 & 0 & 1.0000 & 0 \\ 0 & 1.0000 & 0 & 0 \\ 1.0000 & 0 & 0 & 0 \end{bmatrix}$$

The Controllability Canonical Form (CCF) Transformation matrix is:

Ptran =

$$\begin{bmatrix} 0 & 0 & 0.3333 & 0 \\ 0 & 0 & 0 & 0.3333 \\ -3.2667 & 0 & 0.3333 & 0 \\ 0 & -3.2667 & 0 & 0.3333 \end{bmatrix}$$

The transformed matrices using CCF are:

Abar =

$$\begin{bmatrix} 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 1.0000 \\ 0 & 0 & 8.1667 & 0 \end{bmatrix}$$

Bbar =

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Cbar =

$$\begin{bmatrix} -3.2667 & 0 & 0.6667 & 0 \end{bmatrix}$$

Dbar =

$$0$$

Note incorporating $-K$ in A_{bar} :

$A_{bar} K =$

$$\begin{bmatrix} 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 1.0000 \\ -k_1 & -k_2 & 8.1667-k_3 & -k_4 \end{bmatrix}$$

System Characteristic equation is:

$$-k_4 s^4 + (8.1667 - k_3) s^3 - k_2 s^2 - k_1 s = 0$$

From desired poles we have:

$$>> \text{collect}((s-210)*(s-210)*(s+20)*(s-12))$$

ans =

$$-10584000 + s^4 - 412s^3 + 40500s^2 + 453600s$$

Hence: $k_1 = 10584000$, $k_2 = 40500$, $k_3 = 412 + 8.1667$ and $k_4 = 1$

8-65) If $t_p = 3$ and $\xi = 0.707$, then $\omega_n = 1.414$. The 2nd order desired characteristic equation of the system is:

$$s^2 + 2s + 2 = 0 \quad (1)$$

On the other hand:

$$\dot{X} = (A - BK)x = \begin{bmatrix} 0 & 1 \\ -6 - K_1 & -5 - K_2 \end{bmatrix} x$$

where the characteristic equation would be:

$$s^2 + (5 + K_2)s + (6 + K_1) = 0 \quad (2)$$

Comparing equation (1) and (2) gives:

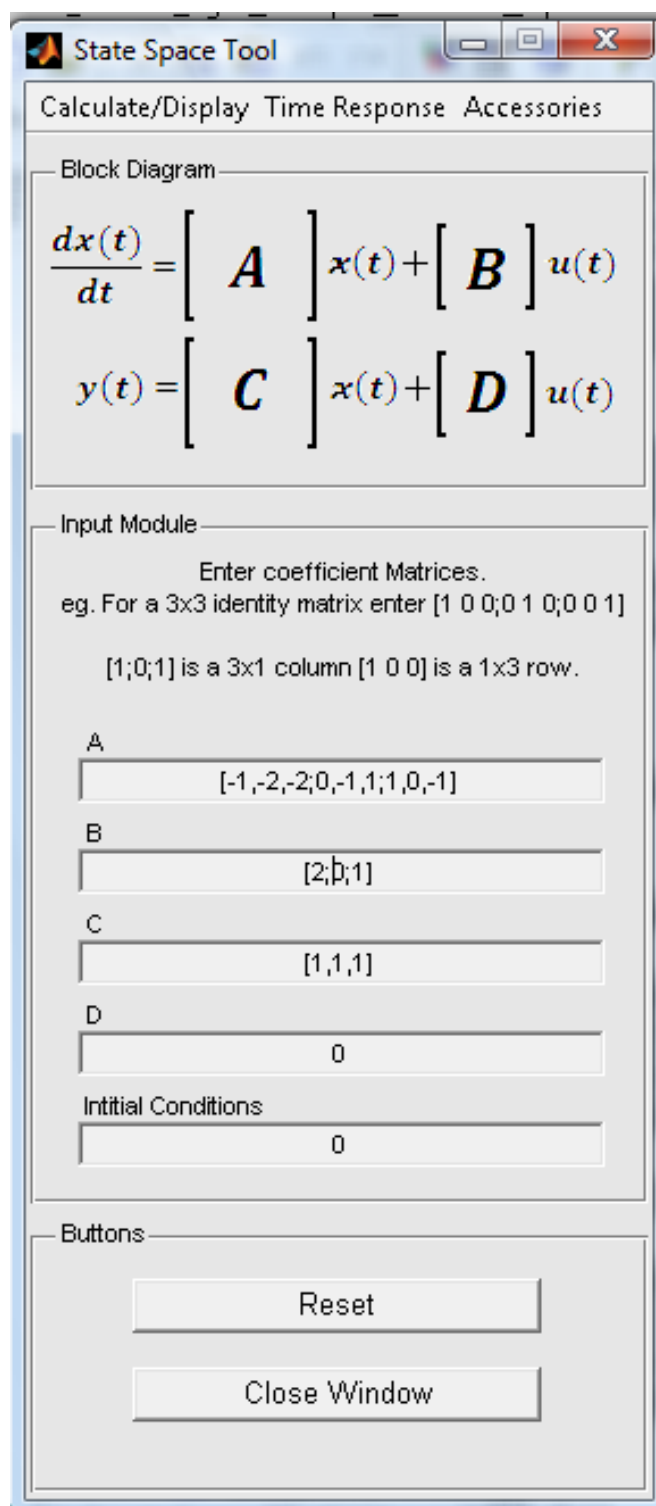
$$\begin{cases} 5 + K_2 = 2 \\ 6 + K_1 = 2 \end{cases}$$

which means $K_1 = -4$ and $K_2 = -3$

8-66) Using ACSYS we can convert the system into transfer function form.

USE ACSYS as illustrated in section 8-19-1

- 1) Activate MATLAB
- 2) Go to the folder containing ACSYS
- 3) Type in Acsys
- 4) Click the “State Space” pushbutton
- 5) Enter the A,B,C, and D values. Note C must be entered here and must have the same number of columns as A. We use [1,1] arbitrarily as it will not affect the eigenvalues.
- 6) Use the “Calculate/Display” menu and find the eigenvalues.
- 7) Next use the “Calculate/Display” menu and conduct State space calculations.
- 8) Next verify Controllability and find the \bar{A} matrix
- 9) Follow the design procedures in section 8-17 (pole placement)



The image shows a MATLAB State Space Tool window. It has a title bar 'State Space Tool' and three tabs: 'Calculate/Display', 'Time Response', and 'Accessories'. The 'Calculate/Display' tab is active. Under the 'Block Diagram' section, the state-space equations are displayed: $\frac{dx(t)}{dt} = \begin{bmatrix} A \end{bmatrix} x(t) + \begin{bmatrix} B \end{bmatrix} u(t)$ and $y(t) = \begin{bmatrix} C \end{bmatrix} x(t) + \begin{bmatrix} D \end{bmatrix} u(t)$. The 'Input Module' section contains instructions to enter coefficient matrices, with an example for a 3x3 identity matrix. It also provides a note on matrix dimensions: '[1;0;1] is a 3x1 column [1 0 0] is a 1x3 row.' Below this, there are input fields for matrices A, B, C, and D, and an 'Initial Conditions' field. Matrix A is set to [-1,-2,-2;0,-1,1;1,0,-1], B to [2;0;1], C to [1,1,1], D to 0, and Initial Conditions to 0. The 'Buttons' section at the bottom contains 'Reset' and 'Close Window' buttons.

State Space Tool

Calculate/Display Time Response Accessories

Block Diagram

$$\frac{dx(t)}{dt} = \begin{bmatrix} A \end{bmatrix} x(t) + \begin{bmatrix} B \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} C \end{bmatrix} x(t) + \begin{bmatrix} D \end{bmatrix} u(t)$$

Input Module

Enter coefficient Matrices.
eg. For a 3x3 identity matrix enter [1 0 0;0 1 0;0 0 1]

[1;0;1] is a 3x1 column [1 0 0] is a 1x3 row.

A

[-1,-2,-2;0,-1,1;1,0,-1]

B

[2;0;1]

C

[1,1,1]

D

0

Initial Conditions

0

Buttons

Reset

Close Window

The A matrix is:

A_{mat} =

$$\begin{matrix} -1 & -2 & -2 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{matrix}$$

$$\begin{matrix} 0 & -1 & 1 \\ 1 & 0 & -1 \end{matrix}$$

$$\begin{matrix} 1 & 0 & -1 \end{matrix}$$

Characteristic Polynomial:

ans =

$$s^3 + 3s^2 + 5s + 5$$

Eigenvalues of A = Diagonal Canonical Form of A is:

A_{bar} =

$$\begin{matrix} -0.6145 + 1.5639i & 0 & 0 \\ 0 & -0.6145 - 1.5639i & 0 \\ 0 & 0 & -1.7709 \end{matrix}$$

$$\begin{matrix} 0 & -0.6145 - 1.5639i & 0 \\ 0 & 0 & -1.7709 \end{matrix}$$

$$\begin{matrix} 0 & 0 & -1.7709 \end{matrix}$$

Eigen Vectors are

T =

$$\begin{matrix} -0.8074 & -0.8074 & -0.4259 \\ 0.2756 + 0.1446i & 0.2756 - 0.1446i & -0.7166 \\ -0.1200 + 0.4867i & -0.1200 - 0.4867i & 0.5524 \end{matrix}$$

$$\begin{matrix} 0.2756 + 0.1446i & 0.2756 - 0.1446i & -0.7166 \\ -0.1200 + 0.4867i & -0.1200 - 0.4867i & 0.5524 \end{matrix}$$

$$\begin{matrix} -0.1200 + 0.4867i & -0.1200 - 0.4867i & 0.5524 \end{matrix}$$

State-Space Model is:

a =

$$\begin{matrix} x1 & x2 & x3 \end{matrix}$$

$$\begin{matrix} x1 & -1 & -2 & -2 \end{matrix}$$

$$\begin{matrix} x2 & 0 & -1 & 1 \end{matrix}$$

$$\begin{matrix} x3 & 1 & 0 & -1 \end{matrix}$$

$\mathbf{b} =$

$$u_1$$

$$x_1 \quad 2$$

$$x_2 \quad 0$$

$$x_3 \quad 1$$

$\mathbf{c} =$

$$x_1 \quad x_2 \quad x_3$$

$$y_1 \quad 1 \quad 1 \quad 1$$

$\mathbf{d} =$

$$u_1$$

$$y_1 \quad 0$$

Continuous-time model.

Characteristic Polynomial:

ans =

$$s^3 + 3s^2 + 5s + 5$$

Equivalent Transfer Function Model is:

Transfer function:

$$\frac{3s^2 + 7s + 4}{s^3 + 3s^2 + 5s + 5}$$

$$s^3 + 3s^2 + 5s + 5$$

Pole, Zero Form:

Zero/pole/gain:

$$3(s+1.333)(s+1)$$

$$(s+1.771)(s^2 + 1.229s + 2.823)$$

The Controllability Matrix $[B \ AB \ A^2B \ \dots]$ is =

Smat =

$$\begin{bmatrix} 2 & -4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -5 \end{bmatrix}$$

The system is therefore Controllable, rank of S Matrix is =

rankS =

$$3$$

Mmat =

$$\begin{bmatrix} 5 & 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

The Controllability Canonical Form (CCF) Transformation matrix is:

Ptran =

$$\begin{bmatrix} -2 & 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 & 1 \end{bmatrix}$$

The transformed matrices using CCF are:

Abar =

$$\begin{bmatrix} 0 & 1.0000 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1.0000 \end{bmatrix}$$

$$\begin{bmatrix} -5.0000 & -5.0000 & -3.0000 \end{bmatrix}$$

Bbar =

0

0

1

Cbar =

4 7 3

Dbar =

0

Using Equation (8-324) we get:

$$|sI - (A - BK)| = s^3 + (3 + k_3)s^2 + (5 + k_2)s + (5 + k_1) = 0$$

Using a 2nd order prototype system, for $t_s \leq 5$, then $\xi\omega_n = 1$. For overshoot of 4.33%, $\xi = 0.707$. Then the desired 2nd order system will have a characteristic equation:

$$s^2 + 2\xi\omega_n s + \omega_n^2 = s^2 + 2s + 2 = 0$$

The above system poles are: $s_{1,2} = -1 \pm j$

One approach is to pick $K=[k_1 \ k_2 \ k_3]$ values so that two poles of the system are close to the desired second order poles and the third pole reduces the effect of the two system zeros that are at $z=-1.333$ and $z=-1$. Let's set the third pole at $s=-1.333$. Hence

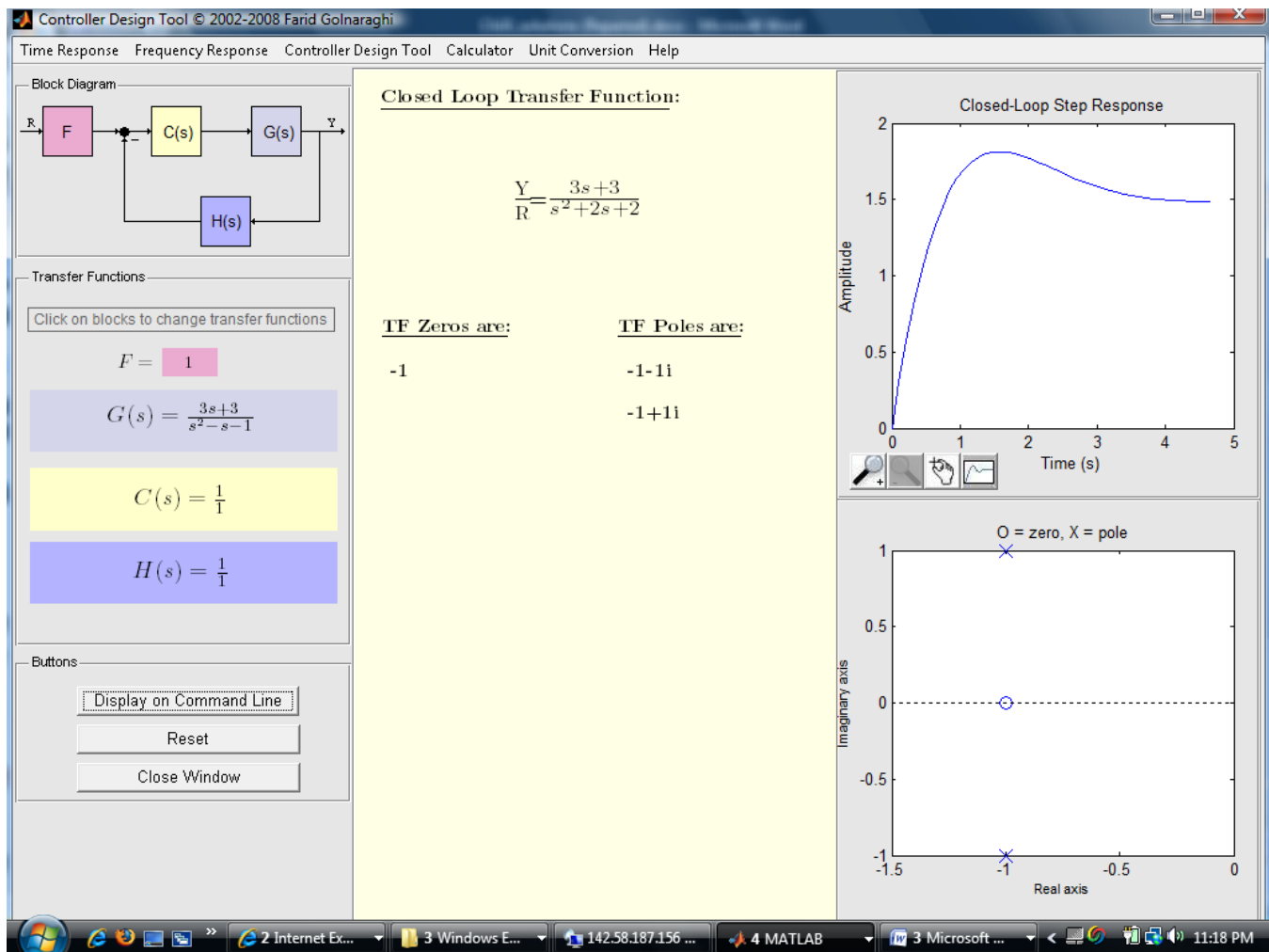
$$(s+1.333)*(s^2+2s+2) = s^3+3.33s^2+4.67s+2.67$$

and $K=[-2.37 \ -0.37 \ 0.33]$.

$$\frac{Y}{R} = \frac{3(s+1)}{s^2 + 2s + 2}$$

Use ACSYS control tool to find the time response. First convert the transfer function to a unity feedback system to make compatible to the format used in the Control toolbox.

$$G = \frac{3(s+1)}{s^2 - s - 1}$$



Overshoot is about 2%. You can adjust K values to obtain alternative results by repeating this process.

8-67) a) According to the circuit:

$$L \frac{di_2}{dt} = R_2 i_2 = v_c + R_1 C \frac{dv_c}{dt}$$

$$\frac{dv_c}{dt} = i(t) - i_2$$

$$y = (i(t) - i_2)R_2$$

If $i_2 = x_1$, $v_c = x_2$ and $i(t) = u$, then

$$\begin{cases} L\dot{x}_1 + R_2x_1 = x_2 + R_1C\dot{x}_2 \\ \dot{x}_2 = \frac{1}{C}(u - x_1) \\ y = (u - x_1)R_2 \end{cases}$$

or

$$\begin{cases} \dot{x}_1 = -\frac{2R_2}{L}x_1 + \frac{1}{L}x_2 + \frac{R_1}{L}(u - x_1) \\ \dot{x}_2 = \frac{1}{C}(u - x_1) \\ y = (u - x_1)R_2 \end{cases}$$

Therefore:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{2R_2}{L} & \frac{1}{L} \\ -\frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{R_1}{L} \\ \frac{1}{C} \end{bmatrix} u$$

$$y = [-R_2 \quad 0]x + R_2u$$

b) Uncontrollability condition is:

$$\det[B \ AB] = \det(C) = 0$$

According to the state-space of the system, C is calculated as:

$$C = \begin{bmatrix} \frac{R_1}{L} & -\frac{2R_1R_2}{L^2} + \frac{1}{LC} \\ \frac{1}{C} & -\frac{R_1}{LC} \end{bmatrix}$$

$$\det(C) = \frac{R_1R_2}{L^2C} - \frac{1}{LC^2}$$

As $\det(C) \neq 0$, because $R_1R_2 \neq RC$, then the system is controllable

c) Unobservability condition is:

$$\det \begin{bmatrix} C \\ CA \end{bmatrix} = \det(H) = 0$$

According to the state-space of the system, C is calculated as:

$$H = \begin{bmatrix} -R_2 & 0 \\ \frac{2R_2}{L} & -\frac{R_2}{L} \end{bmatrix}$$

$$\det(H) = \frac{R_2^2}{L}$$

Since $\det(H) \neq 0$, because $R \neq 0$ or $L \neq \infty$, then the system is observable.

d) The same as part (a)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1 C} & 0 \\ 0 & \frac{R_2}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1 C} \\ \frac{1}{L} \end{bmatrix} u$$

$$y = \begin{bmatrix} -\frac{1}{R_1} & 1 \end{bmatrix} x + \frac{1}{R_1} u$$

For controllability, we define G as:

$$G = [B \ AB] = \begin{bmatrix} \frac{1}{R_1 C} & -\frac{1}{(R_1 C)^2} \\ \frac{1}{L} & -\frac{R_2}{L^2} \end{bmatrix}$$

$$\det(G) = -\frac{R_2}{R_1 C L^2} + \frac{1}{L(R_1 C)^2}$$

If $R_1 R_2 C = L$, and then $\det(G) = 0$, which means the system is not controllable.

For observability, we define H as:

$$H = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1} & 1 \\ \frac{1}{R_1^2 C} & -\frac{R_2}{L} \end{bmatrix}$$

$$\det(H) = \frac{R_2}{R_1 L} - \frac{1}{R_1^2 C}$$

If $R_1 R_2 C = L$, then $\det(H) = 0$, which means the system is not observable.

Chapter 9

9-1 (a) $P(s) = s^4 + 4s^3 + 4s^2 + 8s$ $Q(s) = s + 1$

Finite zeros of $P(s)$: $0, -3.5098, -0.24512 \pm j1.4897$

Finite zeros of $Q(s)$: -1

Asymptotes: **$K > 0$:** $60^\circ, 180^\circ, 300^\circ$ **$K < 0$:** $0^\circ, 120^\circ, 240^\circ$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-3.5 - 0.24512 - 0.24512 - (-1)}{4 - 1} = -1$$

(b) $P(s) = s^3 + 5s^2 + s$ $Q(s) = s + 1$

Finite zeros of $P(s)$: $0, -4.7912, -0.20871$

Finite zeros of $Q(s)$: -1

Asymptotes: **$K > 0$:** $90^\circ, 270^\circ$ **$K < 0$:** $0^\circ, 180^\circ$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-4.7913 - 0.2087 - (-1)}{3 - 1} = -2$$

(c) $P(s) = s^2$ $Q(s) = s^3 + 3s^2 + 2s + 8$

Finite zeros of $P(s)$: $0, 0$

Finite zeros of $Q(s)$: $-3.156, 0.083156 \pm j1.5874$

Asymptotes: **$K > 0$:** 180° **$K < 0$:** 0°

$$(d) \quad P(s) = s^3 + 2s^2 + 3s \quad Q(s) = (s^2 - 1)(s + 3)$$

Finite zeros of $P(s)$: $0, -1 \pm j1.414$

Finite zeros of $Q(s)$: $1, -1, -3$

Asymptotes: **There are no asymptotes, since the number of zeros of $P(s)$ and $Q(s)$ are equal.**

$$(e) \quad P(s) = s^5 + 2s^4 + 3s^3 \quad Q(s) = s^2 + 3s + 5$$

Finite zeros of $P(s)$: $0, 0, 0, -1 \pm j1.414$

Finite zeros of $Q(s)$: $-1.5 \pm j1.6583$

Asymptotes: **$K > 0$:** $60^\circ, 180^\circ, 300^\circ$ **$K < 0$:** $0^\circ, 120^\circ, 240^\circ$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-1 - 1 - (-1.5) - (-1.5)}{5 - 2} = \frac{1}{3}$$

$$(f) \quad P(s) = s^4 + 2s^2 + 10 \quad Q(s) = s + 5$$

Finite zeros of $P(s)$: $-1.0398 \pm j1.4426, 1.0398 \pm j1.4426$

Finite zeros of $Q(s)$: -5

Asymptotes: **$K > 0$:** $60^\circ, 180^\circ, 300^\circ$ **$K < 0$:** $0^\circ, 120^\circ, 240^\circ$

Intersect of Asymptotes:

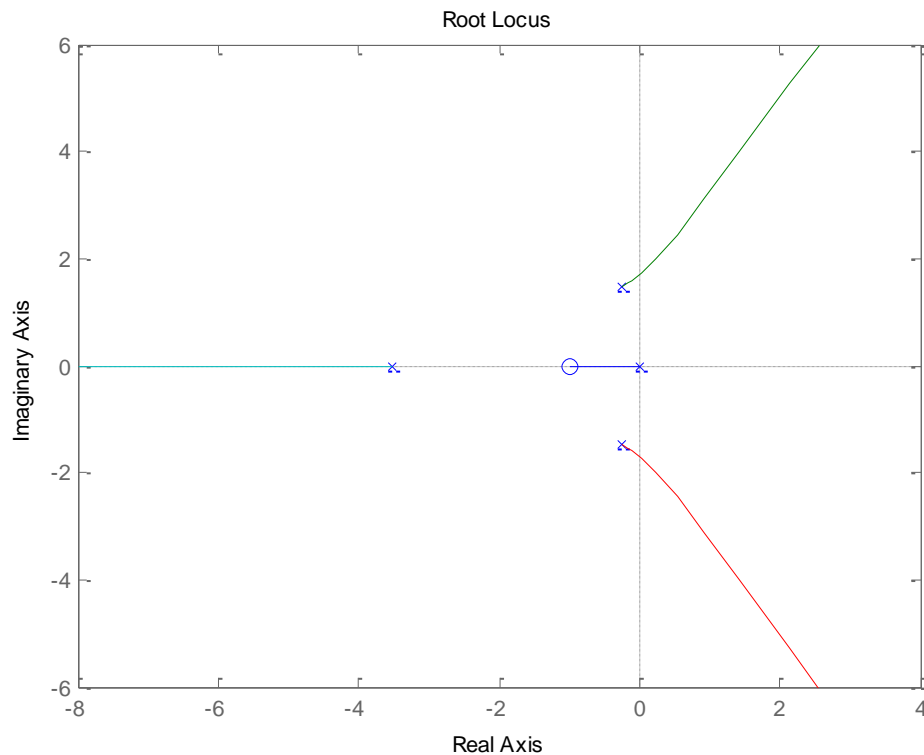
$$\sigma_1 = \frac{-1.0398 - 1.0398 + 1.0398 + 1.0398 - (-5)}{4 - 1} = \frac{-5}{3}$$

9-2(a) MATLAB code:

```

s = tf('s')
num_GH=(s+1);
den_GH=(s^4+4*s^3+4*s^2+8*s);
GH_a=num_GH/den_GH;
figure(1);
rlocus(GH_a)
GH_p=pole(GH_a)
GH_z=zero(GH_a)
n=length(GH_p)    %number of poles of G(s)H(s)
m=length(GH_z)    %number of zeros of G(s)H(s)
%Asymptotes angles:
k=0;
Assymp1_angle=+180*(2*k+1)/(n-m)
Assymp2_angle=-180*(2*k+1)/(n-m)
k=1;
Assymp3_angle=+180*(2*k+1)/(n-m)
%Asymptotes intersection point on real axis:
sigma=(sum(GH_p)-sum(GH_z))/(n-m)

```



Assymp1_angle = 60

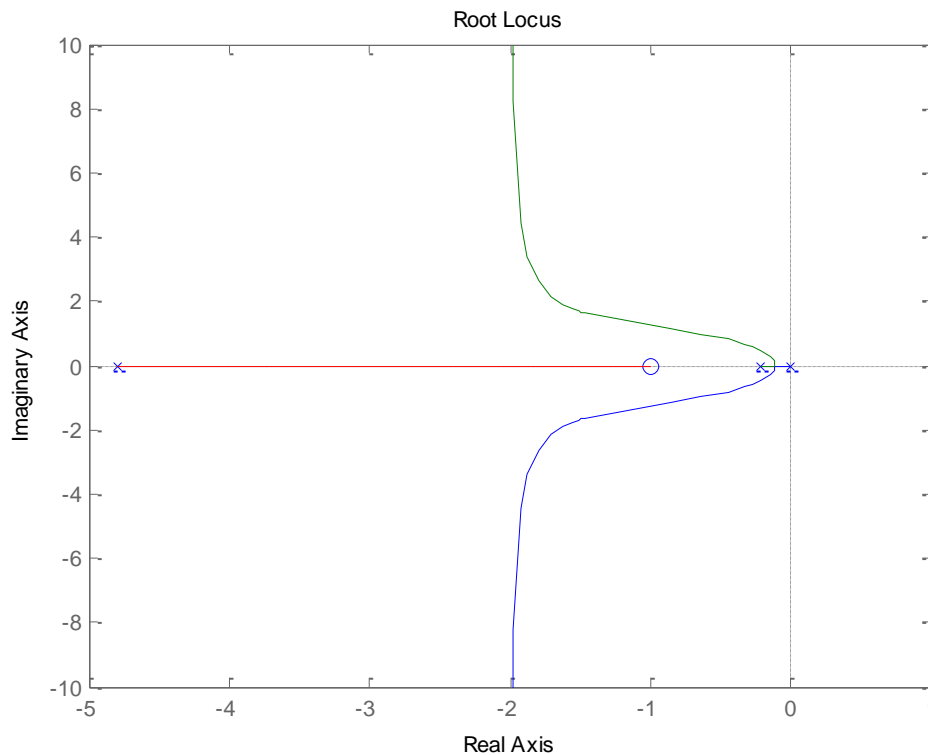
Assymp2_angle = -60

Assymp3_angle = 180

$\sigma = -1.0000$ (intersect of asymptotes)

9-2(b) MATLAB code:

```
s = tf('s')
'Generating the transfer function:'
num_GH=(s+1);
den_GH=(s^3+5*s^2+s);
GH_a=num_GH/den_GH;
figure(1);
rlocus(GH_a)
GH_p=pole(GH_a)
GH_z=zero(GH_a)
n=length(GH_p)    %number of poles of G(s)H(s)
m=length(GH_z)    %number of zeros of G(s)H(s)
%Asymptotes angles:
k=0;
Assymp1_angle=+180*(2*k+1)/(n-m)
Assymp2_angle=-180*(2*k+1)/(n-m)
%Asymptotes intersection point on real axis:
sigma=(sum(GH_p)-sum(GH_z))/(n-m)
```



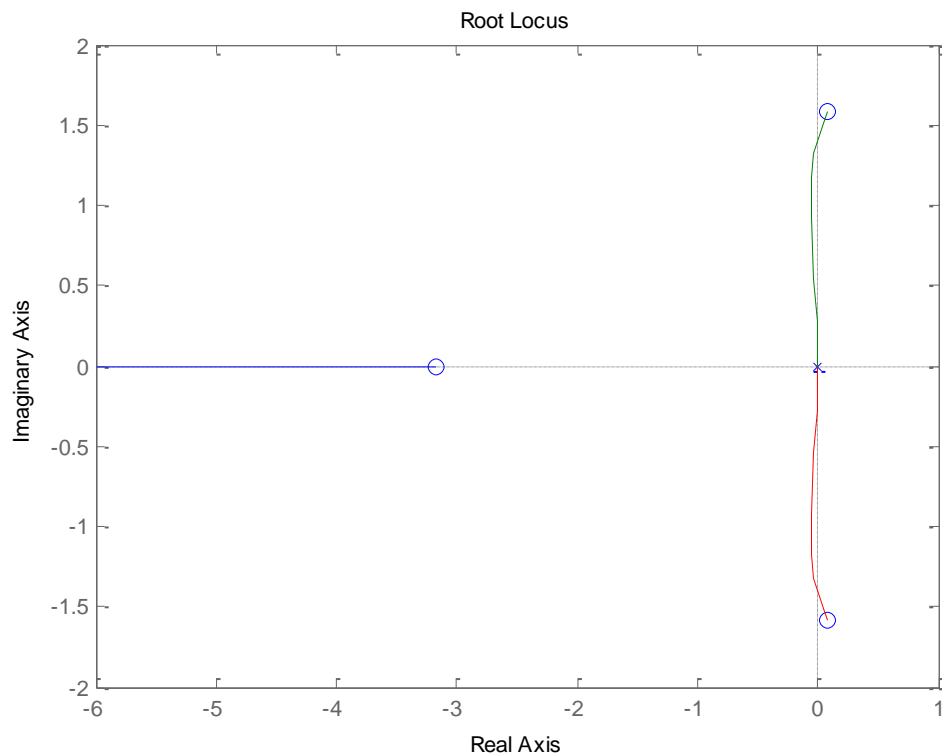
Assymp1_angle = 90

Assymp2_angle = -90

$\sigma = -2$ (intersect of asymptotes)

9-2(c) MATLAB code:

```
s = tf('s')
'Generating the transfer function:'
num_GH=(s^3+3*s^2+2*s+8);
den_GH=(s^2);
GH_a=num_GH/den_GH;
figure(1);
rlocus(GH_a)
GH_p=pole(GH_a)
GH_z=zero(GH_a)
n=length(GH_p)    %number of poles of G(s)H(s)
m=length(GH_z)    %number of zeros of G(s)H(s)
%Asymptotes angles:
k=0;
Assymp1_angle=+180*(2*k+1)/(n-m)
%Asymptotes intersection point on real axis:
sigma=(sum(GH_p)-sum(GH_z))/(n-m)
```

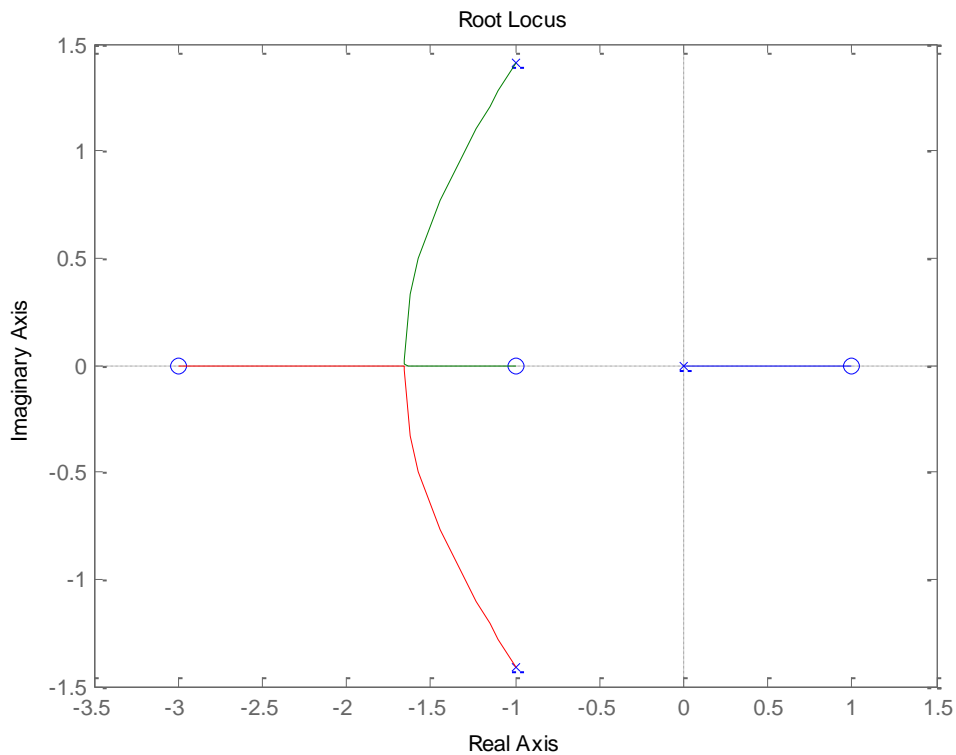


Assymp1_angle = 180

sigma = -3.0000 (intersect of asymptotes)

9-2(d) MATLAB code:

```
s = tf('s')
'Generating the transfer function:'
num_GH = ((s^2-1)*(s+3));
den_GH = (s^3+2*s^2+3*s);
GH_a = num_GH/den_GH;
figure(1);
rlocus(GH_a)
GH_p = pole(GH_a)
GH_z = zero(GH_a)
n = length(GH_p)    %number of poles of G(s)H(s)
m = length(GH_z)    %number of zeros of G(s)H(s)
```



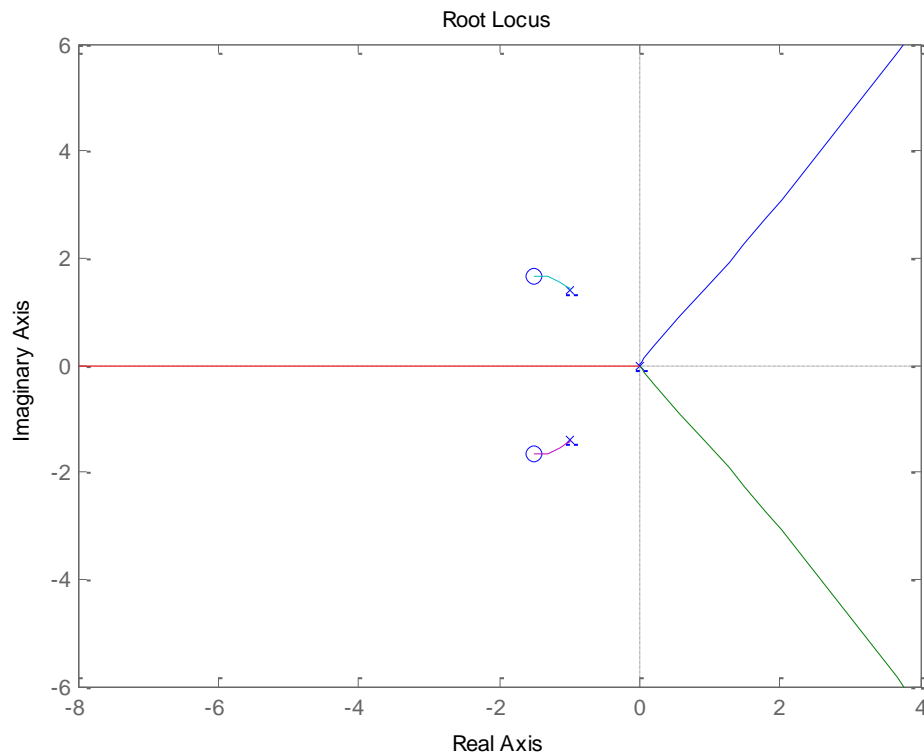
No asymptotes

9-2(e) MATLAB code:

```

s = tf('s')
'Generating the transfer function:'
num_GH=(s^2+3*s+5);
den_GH=(s^5+2*s^4+3*s^3);
GH_a=num_GH/den_GH;
figure(1);
rlocus(GH_a)
GH_p=pole(GH_a)
GH_z=zero(GH_a)
n=length(GH_p)    %number of poles of G(s)H(s)
m=length(GH_z)    %number of zeros of G(s)H(s)
%Asymptotes angles:
k=0;
Assymp1_angle=+180*(2*k+1)/(n-m)
Assymp2_angle=-180*(2*k+1)/(n-m)
k=1;
Assymp3_angle=+180*(2*k+1)/(n-m)
%Asymptotes intersection point on real axis:
sigma=(sum(GH_p)-sum(GH_z))/(n-m)

```



Assymp1_angle = 60

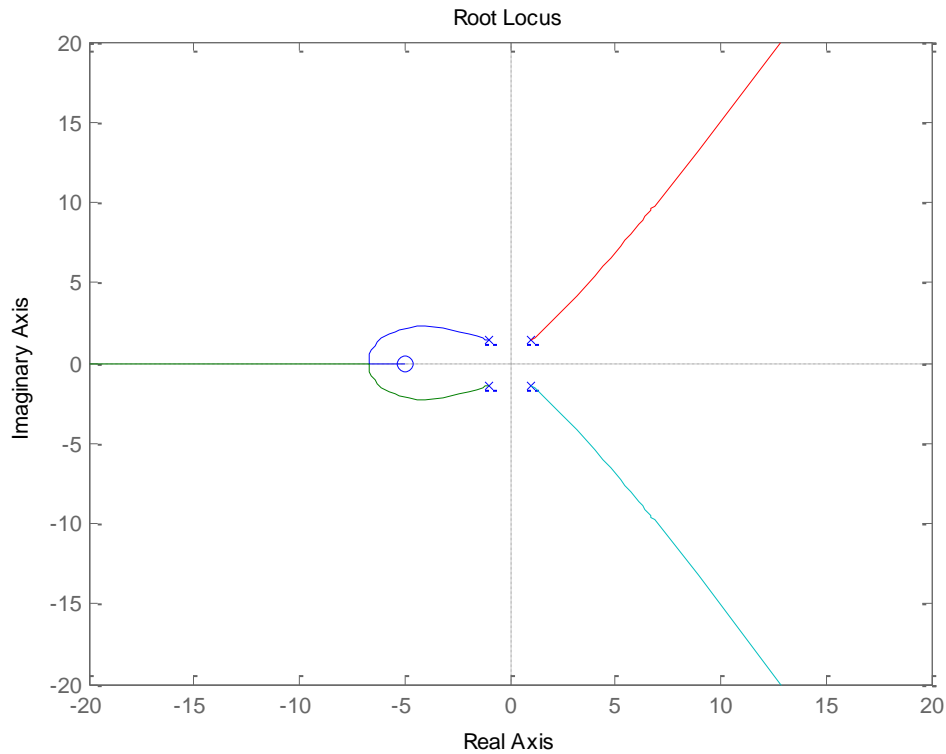
Assymp2_angle = -60

Assymp3_angle = 180

sigma = 0.3333 (intersect of asymptotes)

9-2(f) MATLAB code:

```
s = tf('s')
'Generating the transfer function:'
num_GH=(s+5);
den_GH=(s^4+2*s^2+10);
GH_a=num_GH/den_GH;
figure(1);
rlocus(GH_a)
xlim([-20 20])
ylim([-20 20])
GH_p=pole(GH_a)
GH_z=zero(GH_a)
n=length(GH_p)    %number of poles of G(s)H(s)
m=length(GH_z)    %number of zeros of G(s)H(s)
%Asymptotes angles:
k=0;
Assymp1_angle=+180*(2*k+1)/(n-m)
Assymp2_angle=-180*(2*k+1)/(n-m)
k=1;
Assymp3_angle=+180*(2*k+1)/(n-m)
%Asymptotes intersection point on real axis:
sigma=(sum(GH_p)-sum(GH_z))/(n-m)
```



Assymp1_angle = 60

Assymp2_angle = -60

Assymp3_angle = 180

sigma = 1.6667 (intersect of asymptotes)

9-3) Consider

$$G(s)H(s) = K \frac{Q(s)}{P(s)} = K \frac{\prod_{i=1}^m (s + z_i)}{\prod_{i=1}^n (s + p_i)}$$

As the asymptotes are the behavior of $G(s)H(s)$ when $|s| \rightarrow \infty$, then

$|s| > |z_i|$ for $i = 1, 2, \dots, m$ and $|s| > |p_i|$ for $i = 1, 2, \dots, n$

therefore $\angle G(s)H(s) = m \arg(s) - n \arg(s) = -(n - m) \arg(s)$

According to the condition on angles:

$$\angle G(s)H(s) = \begin{cases} 2(i+1)\pi & K \geq 0 \\ 2i\pi & K \leq 0 \end{cases}$$

If we consider $\arg(s) = \theta_i$, then:

$$\angle G(s)H(s) = \begin{cases} -(n-m)\theta_i = (2i+1)\pi & K \geq 0 \\ -(n-m)\theta_i = 2i\pi & K \leq 0 \end{cases}$$

or

$$\begin{cases} \theta_i = \frac{2i+1}{|n-m|}\pi & K \geq 0 \\ \theta_i = \frac{2i}{|n-m|}\pi & K \leq 0 \end{cases}$$

- 9-4)** If $G(s)H(s) = K \frac{Q(s)}{P(s)}$, then each point on root locus must satisfy the characteristic equation of $P(s) + KQ(s) = 0$

If $P(s) = s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0$ and $Q(s) = s^m + b_{m-1}s^{m-1} + \dots + b_1s + b_0$, then

$$s^n + a_{n-1}s^{n-1} + \dots + a_0 + K(s^m + b_{m-1}s^{m-1} + \dots + b_0) = 0$$

or

$$s^{n-m} + (a_{n-1} - b_{m-1})s^{n-m-1} + \dots + K = 0$$

If the roots of above expression is considered as s_i for $i = 1, 2, \dots, (n-m)$, then

$$a_{n-1} - b_{m-1} = - \sum_{i=1}^{n-m} s_i = - \sum_{i=1}^n s_i - \sum_{i=1}^m s_i = - \sum_{i=1}^n p_i - \sum_{i=1}^m z_i$$

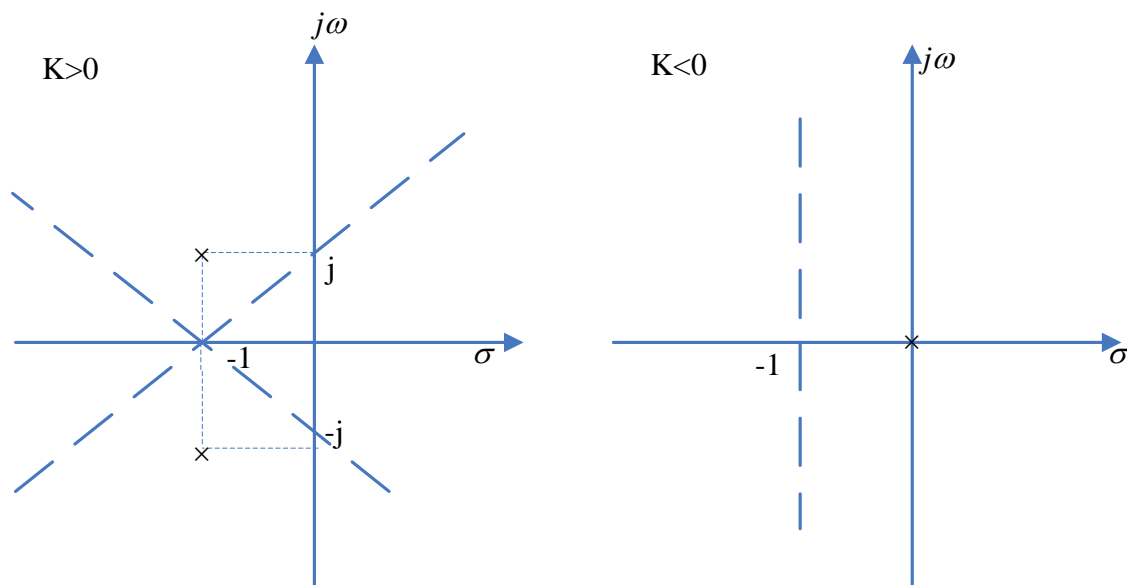
since the intersect of $(n-m)$ asymptotes lies on the real axis of the s -plane and $-(a_{n-1} - b_{m-1})$ is real, therefore

$$\sigma_1 = - \frac{a_{n-1} - b_{m-1}}{n-m} = - \frac{\sum_{i=1}^n p_i - \sum_{i=1}^m z_i}{n-m}$$

- 9-5)** Poles of GH is $s = 0, -2, -1 + j, -1 - j$, therefore the center of asymptotes:

$$\sigma_1 = \frac{\sum p_i - \sum z_i}{n-m} = -1$$

The angles of asymptotes: $\begin{cases} \theta_i = 45^\circ, 135^\circ, 225^\circ, 315^\circ & K > 0 \\ \theta_i = 0^\circ, 90^\circ, 180^\circ, 270^\circ & K < 0 \end{cases}$



9-6 (a) Angles of departure and arrival.

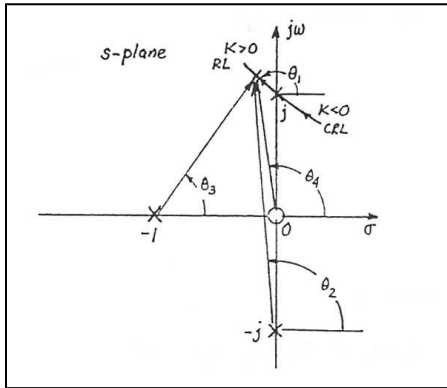
$$K > 0: -\theta_1 - \theta_2 - \theta_3 + \theta_4 = -180^\circ$$

$$-\theta_1 - 90^\circ - 45^\circ + 90^\circ = -180^\circ$$

$$\theta_1 = 135^\circ$$

$$K < 0: -\theta_1 - 90^\circ - 45^\circ + 90^\circ = 0^\circ$$

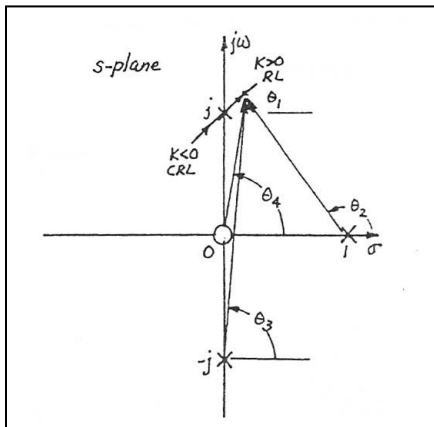
$$\theta_1 = -45^\circ$$

**(b) Angles of departure and arrival.**

$$K > 0: -\theta_1 - \theta_2 - \theta_3 + \theta_4 = -180^\circ$$

$$K < 0: -\theta_1 - 135^\circ - 90^\circ + 90^\circ = 0^\circ$$

$$\theta_1 = -135^\circ$$

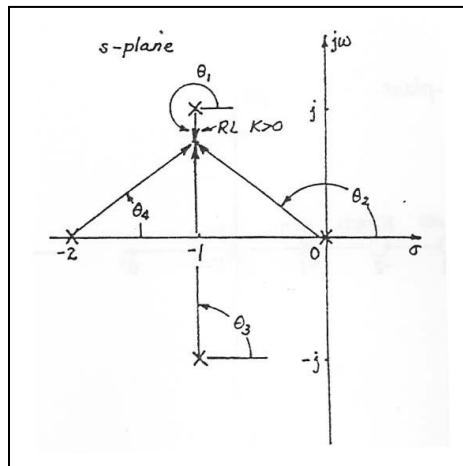


(c) Angle of departure:

$$K > 0: \quad -\theta_1 - \theta_2 - \theta_3 + \theta_4 = -180^\circ$$

$$-\theta_1 - 135^\circ - 90^\circ - 45^\circ = -180^\circ$$

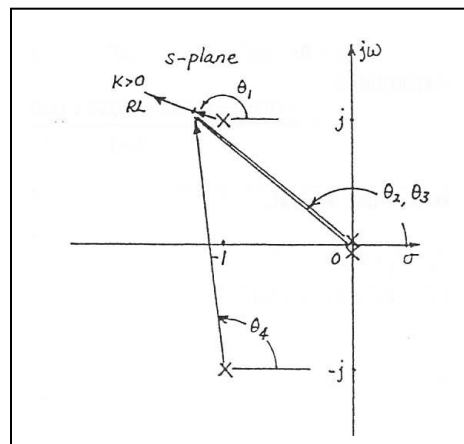
$$\theta_1 = -90^\circ$$

**(d) Angle of departure**

$$K > 0: \quad -\theta_1 - \theta_2 - \theta_3 - \theta_4 = -180^\circ$$

$$-\theta_1 - 135^\circ - 135^\circ - 90^\circ = -180^\circ$$

$$\theta_1 = -180^\circ$$

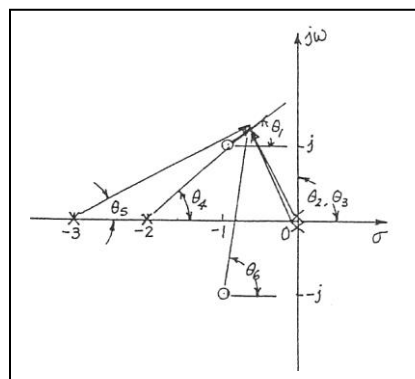


(e) Angle of arrival

$$\mathbf{K} < \mathbf{0}: \quad \theta_1 + \theta_6 - \theta_2 - \theta_3 - \theta_4 - \theta_5 = -360^\circ$$

$$\theta_1 + 90^\circ - 135^\circ - 135^\circ - 45^\circ - 26.565^\circ = -360^\circ$$

$$\theta_1 = -108.435^\circ$$



$$\begin{aligned}
 \text{9-7) a) } \angle G(s)H(s) &= \sum_{i=1}^m \angle(s + z_i) - \sum_{i=1}^n \angle(s + p_i) \\
 &= \sum_{i=1}^m \angle(s + z_i) + \sum_{i=1, i \neq j}^n \angle(s + p_i) - \angle(s + p_j) \\
 &= \angle G(s)H'(s) - \angle(s + p_j) \\
 &= \angle G(s)H'(s) - \theta_D
 \end{aligned}$$

$$\text{we know that } \angle G(s)H(s) = \begin{cases} (2i+1) \times 180 & K \geq 0 \quad i = 0, \pm 1, \dots \\ (2i) \times 180 & K \leq 0 \quad i = 0, \pm 1, \dots \end{cases}$$

therefore

$$\begin{cases} \angle G(s)H'(s) - \theta_D = 180 & K \geq 0 \\ \angle G(s)H'(s) - \theta_D = 0 & K \leq 0 \end{cases}$$

As a result, $\theta_D = \angle G(s)H'(s) - 180^\circ = 180 + \angle G(s)H'(s)$, when $-180^\circ = 180^\circ$

b) Similarly:

$$\begin{aligned}
 \angle G(s)H(s) &= \sum_{i=1}^m \angle(s + z_i) - \sum_{i=1}^n \angle(s + p_i) \\
 &= \sum_{i=1, i \neq j}^m \angle(s + z_i) + \sum_{i=1}^n \angle(s - p_i) + \angle(s + z_j) \\
 &= \angle G(s)H''(s) + \angle(s + z_j) \\
 &= \angle G(s)H''(s) + \theta
 \end{aligned}$$

Therefore:

$$\begin{cases} \angle G(s)H''(s) + \theta = 180 & K \geq 0 \\ \angle G(s)H''(s) + \theta = 0 & K \leq 0 \end{cases}$$

As a result, $\theta = 180 - \angle G(s)H''(s)$

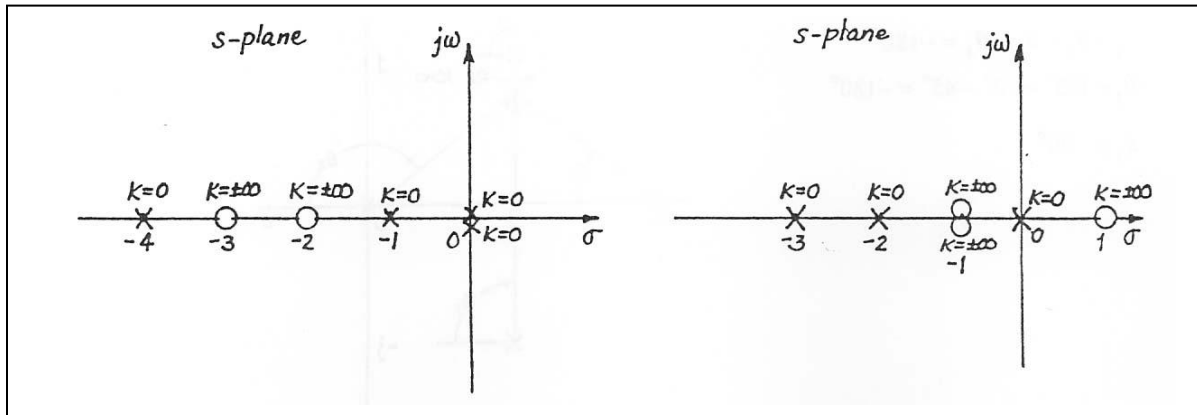
9-8) zeros: $s = -1 - j, -1 + j$ and poles: $s = 0, -2j, +2j$

$$\text{Departure angles from: } \begin{cases} s = 2j & : \quad \theta = 180 - 63.4 = 116.6 \\ s = -2j & : \quad \theta = -198.4 \end{cases}$$

$$\text{Arrival angles at } \begin{cases} s = -1 + j & : \quad \theta = 180 - (-18.4) = 198.4 \\ s = -1 - j & : \quad \theta = -198.4 \end{cases}$$

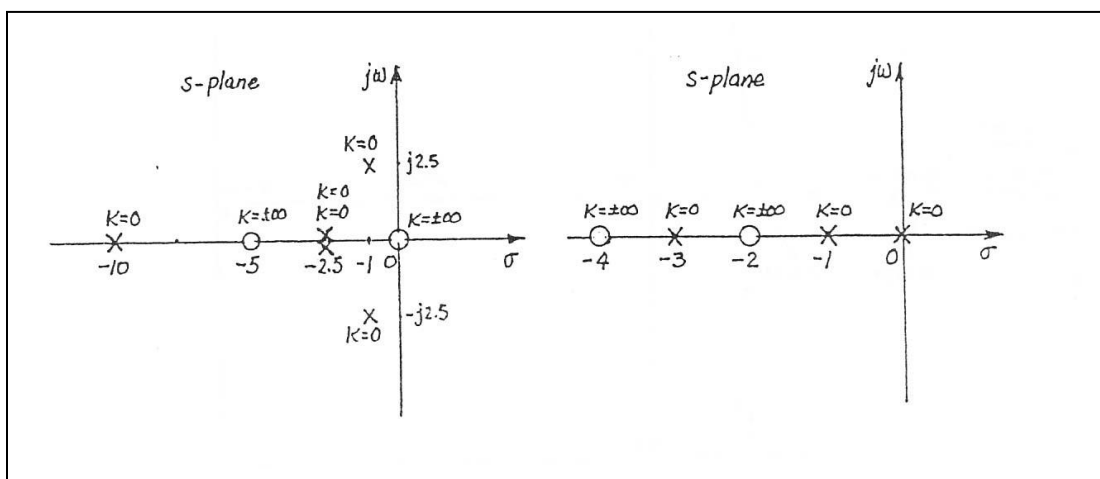
9-9) (a)

(b)



(c)

(d)



9-10) The breaking points are on the real axis of $1 + KG(s)H(s) = 0$ and must satisfy

$$\frac{dG(s)H(s)}{ds} = 0$$

If $G(s)H(s) = \frac{Q(s)}{P(s)}$ and α is a breakaway point, then

$$1 + K \frac{Q(\alpha)}{P(\alpha)} = 0 \rightarrow K = -\frac{P(\alpha)}{Q(\alpha)}$$

Finding α where K is maximum or minimum $\frac{dK}{d\alpha} = 0$, therefore

$$\frac{d}{d\alpha} \left[\frac{P(\alpha)}{Q(\alpha)} \right] = 0$$

or

$$\frac{d}{d\alpha} \left[\frac{(\alpha + p_1)(\alpha + p_2) \dots (\alpha + p_n)}{(\alpha + z_1)(\alpha + z_2) \dots (\alpha + z_m)} \right] = 0$$

$$\sum_{i=1}^n \frac{1}{\alpha + p_i} \left[\frac{P(\alpha)}{Q(\alpha)} \right] - \sum_{i=1}^m \frac{1}{\alpha + z_i} \left[\frac{P(\alpha)}{Q(\alpha)} \right] = 0$$

$$\sum_{i=1}^n \frac{1}{\alpha + p_i} = \sum_{i=1}^m \frac{1}{\alpha + z_i}$$

9-11) (a) Breakaway-point Equation: $2s^5 + 20s^4 + 74s^3 + 110s^2 + 48s = 0$

Breakaway Points: $-0.7275, -2.3887$

(b) Breakaway-point Equation: $3s^6 + 22s^5 + 65s^4 + 100s^3 + 86s^2 + 44s + 12 = 0$

Breakaway Points: $-1, -2.5$

(c) Breakaway-point Equation: $3s^6 + 54s^5 + 347.5s^4 + 925s^3 + 867.2s^2 - 781.25s - 1953 = 0$

Breakaway Points: $-2.5, 1.09$

(d) Breakaway-point Equation: $-s^6 - 8s^5 - 19s^4 + 8s^3 + 94s^2 + 120s + 48 = 0$

Breakaway Points: $-0.6428, 2.1208$

9-12) (a)

$$G(s)H(s) = \frac{K(s+8)}{s(s+5)(s+6)}$$

Asymptotes: $K > 0$: 90° and 270° $K < 0$: 0° and 180°

Intersect of Asymptotes:

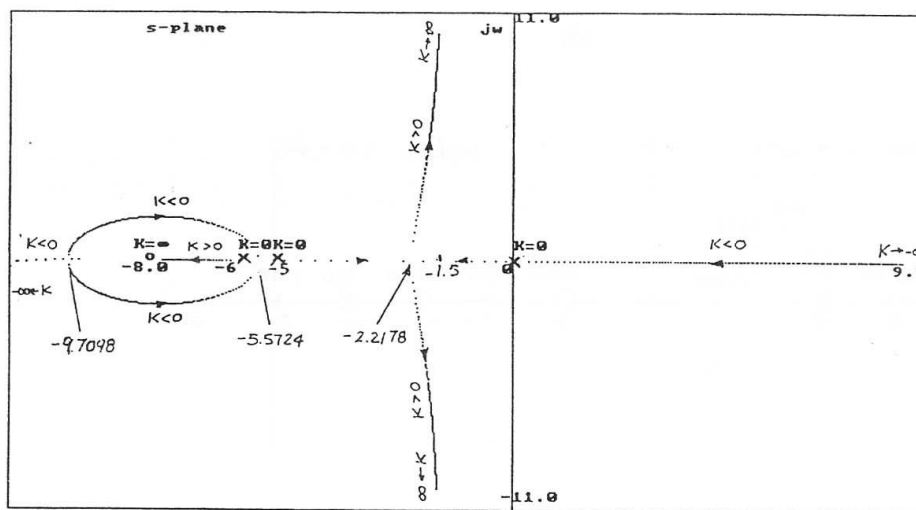
$$\sigma_1 = \frac{0 - 5 - 6 - (-8)}{3 - 1} = -1.5$$

Breakaway-point Equation:

$$2s^3 + 35s^2 + 176s + 240 = 0$$

Breakaway Points: $-2.2178, -5.5724, -9.7098$

Root Locus Diagram:



9-12 (b)

$$G(s)H(s) = \frac{K}{s(s+1)(s+3)(s+4)}$$

Asymptotes: $K > 0$: $45^\circ, 135^\circ, 225^\circ, 315^\circ$ $K < 0$: $0^\circ, 90^\circ, 180^\circ, 270^\circ$

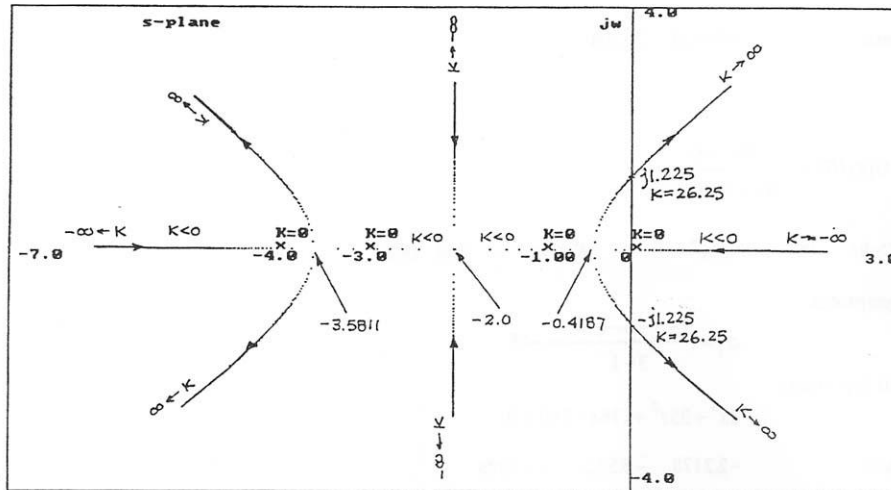
Intersect of Asymptotes:

$$\sigma_1 = \frac{0-1-3-4}{4} = -2$$

Breakaway-point Equation: $4s^3 + 24s^2 + 38s + 12 = 0$

Breakaway Points: $-0.4189, -2, -3.5811$

Root Locus Diagram:



9-12 (c)

$$G(s)H(s) = \frac{K(s+4)}{s^2(s+2)^2}$$

Asymptotes: $K > 0$: $60^\circ, 180^\circ, 300^\circ$

$K < 0$: $0^\circ, 120^\circ, 240^\circ$

Intersect of Asymptotes:

$$\sigma_1 = \frac{0+0-2-2-(-4)}{4-1} = 0$$

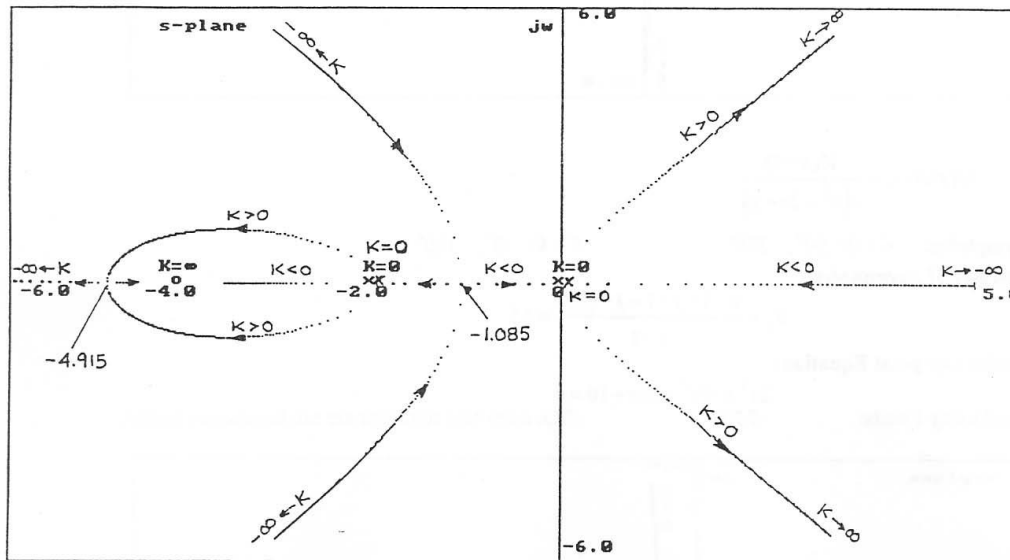
Breakaway-point Equation:

$$3s^4 + 24s^3 + 52s^2 + 32s = 0$$

Breakaway Points:

$$0, \quad -1.085, \quad -2, \quad -4.915$$

Root Locus Diagram:



9-12 (d)

$$G(s)H(s) = \frac{K(s+2)}{s(s^2 + 2s + 2)}$$

Asymptotes: $K > 0$: $90^\circ, 270^\circ$ $K < 0$: $0^\circ, 180^\circ$

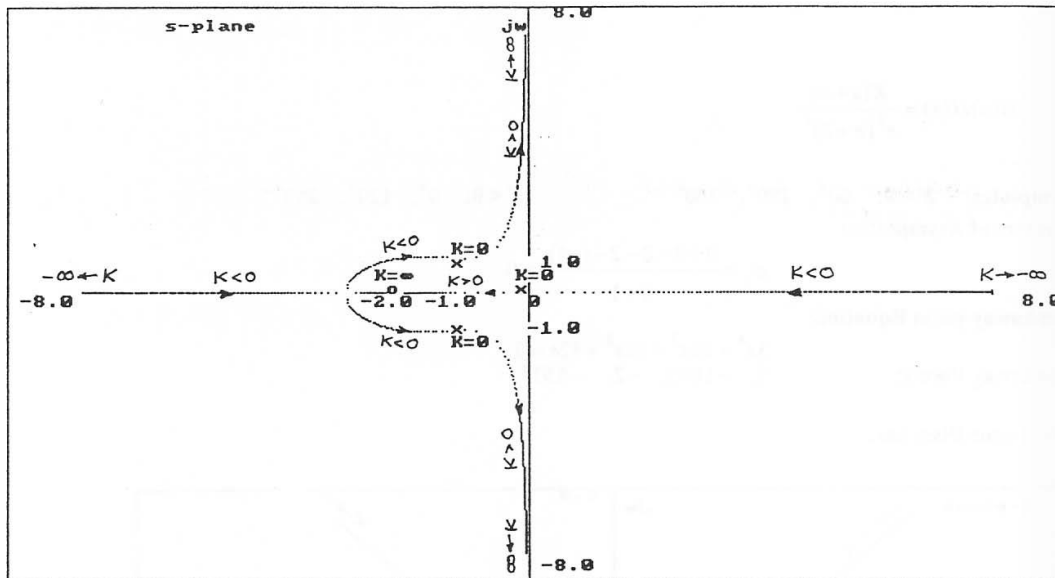
Intersect of Asymptotes:

$$\sigma_1 = \frac{0 - 1 - j - 1 - j - (-2)}{3 - 1} = 0$$

Breakaway-point Equation: $2s^3 + 8s^2 + 8s + 4 = 0$

Breakaway Points: -2.8393 The other two solutions are not breakaway points.

Root Locus Diagram



9-12 (e)

$$G(s)H(s) = \frac{K(s+5)}{s(s^2 + 2s + 2)}$$

Asymptotes: $K > 0$: $90^\circ, 270^\circ$ $K < 0$: $0^\circ, 180^\circ$

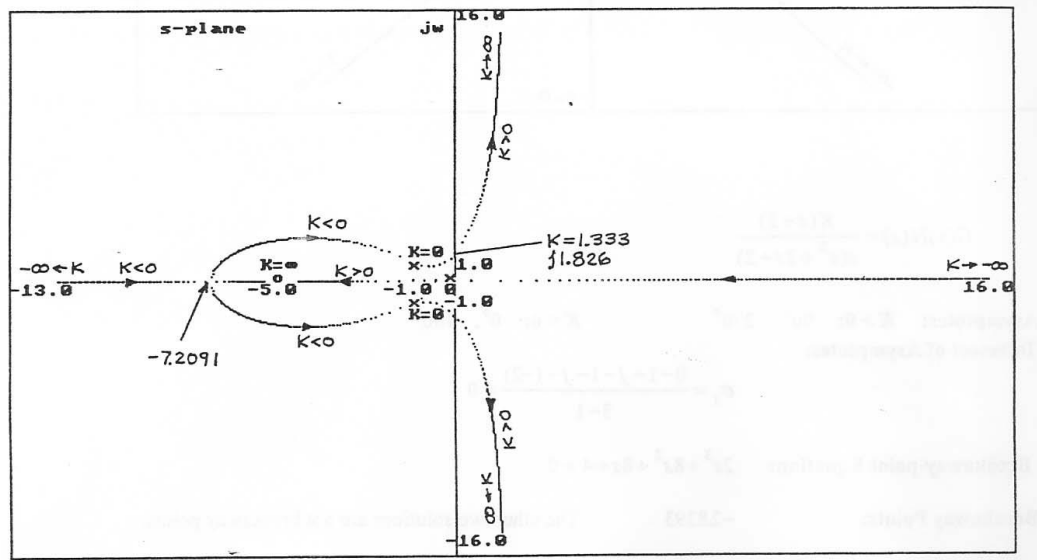
Intersect of Asymptotes:

$$\sigma_1 = \frac{0 - 1 - j - 1 - j - (-5)}{3 - 1} = 1.5$$

Breakaway-point Equation:

$$2s^3 + 17s^2 + 20s + 10 = 0$$

Breakaway Points: -7.2091 The other two solutions are not breakaway points.



9-12 (f)

$$G(s)H(s) = \frac{K}{s(s+4)(s^2+2s+2)}$$

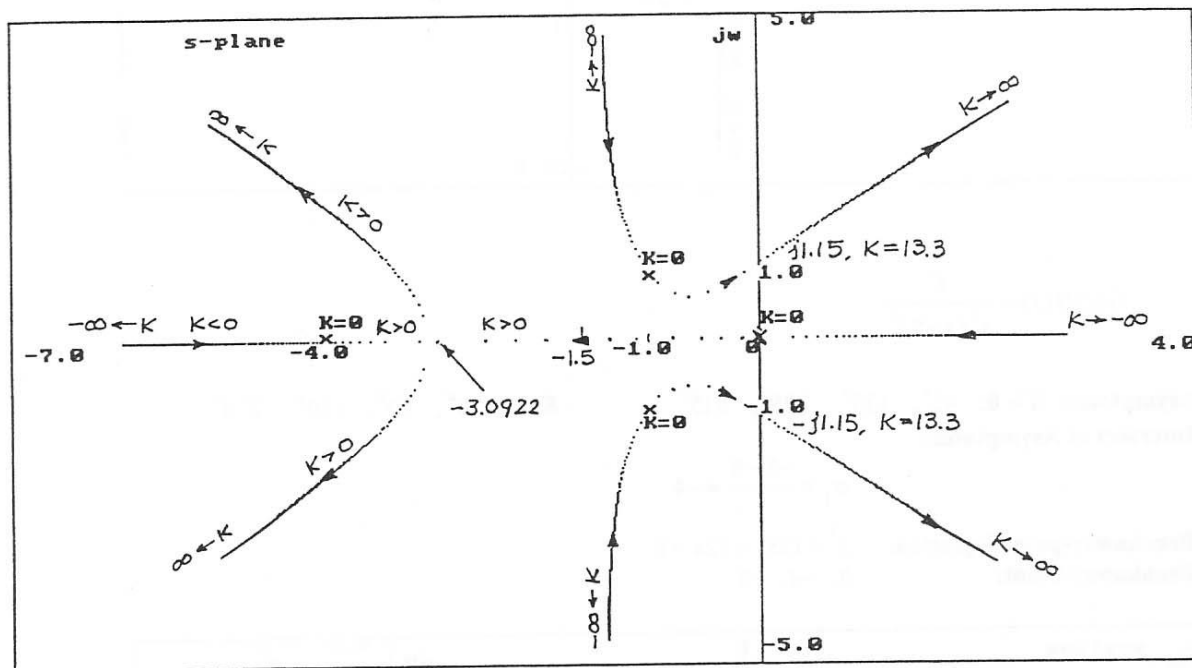
Asymptotes: $K > 0$: $45^\circ, 135^\circ, 225^\circ, 315^\circ$ $K < 0$: $0^\circ, 90^\circ, 180^\circ, 270^\circ$

Intersect of Asymptotes:

$$\sigma_1 = \frac{0 - 1 - 1 + j - 1 + j - 4}{4} = -1.5$$

Breakaway-point Equation: $4s^3 + 18s^2 + 20s + 8 = 0$

Breakaway Point: -3.0922 The other solutions are not breakaway points.



9-12 (g)

$$G(s)H(s) = \frac{K(s+4)^2}{s^2(s+8)^2}$$

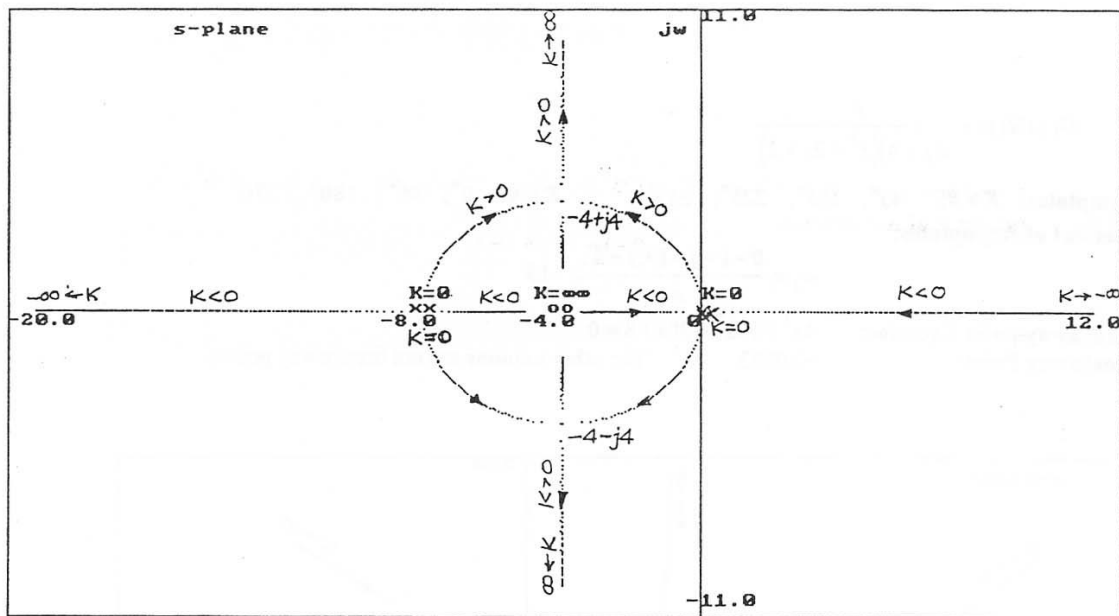
Asymptotes: $K > 0$: $90^\circ, 270^\circ$ $K < 0$: $0^\circ, 180^\circ$

Intesect of Asymptotes:

$$\sigma_1 = \frac{0+0-8-8-(-4)-(-4)}{4-2}$$

Breakaway-point Equation: $s^5 + 20s^4 + 160s^3 + 640s^2 + 1040s = 0$

Breakaway Points: $0, -4, -8, -4-j4, -4+j4$



9-12 (h)

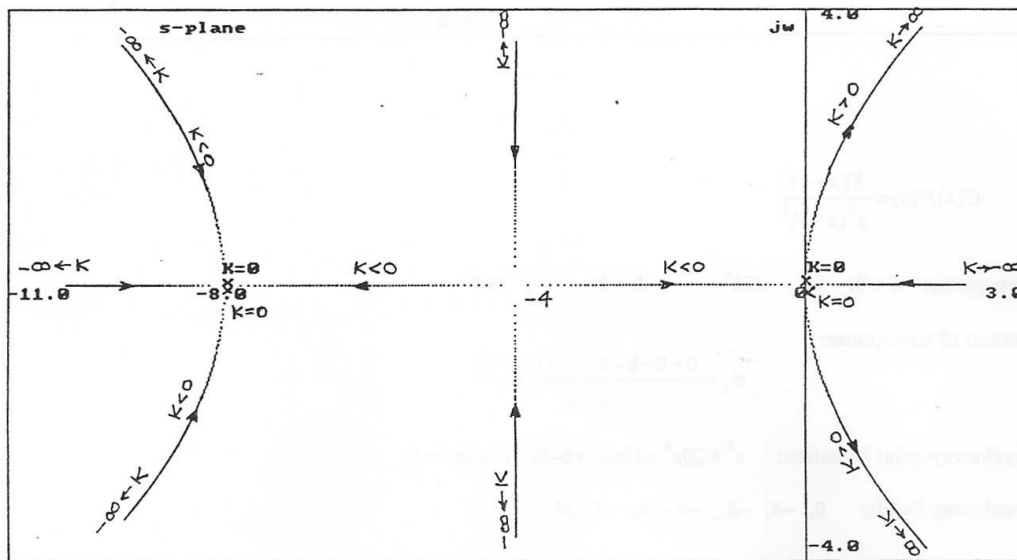
$$G(s)H(s) = \frac{K}{s^2(s+8)^2}$$

Asymptotes: $K > 0$: 45° , 135° , 225° , 315° $K < 0$: 0° , 90° , 180° , 270°

Intersect of Asymptotes:

$$\sigma_1 = \frac{-8-8}{4} = -4$$

Breakaway-point Equation: $s^3 + 12s^2 + 32s = 0$ Breakaway Point: $0, -4, -8$



9-12 (i)

$$G(s)H(s) = \frac{K(s^2 + 8s + 20)}{s^2(s+8)^2}$$

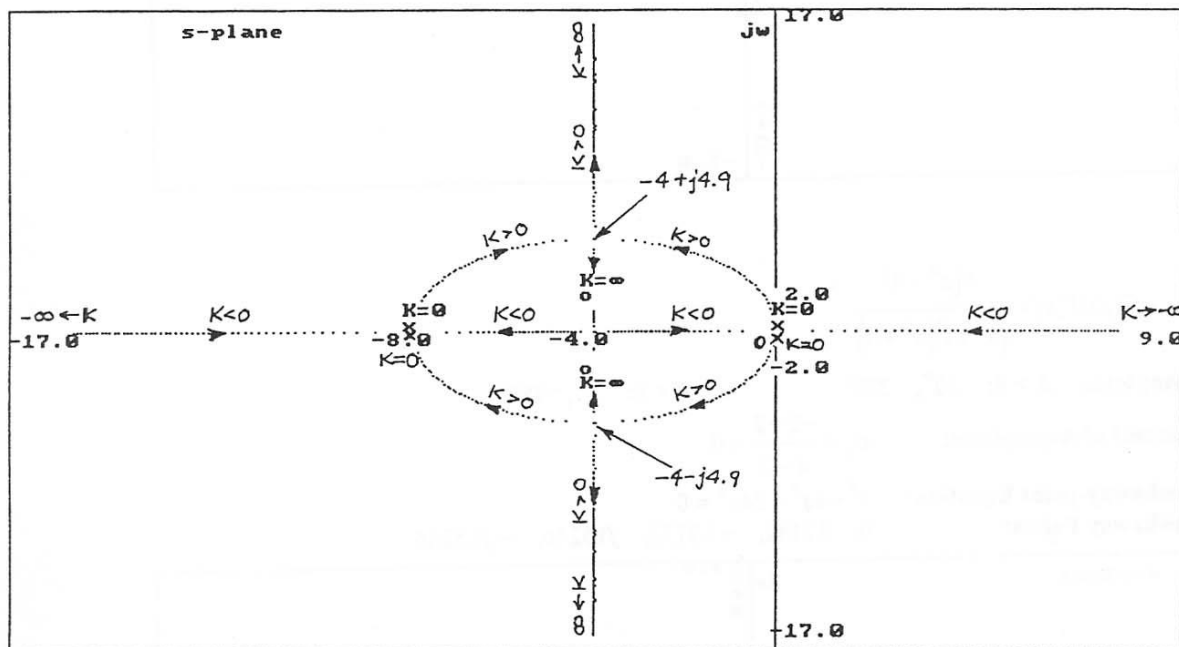
Asymptotes: $K > 0$: 90° , 270° $K < 0$: 0° , 180°

Intersect of Asymptotes:

$$\sigma_1 = \frac{-8 - 8 - (-4) - (-4)}{4 - 2} = -4$$

Breakaway-point Equation: $s^5 + 20s^4 + 128s^3 + 736s^2 + 1280s = 0$

Breakaway Points: -4 , -8 , $-4 + j4.9$, $-4 - j4.9$



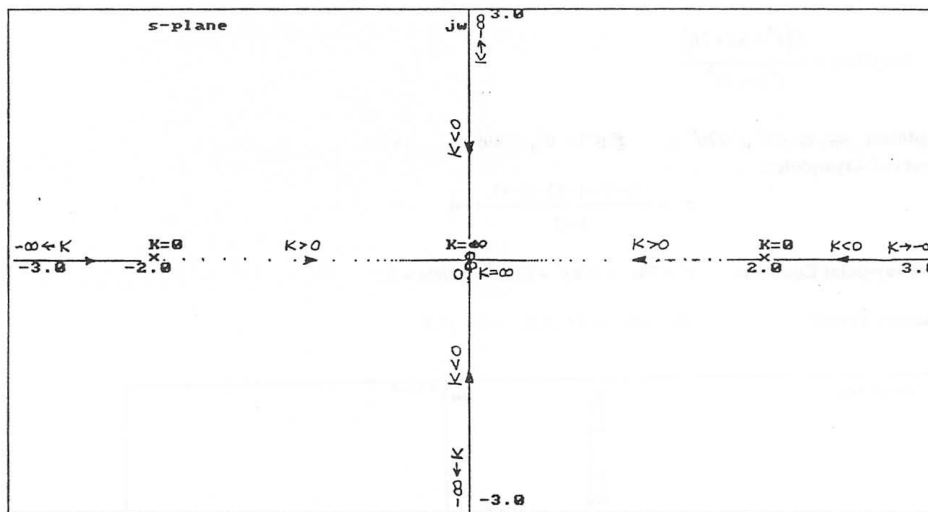
(j)

$$G(s)H(s) = \frac{Ks^2}{(s^2 - 4)}$$

Since the number of finite poles and zeros of $G(s)H(s)$ are the same, there are no asymptotes.

Breakaway-point Equation: $8s = 0$

Breakaway Points: $s = 0$



9-12 (k)

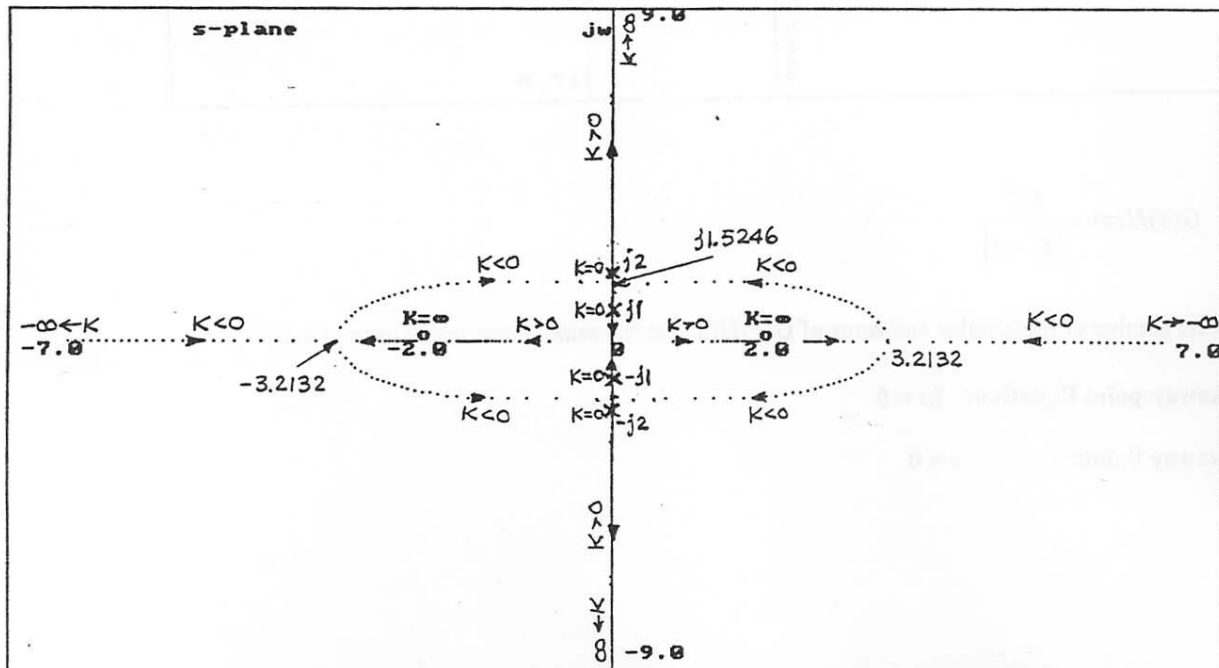
$$G(s)H(s) = \frac{K(s^2 - 4)}{(s^2 + 1)(s^2 + 4)}$$

Asymptotes: $K > 0$: $90^\circ, 270^\circ$ $K < 0$: $0^\circ, 180^\circ$

Intersect of Asymptotes: $\sigma_1 = \frac{-2+2}{4-2} = 0$

Breakaway-point Equation: $s^6 - 8s^4 - 24s^2 = 0$

Breakaway Points: $0, 3.2132, -3.2132, j1.5246, -j1.5246$



9-12 (I)

$$G(s)H(s) = \frac{K(s^2 - 1)}{(s^2 + 1)(s^2 + 4)}$$

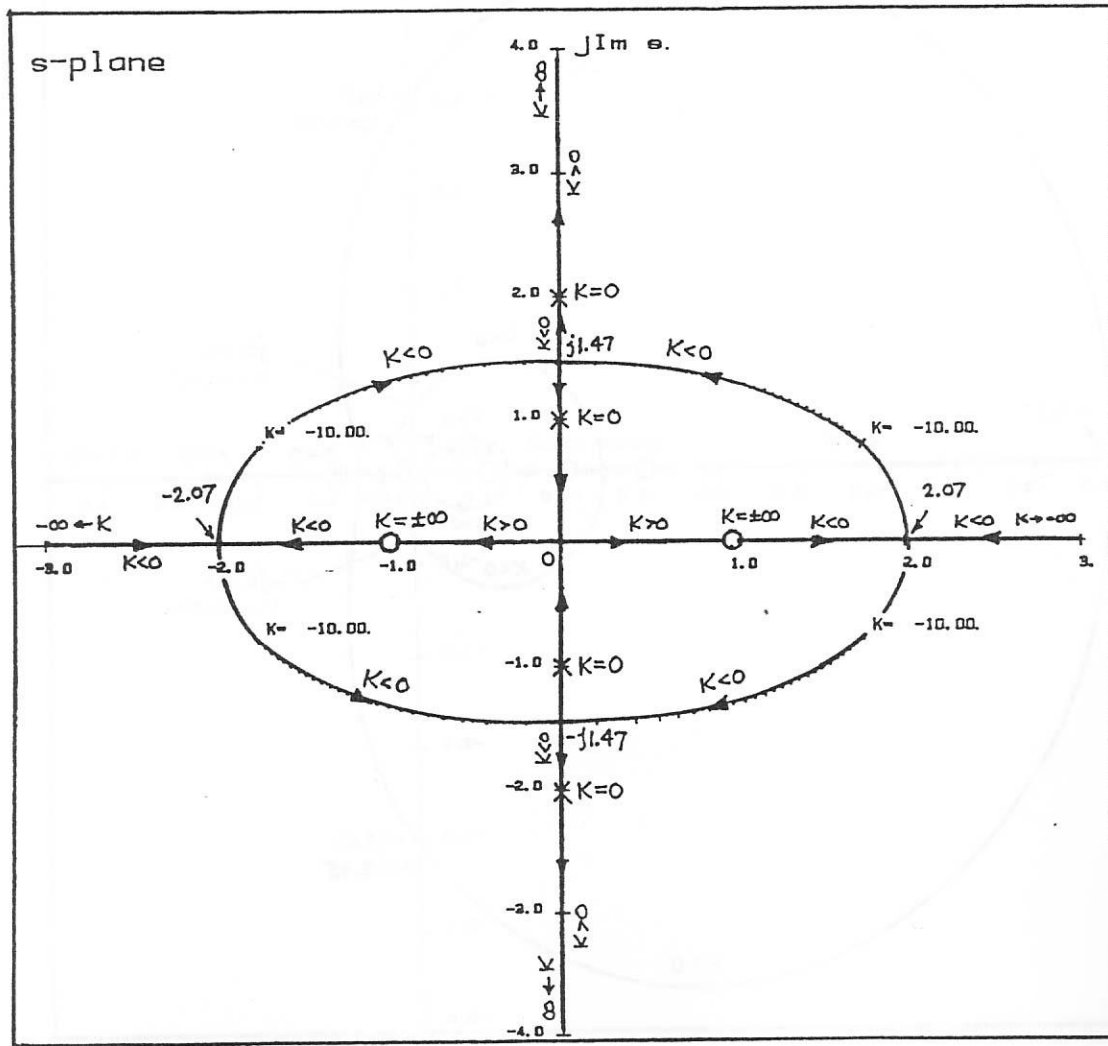
Asymptotes: $K > 0$: $90^\circ, 270^\circ$ $K < 0$: $0^\circ, 180^\circ$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-1+1}{4-2} = 0$$

Breakaway-point Equation: $s^5 - 2s^3 - 9s = 0$

Breakaway Points: $-2.07, 2.07, -j1.47, j1.47$



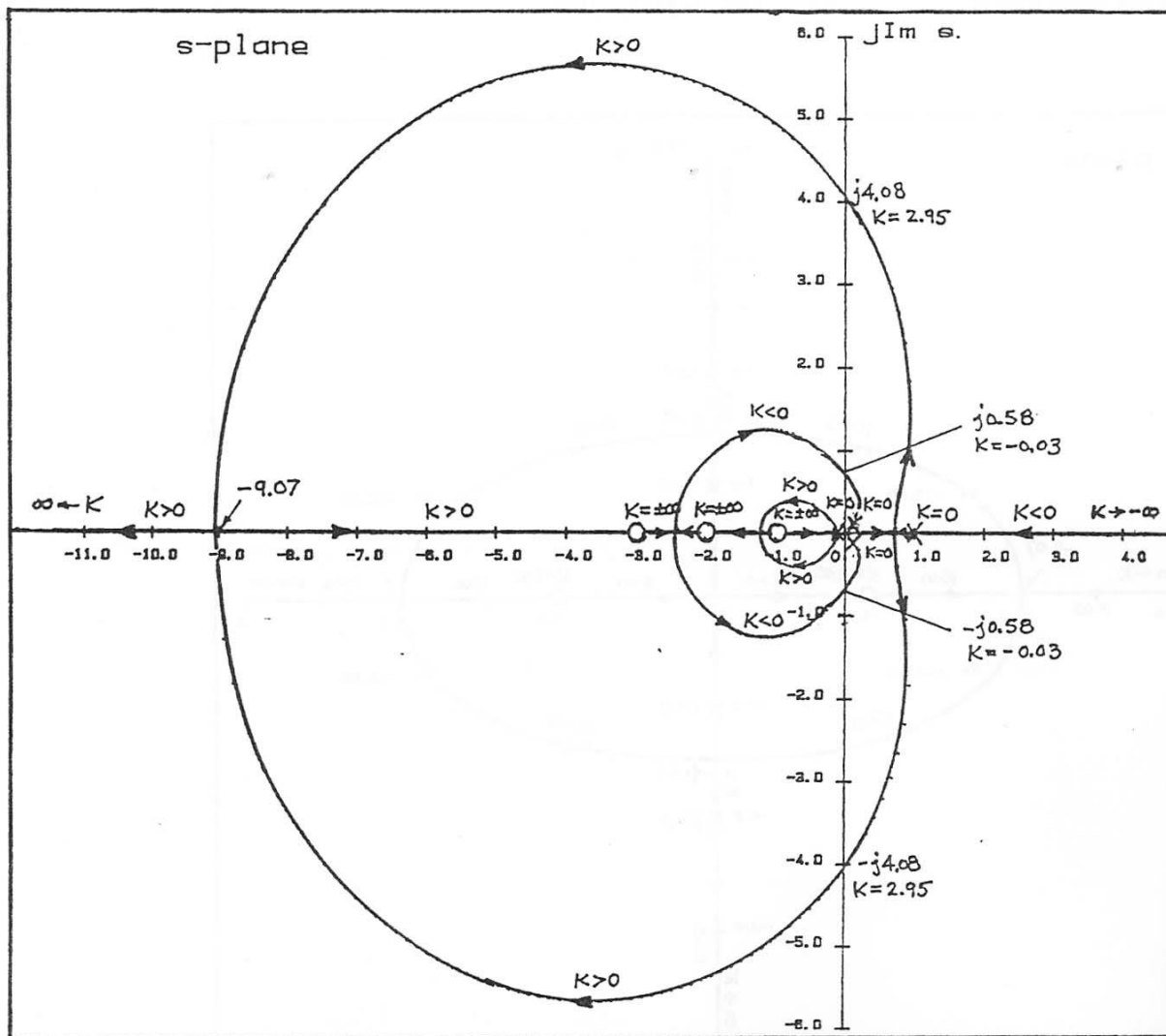
(m)

$$G(s)H(s) = \frac{K(s+1)(s+2)(s+3)}{s^3(s-1)}$$

Asymptotes: $K > 0$: 180° $K < 0$: 0°

Breakaway-point Equation: $s^6 + 12s^5 + 27s^4 + 2s^3 - 18s^2 = 0$

Breakaway Points: $-1.21, -2.4, -9.07, 0.683, 0, 0$



(n)

$$G(s)H(s) = \frac{K(s+5)(s+40)}{s^3(s+250)(s+1000)}$$

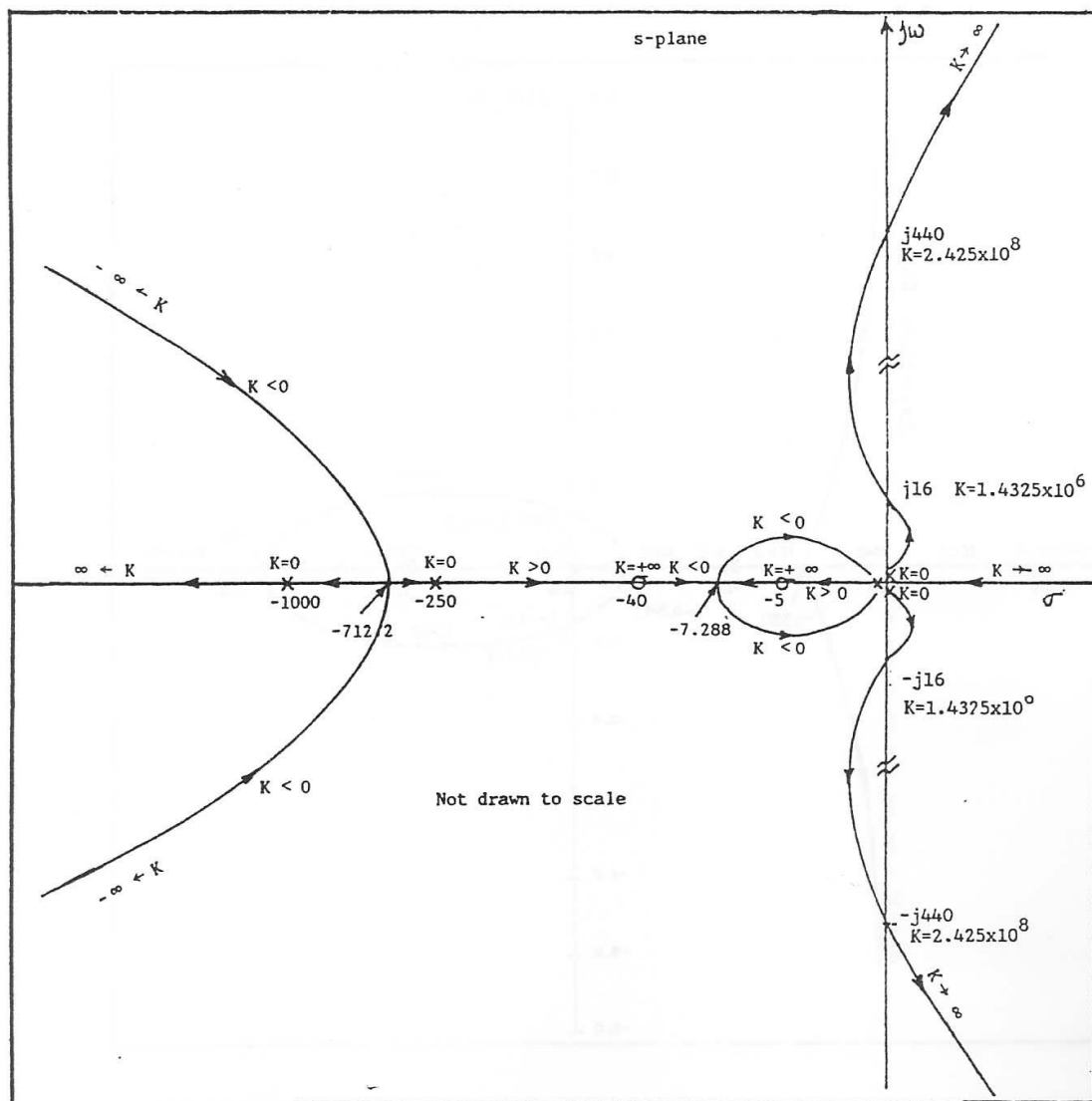
Asymptotes: $K > 0$: 60° , 180° , 300° $K < 0$: 0° , 120° , 240°

Intersect of asymptotes:

$$\sigma_1 = \frac{0+0+0-250-1000-(-5)-(-40)}{5-2} = -401.67$$

Breakaway-point Equation: $3750s^6 + 335000s^5 + 5.247 \times 10^8 s^4 + 2.9375 \times 10^{10} s^3 + 1.875 \times 10^{11} s^2 = 0$

Breakaway Points: -7.288 , -712.2 , 0 , 0



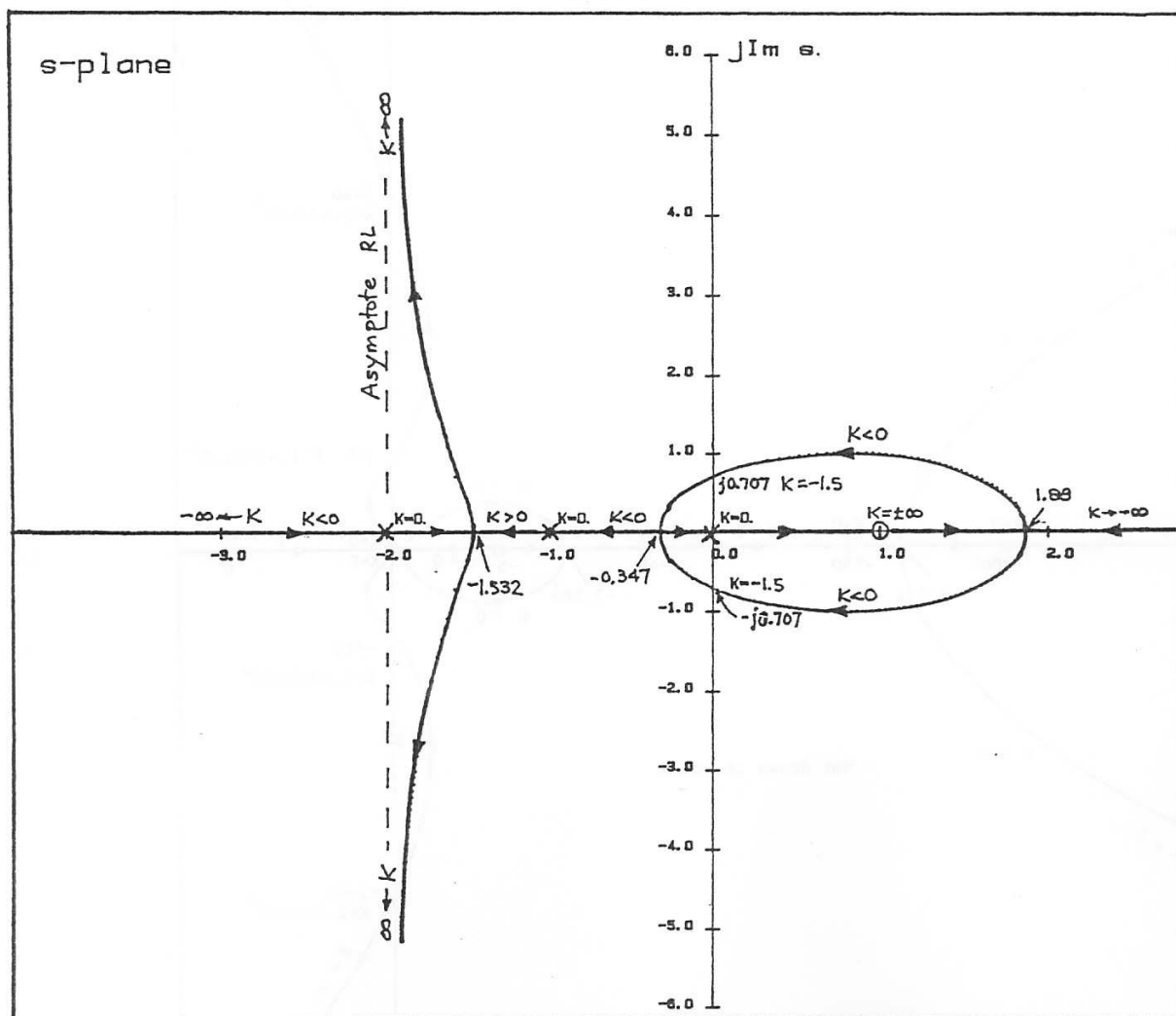
9-12 (o)

$$G(s)H(s) = \frac{K(s-1)}{s(s+1)(s+2)}$$

Asymptotes: $K > 0$: $90^\circ, 270^\circ$ $K < 0$: $0^\circ, 180^\circ$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-1-2-1}{3-1} = -2$$

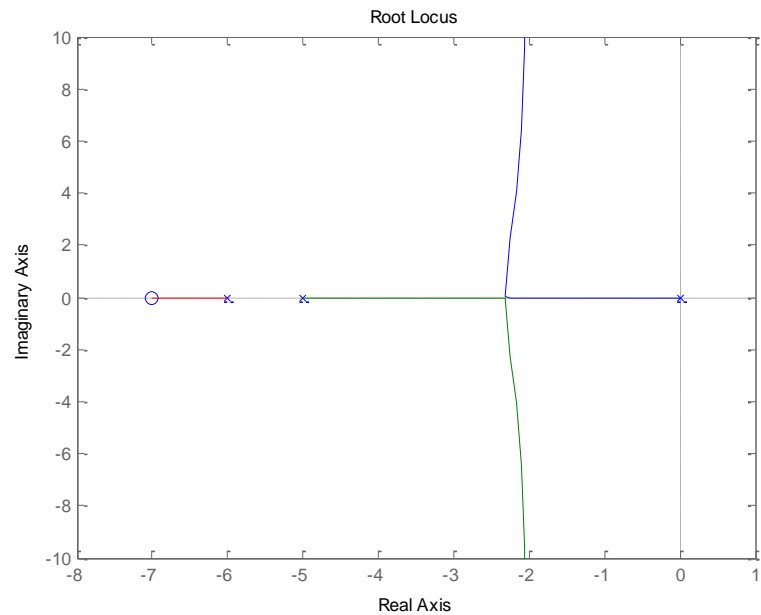
Breakaway-point Equation: $s^3 - 3s - 1 = 0$ Breakaway Points; $-0.3473, -1.532, 1.879$ 

9-13(a) MATLAB code:

```

num=[1 7];
den=conv([1 0],[1 5]);
den=conv(den,[1 6]);
mysys=tf(num,den)
rlocus(mysys);

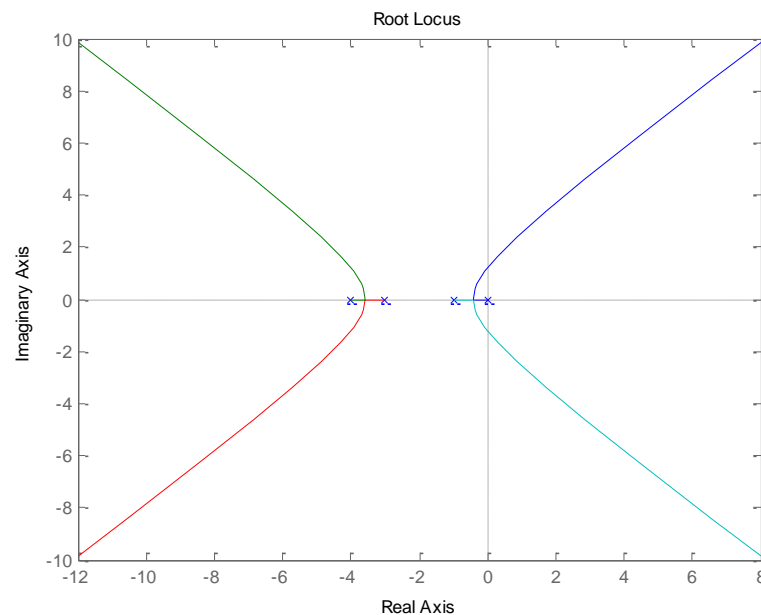
```

**9-13(b) MATLAB code:**

```

num=[0 1];
den=conv([1 0],[1 1]);
den=conv(den,[1 3]);
den=conv(den,[1 4]);
mysys=tf(num,den)
rlocus(mysys);

```

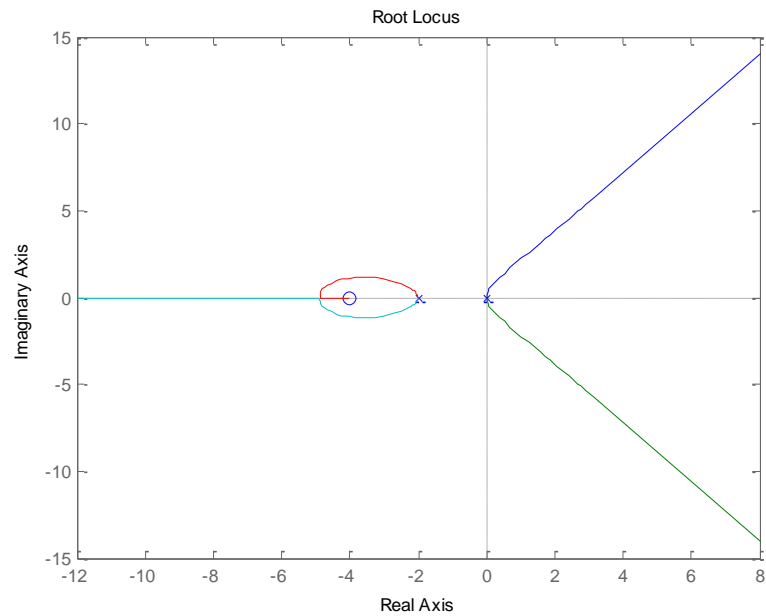


9-13(c) MATLAB code:

```

num=[1 4];
den=conv([1 0],[1 0]);
den=conv(den,[1 2]);
den=conv(den,[1 2]);
mysys=tf(num,den)
rlocus(mysys);

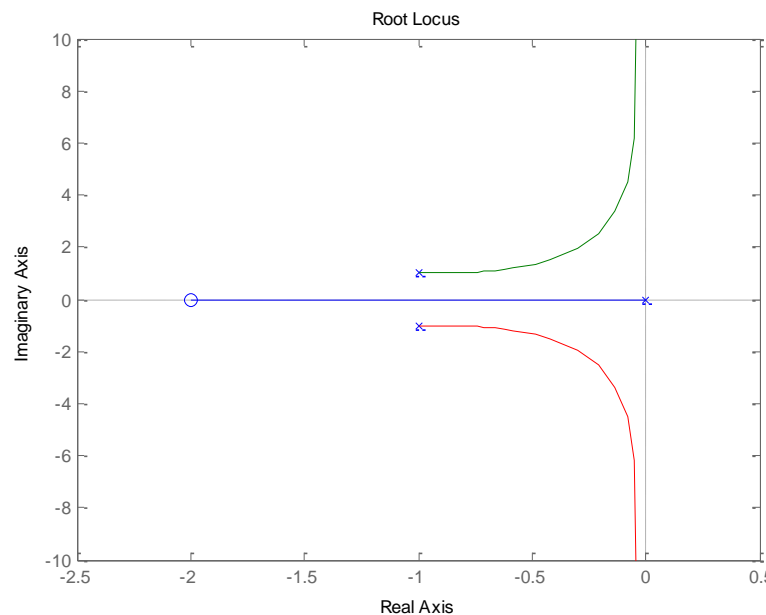
```

**9-13(d) MATLAB code:**

```

num=[1 2];
den=conv([1 0],[1
(1+j)]);
den=conv(den,[1 (1-
j)]);
mysys=tf(num,den)
rlocus(mysys);

```

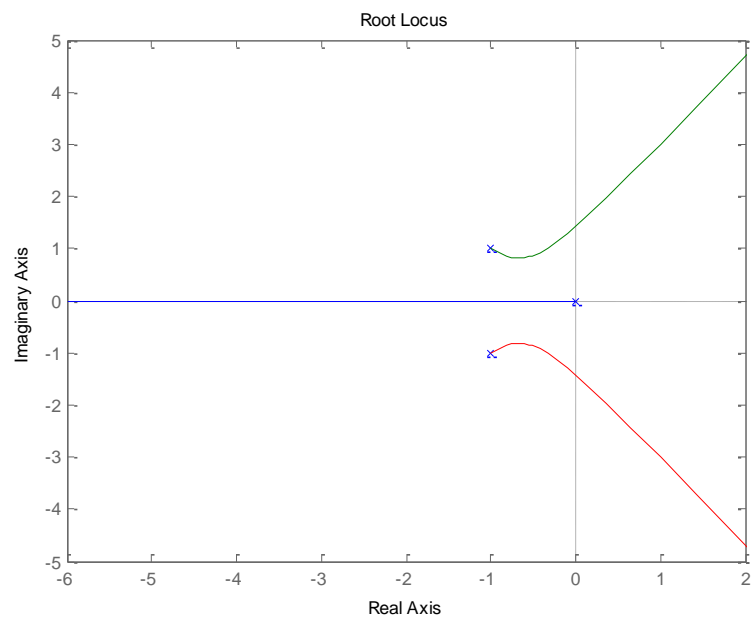
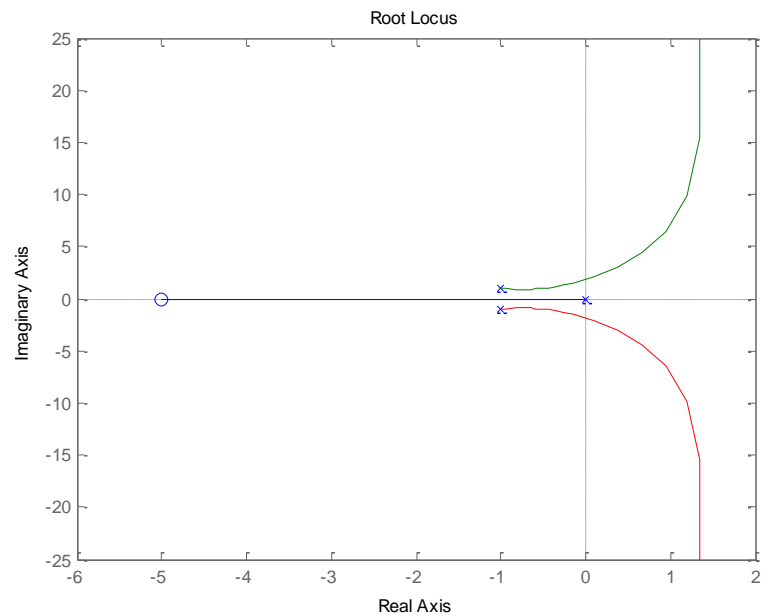


9-13(e) MATLAB code:

```

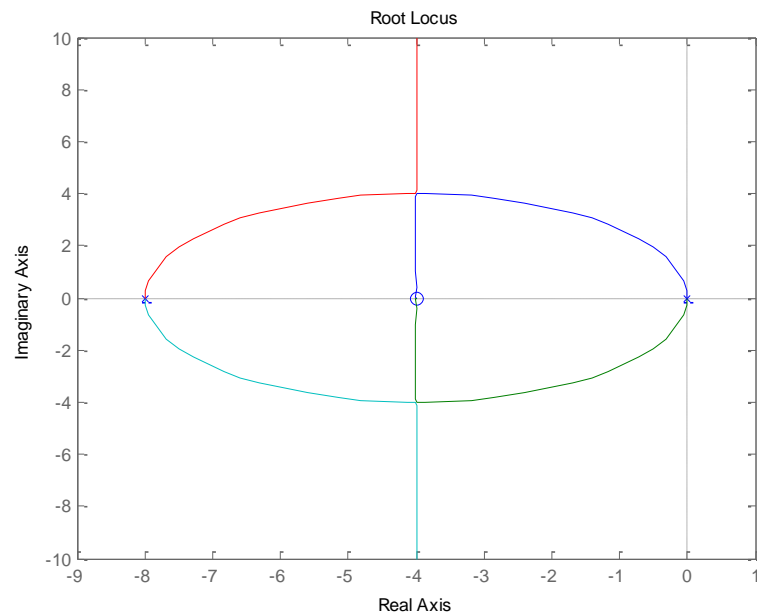
num=[1 5];
den=conv([1 0],[1
(1+j)]);
den=conv(den,[1 (1-
j)]);
mysys=tf(num,den)
rlocus(mysys);

```



9-13(f) MATLAB code:

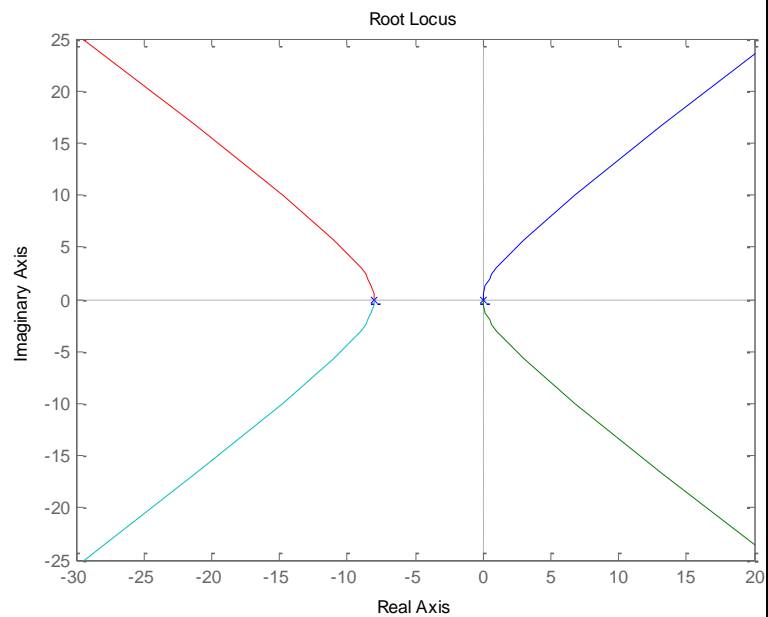
```
num=conv([1 4],[1 4]);  
den=conv([1 0],[1 0]);  
den=conv(den,[1 8]);  
den=conv(den,[1 8]);  
mysys=tf(num,den)  
rlocus(mysys);
```

**9-13(g)** MATLAB code:

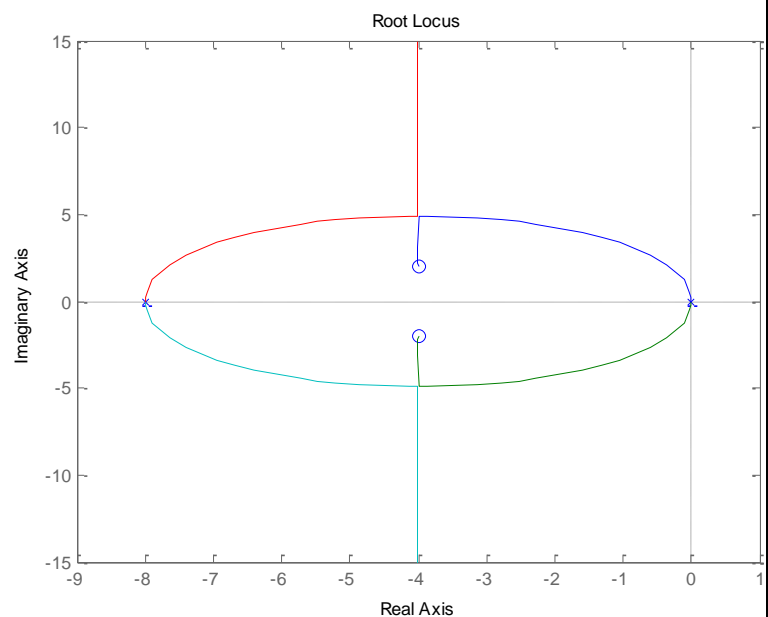
```
num=conv([1 4],[1 4]);  
den=conv([1 0],[1 0]);  
den=conv(den,[1 8]);  
den=conv(den,[1 8]);  
mysys=tf(num,den)  
rlocus(mysys);
```

9-13(h) MATLAB code:

```
num=[0 1];
den=conv([1 0],[1 0]);
den=conv(den,[1 8]);
den=conv(den,[1 8]);
mysys=tf(num,den)
rlocus(mysys);
```

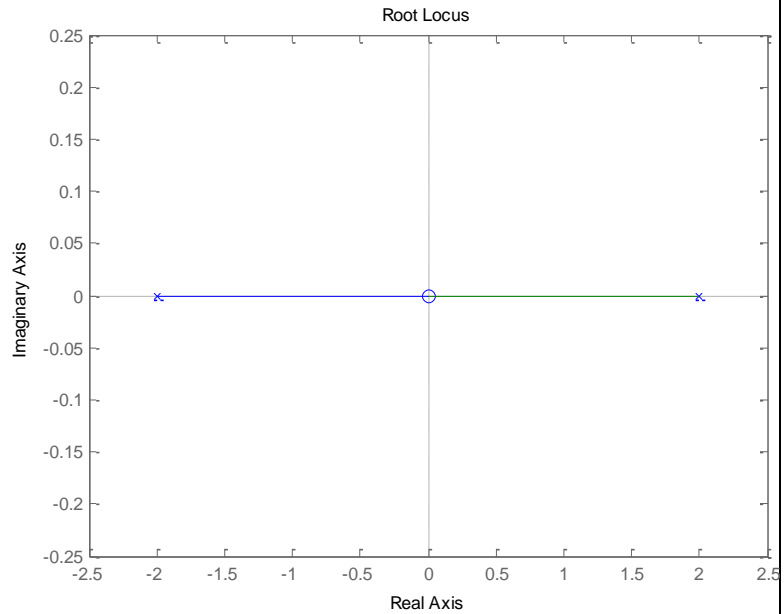
**9-13(i)** MATLAB code:

```
num=conv([1 4-2j],[1 4+2j]);
den=conv([1 0],[1 0]);
den=conv(den,[1 8]);
den=conv(den,[1 8]);
mysys=tf(num,den)
rlocus(mysys);
```

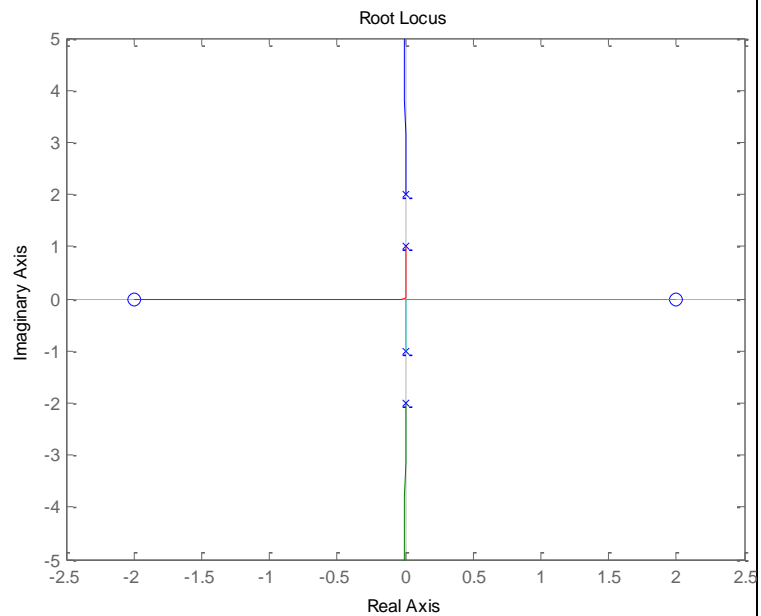


9-13(j) MATLAB code:

```
num=conv([1 0],[1 0]);
den=conv([1 2],[1 -2]);
mysys=tf(num,den)
rlocus(mysys);
```

**9-13(k)** MATLAB code:

```
num=conv([1 2],[1 -2]);
den=conv([1 -j],[1 j]);
den=conv(den,[1 -2j]);
den=conv(den,[1 2j]);
mysys=tf(num,den)
rlocus(mysys);
```

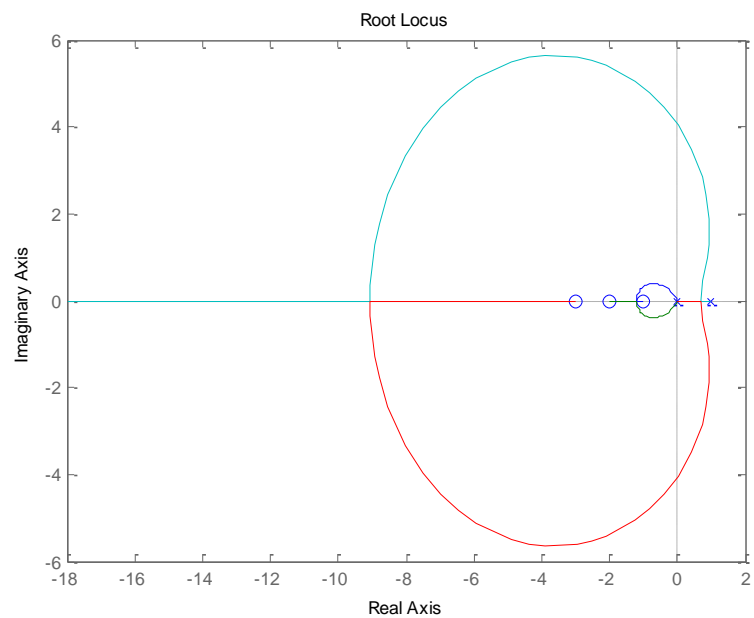
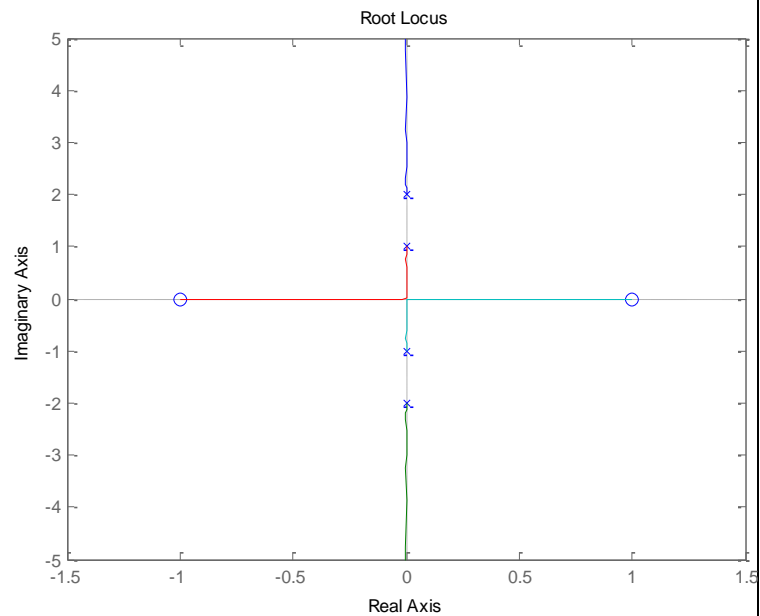


9-13(l) MATLAB code:

```

num=conv([1 1],[1 -1]);
den=conv([1 -j],[1 j]);
den=conv(den,[1 -2j]);
den=conv(den,[1 2j]);
mysys=tf(num,den)
rlocus(mysys);

```

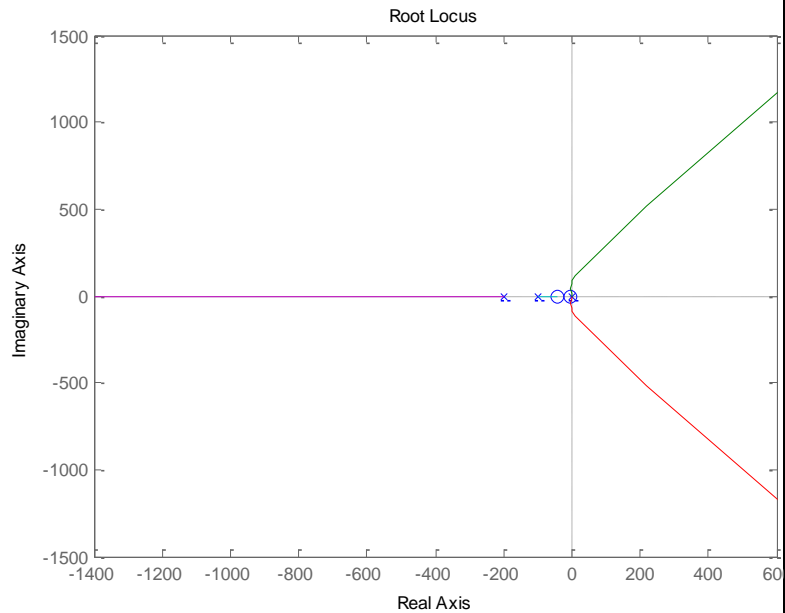


9-13(m) MATLAB code:

```

num=conv([1 1],[1 2]);
num=conv(num,[1 3]);
den=conv([1 0],[1 0]);
den=conv(den,[1 0]);
den=conv(den,[1 -1]);
mysys=tf(num,den)
rlocus(mysys);

```

**9-13(n)** MATLAB code:

```

num=conv([1 5],[1 40]);
den=conv([1 0],[1 0]);
den=conv(den,[1 0]);
den=conv(den,[1 100]);
den=conv(den,[1 200]);
mysys=tf(num,den)
rlocus(mysys);

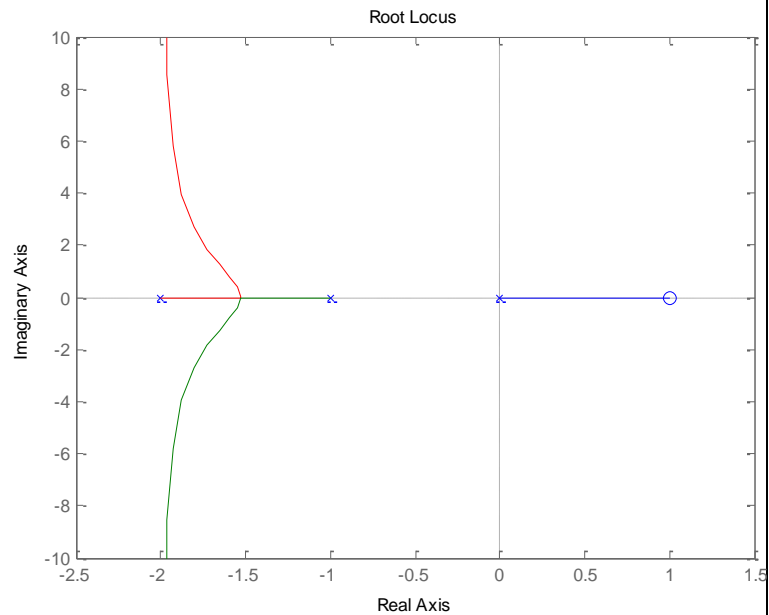
```

9-13(o) MATLAB code:

```

num=conv([1 5],[1 40]);
den=conv([1 0],[1 0]);
den=conv(den,[1 0]);
den=conv(den,[1 100]);
den=conv(den,[1 200]);
mysys=tf(num,den)
rlocus(mysys);

```



9-14) (a) $Q(s) = s + 5$ $P(s) = s(s^2 + 3s + 2) = s(s+1)(s+2)$

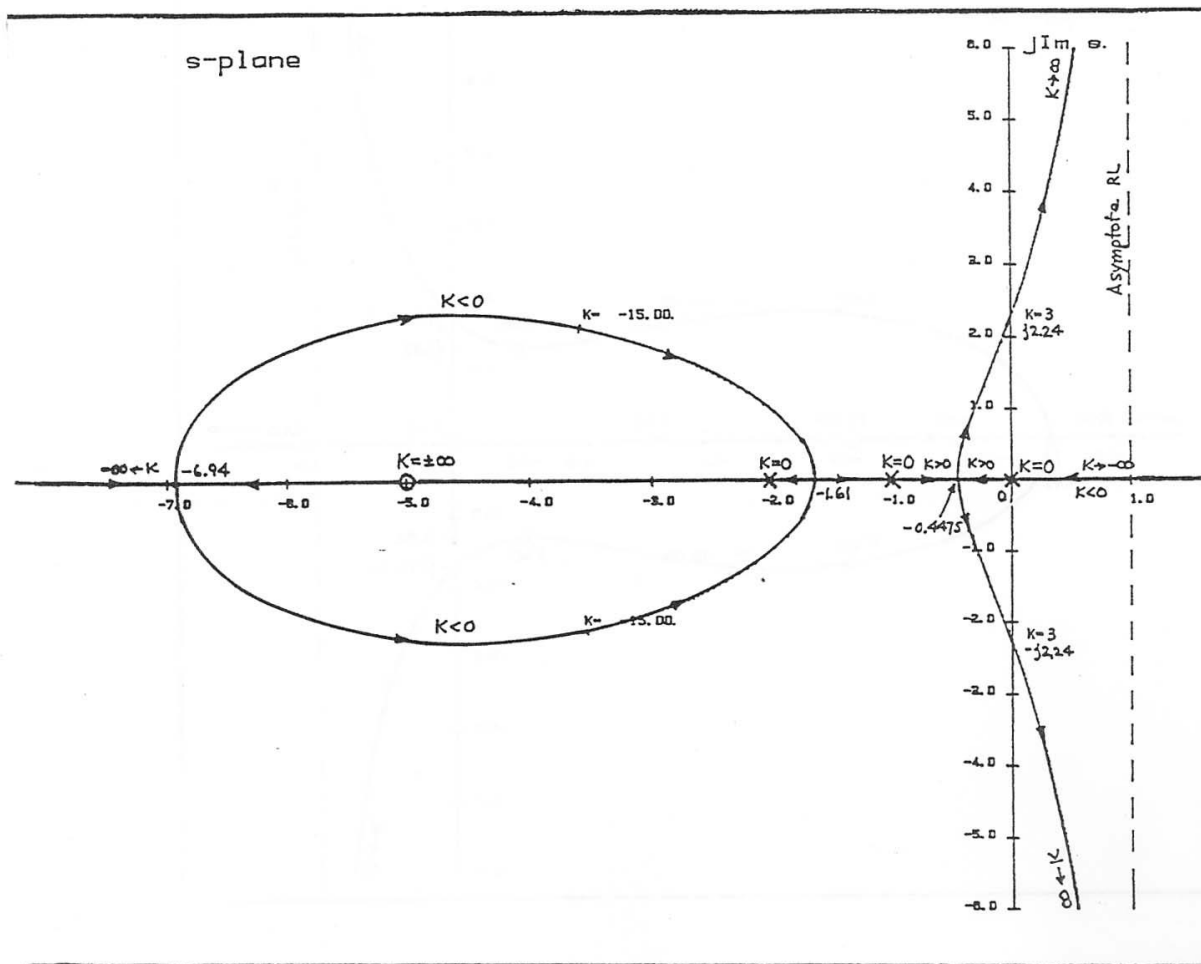
Asymptotes: $K > 0$: $90^\circ, 270^\circ$ $K < 0$: $0^\circ, 180^\circ$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-1 - 2 - (-5)}{3 - 1} = 1$$

Breakaway-point Equation: $s^3 + 9s^2 + 15s + 5 = 0$

Breakaway Points: $-0.4475, -1.609, -6.9434$



9-14 (b) $Q(s) = s + 3$ $P(s) = s^2 + s + 2$

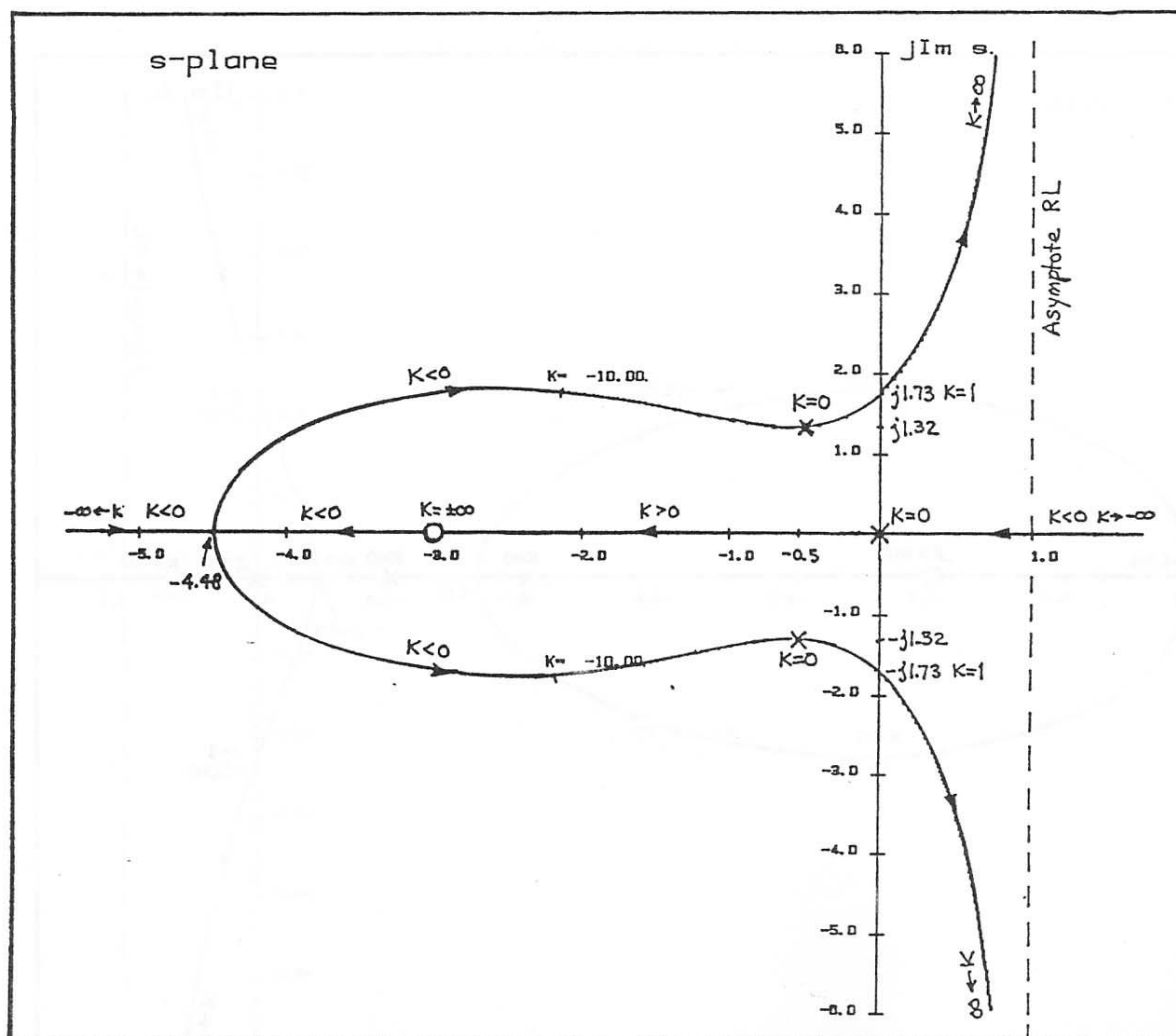
Asymptotes: $K > 0$: $90^\circ, 270^\circ$ $K < 0$: $0^\circ, 180^\circ$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-1 - (-3)}{3 - 1} = 1$$

Breakaway-point Equation: $s^3 + 5s^2 + 3s + 3 = 0$

Breakaway Points: -4.4798 The other solutions are not breakaway points.

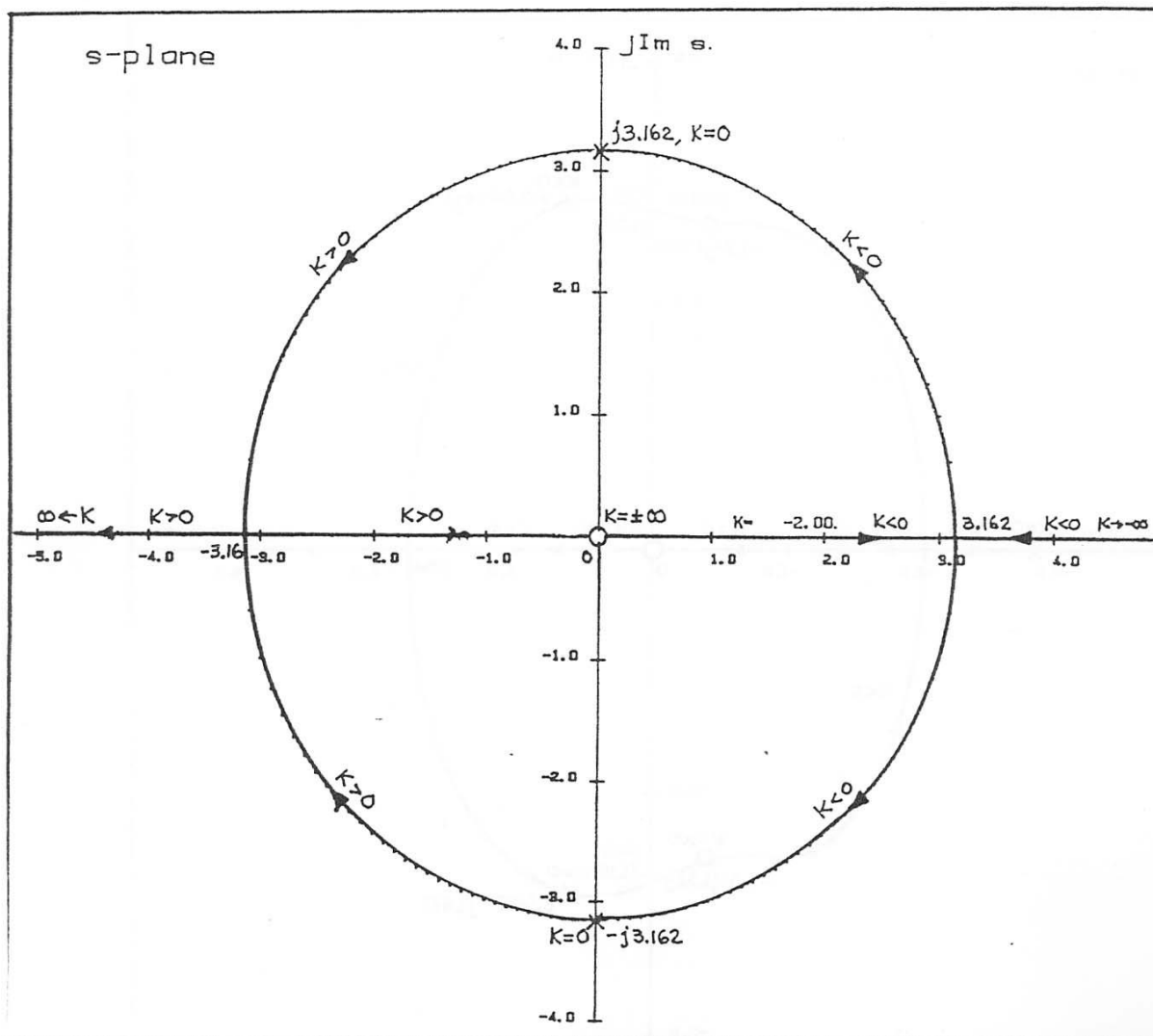


9-14 (c) $Q(s) = 5s$ $P(s) = s^2 + 10$

Asymptotes: $K > 0$: 180° $K < 0$: 0°

Breakaway-point Equation: $5s^2 - 50 = 0$

Breakaway Points: $-3.162, 3.162$

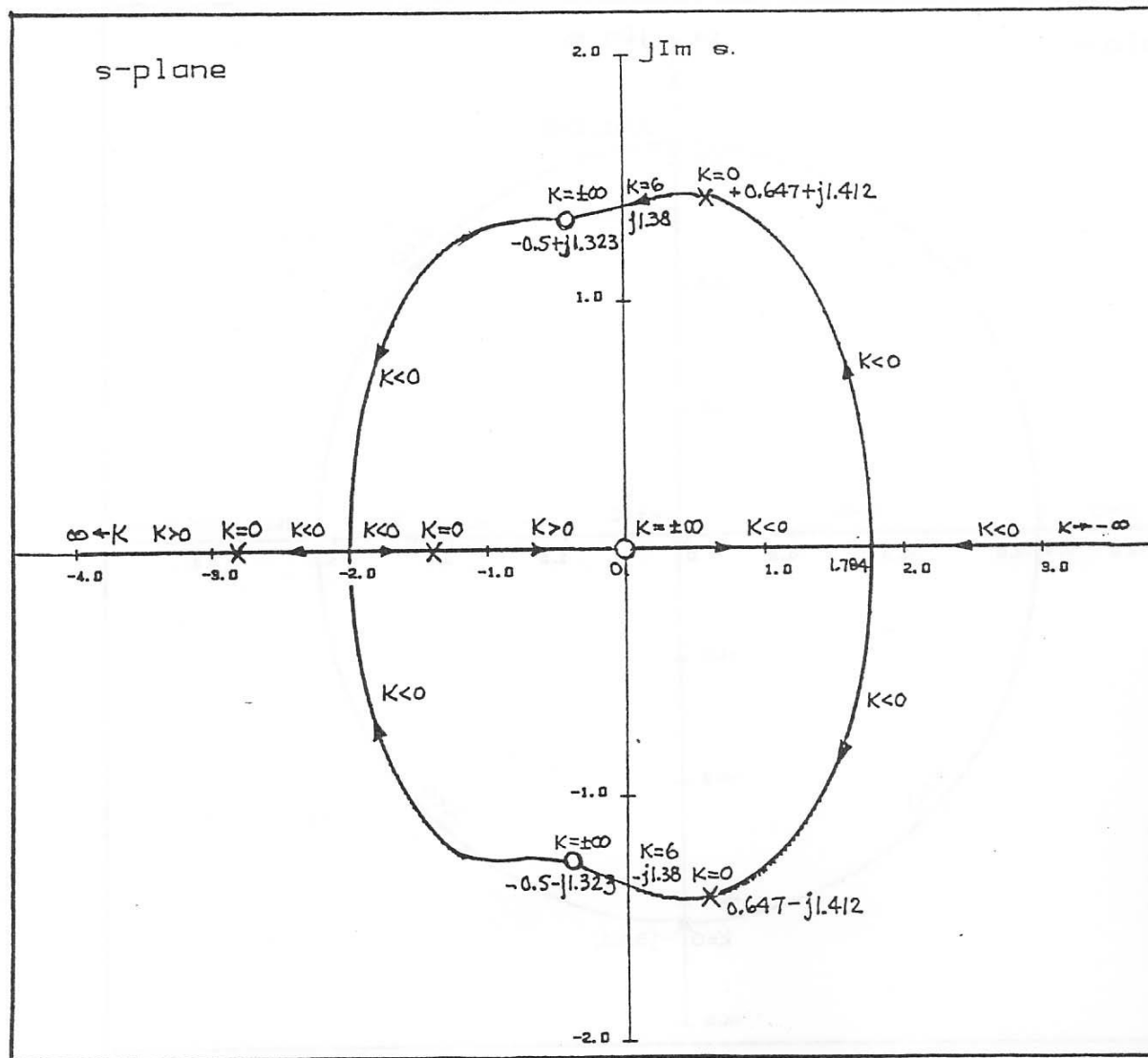


9-14 (d) $Q(s) = s(s^2 + s + 2)$ $P(s) = s^4 + 3s^3 + s^2 + 5s + 10$

Asymptotes: $K > 0$: 180° $K < 0$: 0°

Breakaway-point Equation: $s^6 + 2s^5 + 8s^4 + 2s^3 - 33s^2 - 20s - 20 = 0$

Breakaway Points: $-2, 1.784$. The other solutions are not breakaway points.

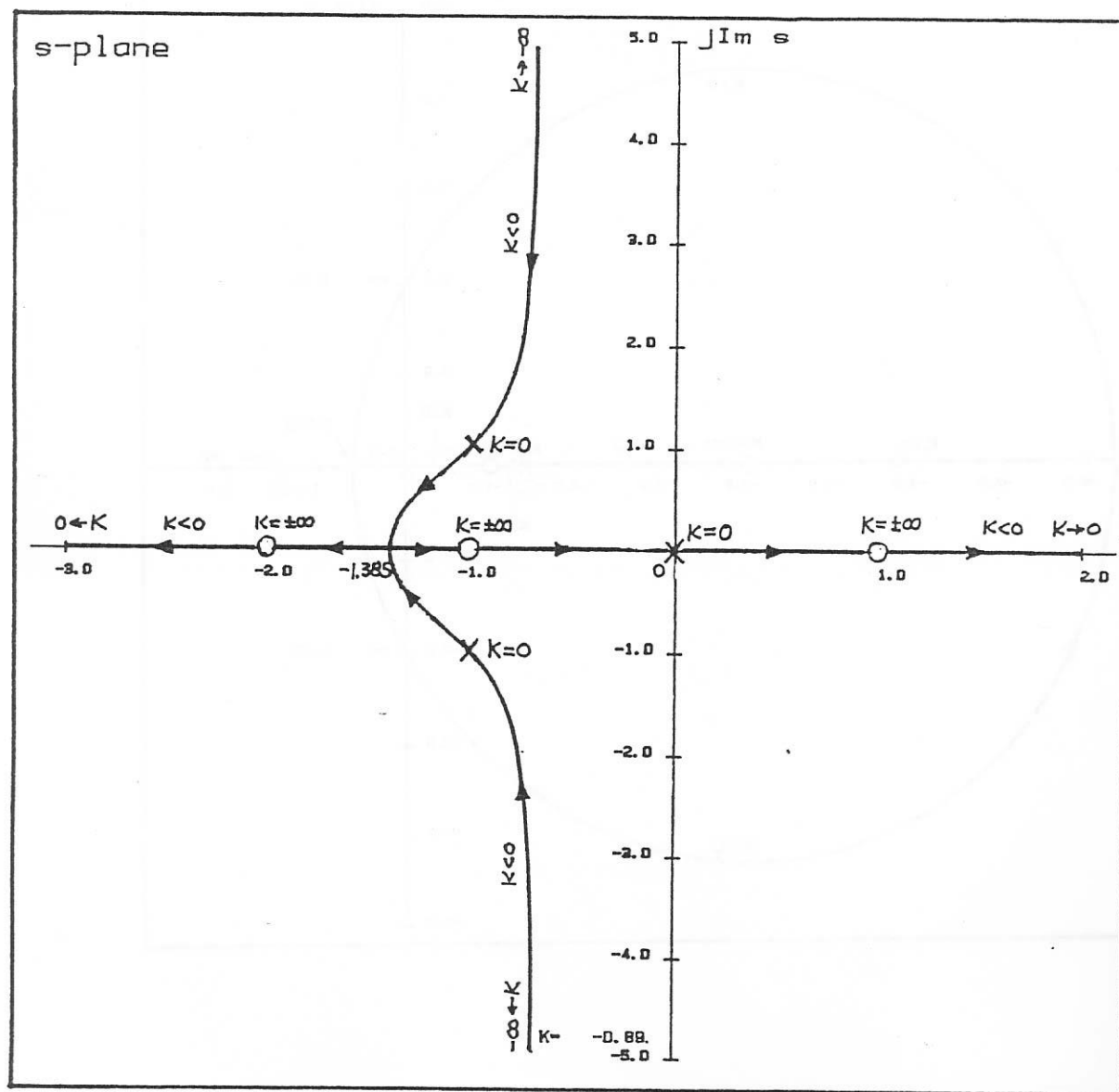


9-14 (e) $Q(s) = (s^2 - 1)(s + 2)$ $P(s) = s(s^2 + 2s + 2)$

Since $Q(s)$ and $P(s)$ are of the same order, there are no asymptotes.

Breakaway-point Equation: $6s^3 + 12s^2 + 8s + 4 = 0$

Breakaway Points: -1.3848

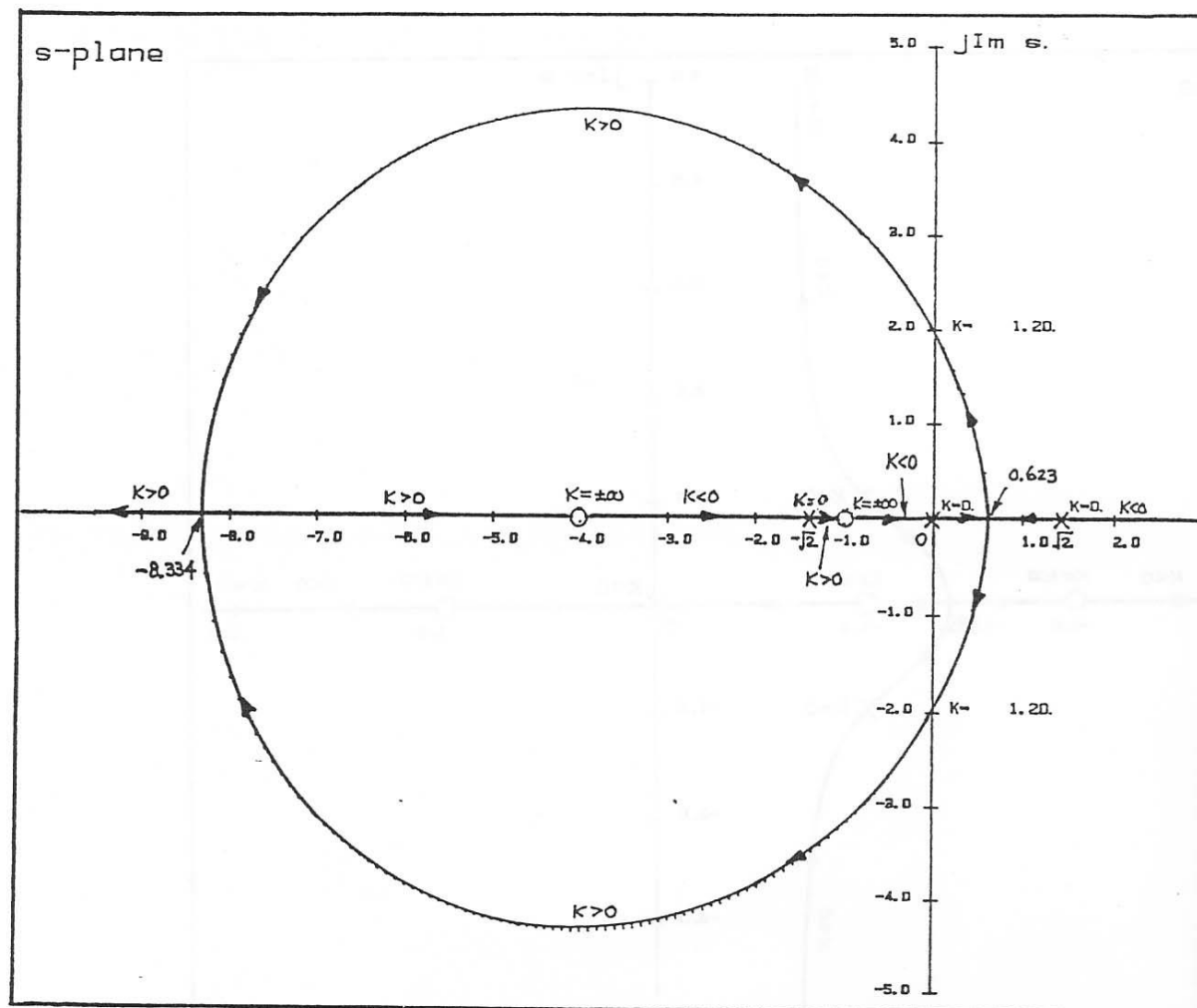


9-14 (f) $Q(s) = (s+1)(s+4)$ $P(s) = s(s^2 - 2)$

Asymptotes: $K > 0$: 180° $K < 0$: 0°

Breakaway-point equations: $s^4 + 10s^3 + 14s^2 - 8 = 0$

Breakaway Points: $-8.334, 0.623$



9-14 (g) $Q(s) = s^2 + 4s + 5$ $P(s) = s^2 (s^2 + 8s + 16)$

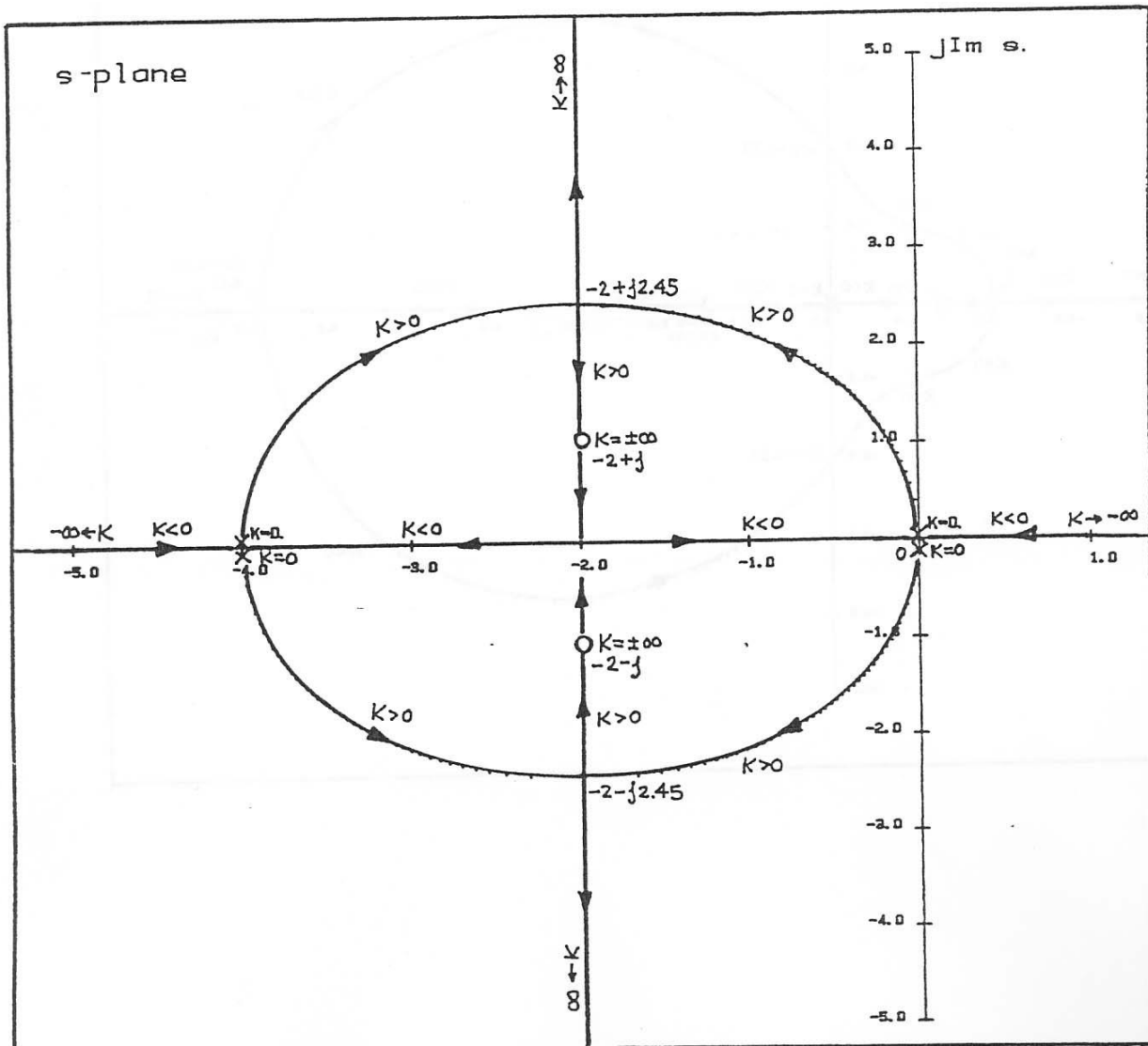
Asymptotes: $K > 0$: $90^\circ, 270^\circ$ $K < 0$: $0^\circ, 180^\circ$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-8 - (-4)}{4 - 2} = -2$$

Breakaway-point Equation: $s^5 + 10s^4 + 42s^3 + 92s^2 + 80s = 0$

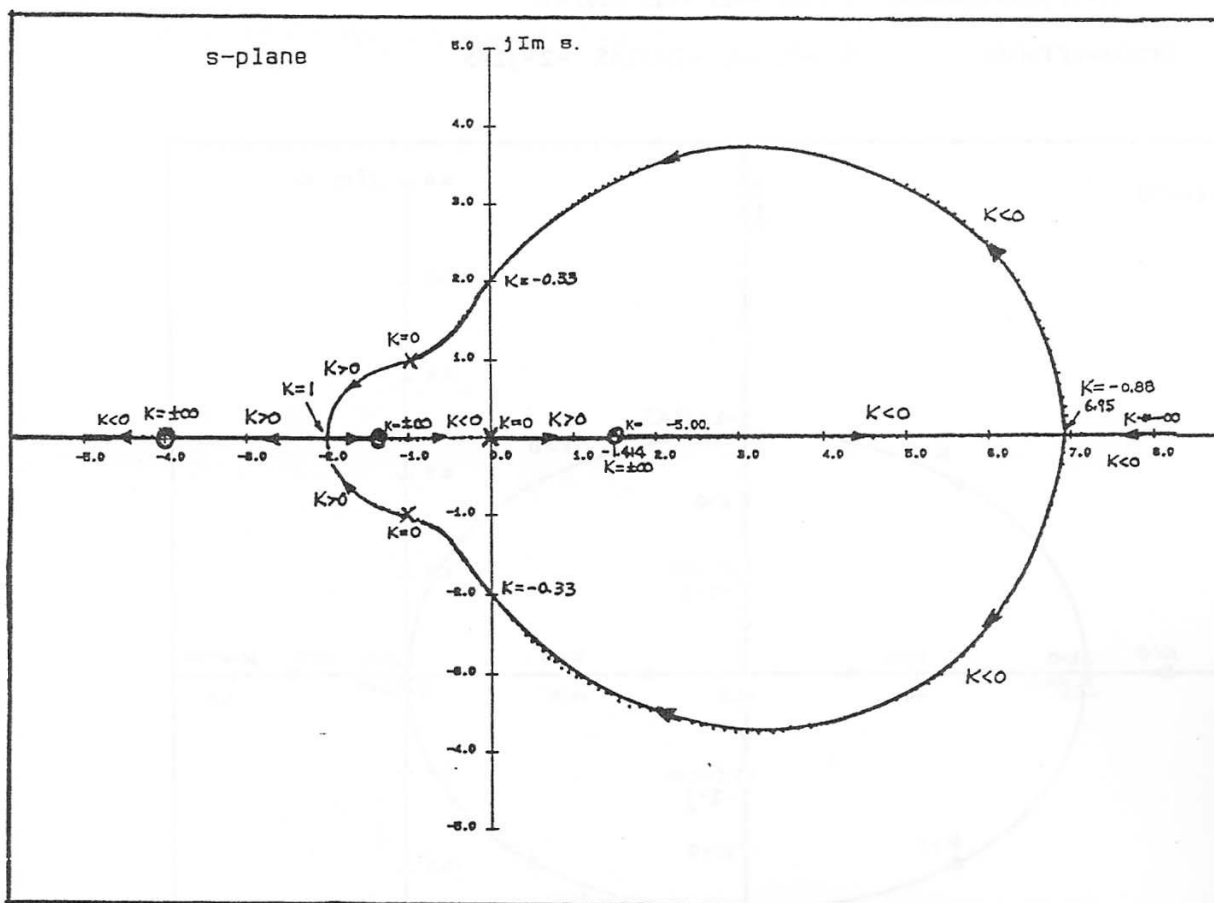
Breakaway Points: $0, -2, -4, -2 + j2.45, -2 - j2.45$



9-14 (h) $Q(s) = (s^2 - 2)(s + 4)$ $P(s) = s(s^2 + 2s + 2)$

Since $Q(s)$ and $P(s)$ are of the same order, there are no asymptotes.

Breakaway Points: $-2, 6.95$

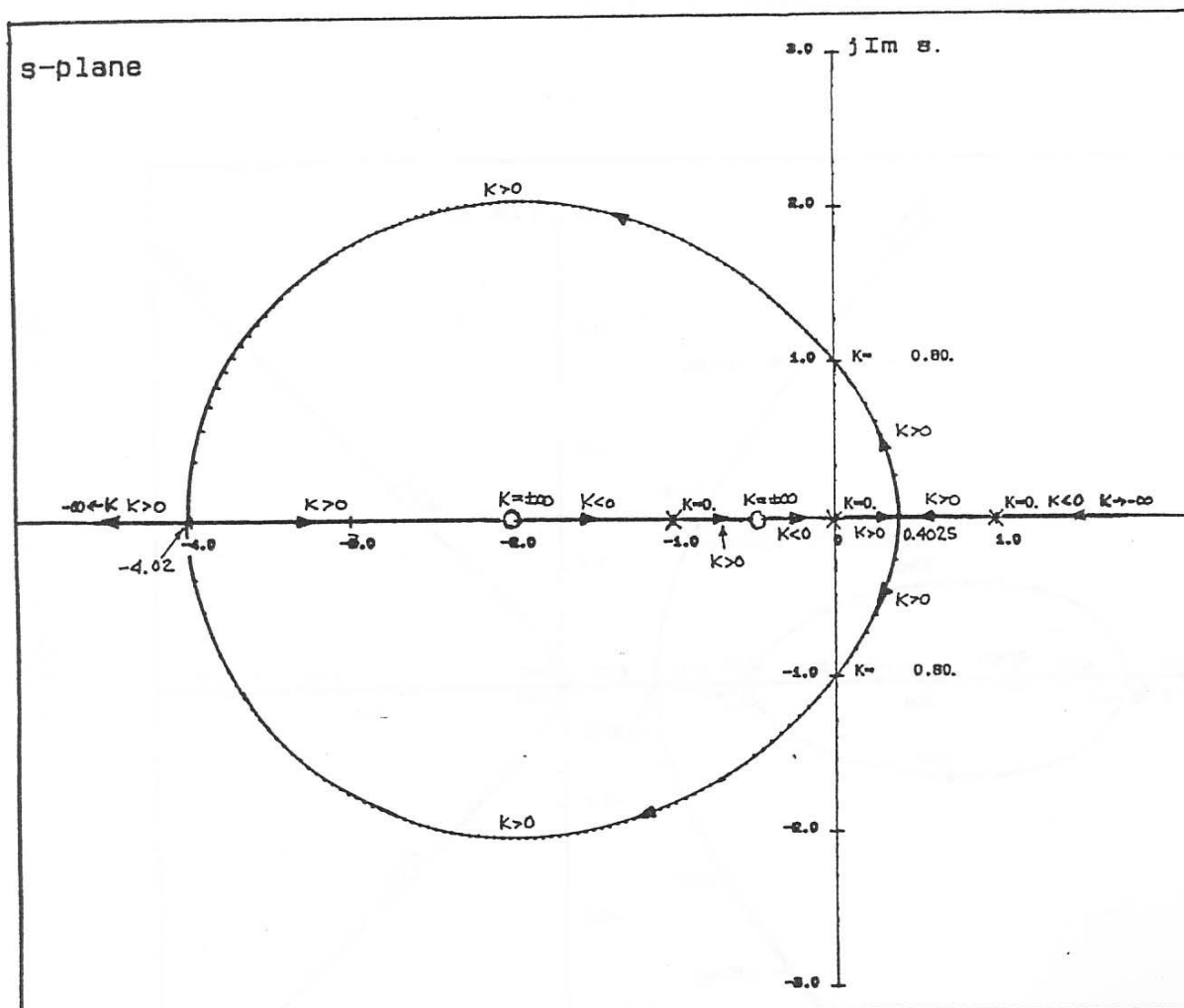


9-14 (i) $Q(s) = (s + 2)(s + 0.5)$ $P(s) = s^3 - 1$

Asymptotes: $K > 0:$ 180° $K < 0:$ 0°

Breakaway-point Equation: $s^4 + 5s^3 + 4s^2 - 1 = 0$

Breakaway Points: $-4.0205, 0.40245$ The other solutions are not breakaway points.



9-14 (j)

$$Q(s) = 2s + 5 \quad P(s) = s^2 (s^2 + 2s + 1) = s^2 (s + 1)^2$$

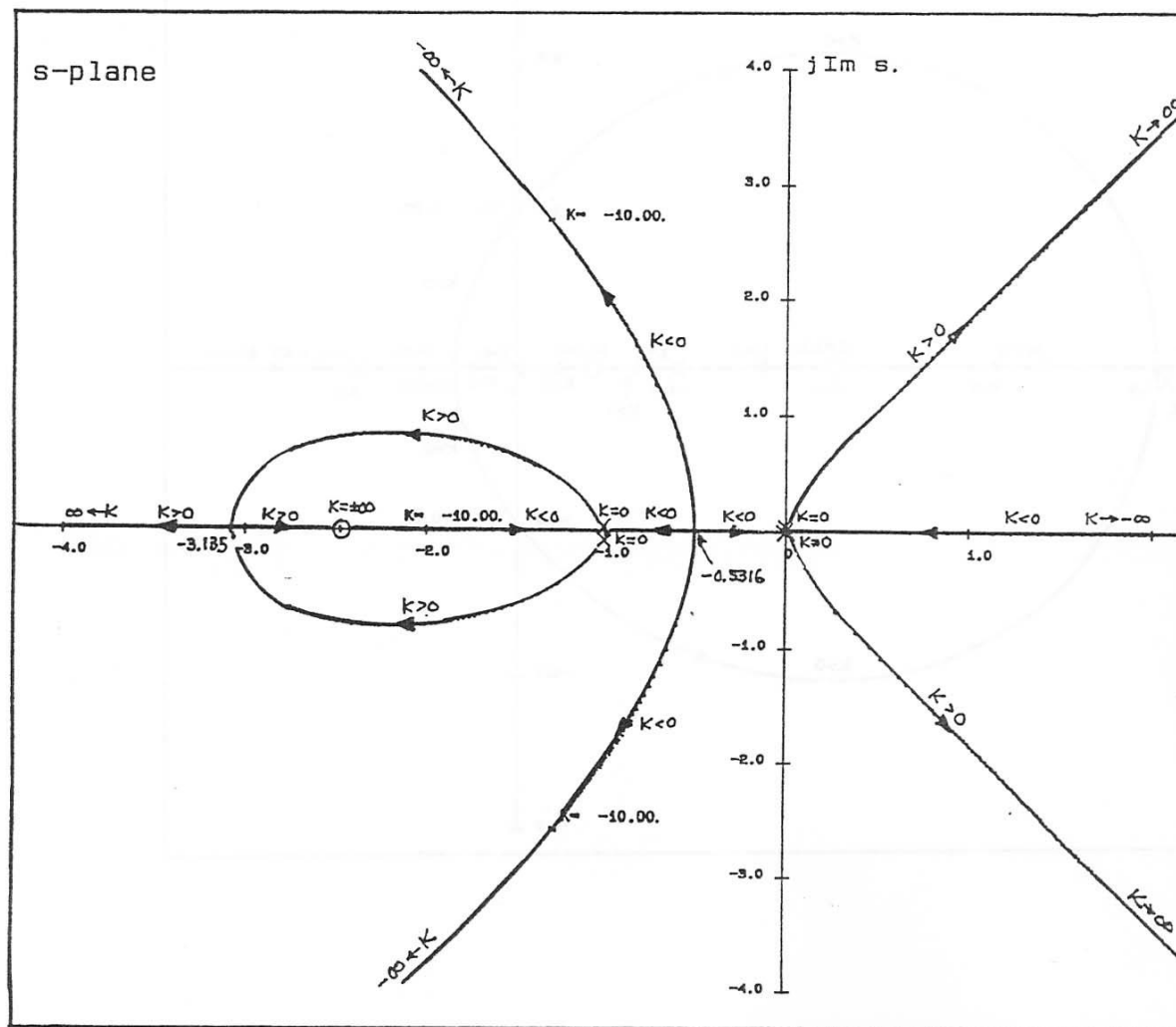
Asymptotes: $K > 0$: 60° , 180° , 300° $K < 0$: 0° , 120° , 240°

Intersect of Asymptotes;

$$\sigma_1 = \frac{0 + 0 - 1 - 1 - (-2.5)}{4 - 1} = \frac{0.5}{3} = 0.167$$

Breakaway-point Equation: $6s^4 + 28s^3 + 32s^2 + 10s = 0$

Breakaway Points: 0 , -0.5316 , -1 , -3.135



9-15) MATLAB code:

```
clear all;
close all;
s = tf('s')

%a)
num_GH_a=(s+5);
den_GH_a=(s^3+3*s^2+2*s);
GH_a=num_GH_a/den_GH_a;
figure(1);
rlocus(GH_a)
```

```
%b)
num_GH_b=(s+3);
den_GH_b=(s^3+s^2+2*s);
GH_b=num_GH_b/den_GH_b;
figure(2);
rlocus(GH_b)

%c)
num_GH_c= 5*s^2;
den_GH_c=(s^3+10);
GH_c=num_GH_c/den_GH_c;
figure(3);
rlocus(GH_c)

%d)
num_GH_d=(s^3+s^2+2);
den_GH_d=(s^4+3*s^3+s^2+15);
GH_d=num_GH_d/den_GH_d;
figure(4);
rlocus(GH_d)

%e)
num_GH_e=(s^2-1)*(s+2);
den_GH_e=(s^3+2*s^2+2*s);
GH_e=num_GH_e/den_GH_e;
figure(5);
rlocus(GH_e)

%f)
num_GH_f=(s+4)*(s+1);
den_GH_f=(s^3-2*s);
GH_f=num_GH_f/den_GH_f;
figure(6);
rlocus(GH_f)

%g)
num_GH_g=(s^2+4*s+5);
den_GH_g=(s^4+6*s^3+9*s^2);
GH_g=num_GH_g/den_GH_g;
figure(7);
rlocus(GH_g)

%h)
num_GH_h=(s^2-2)*(s+4);
den_GH_h=(s^3+2*s^2+2*s);
GH_h=num_GH_h/den_GH_h;
```

```

figure(8);
rlocus(GH_h)

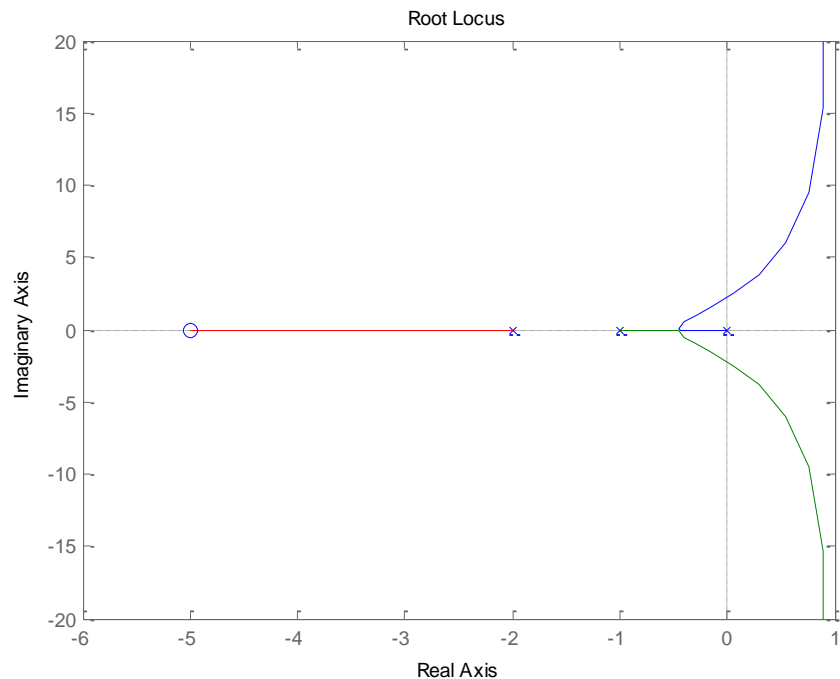
%i)
num_GH_i=(s+2)*(s+0.5);
den_GH_i=(s^3-s);
GH_i=num_GH_i/den_GH_i;
figure(9);
rlocus(GH_i)

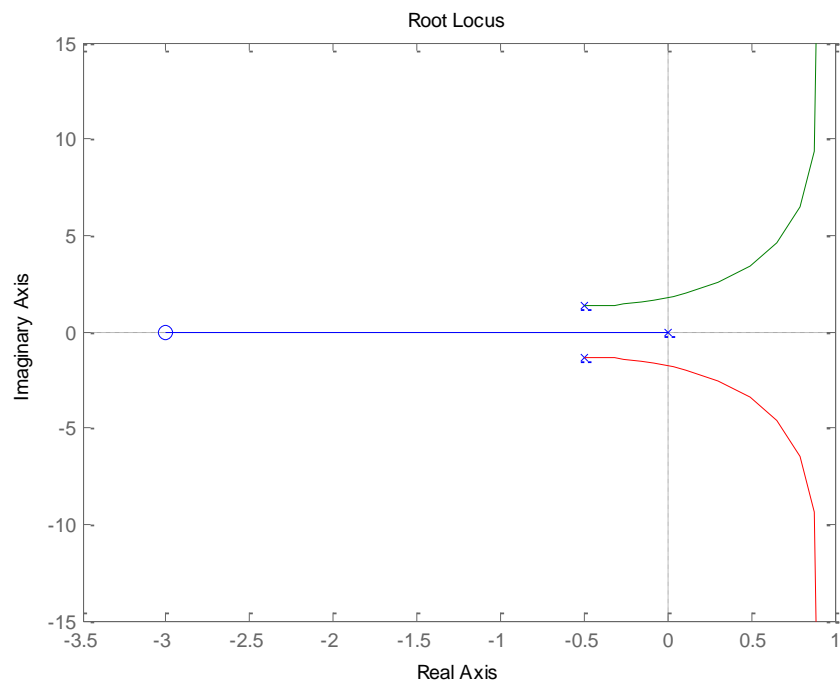
%j)
num_GH_j=(2*s+5);
den_GH_j=(s^4+2*s^3+2*s^2);
GH_j=num_GH_j/den_GH_j;
figure(10);
rlocus(GH_j)

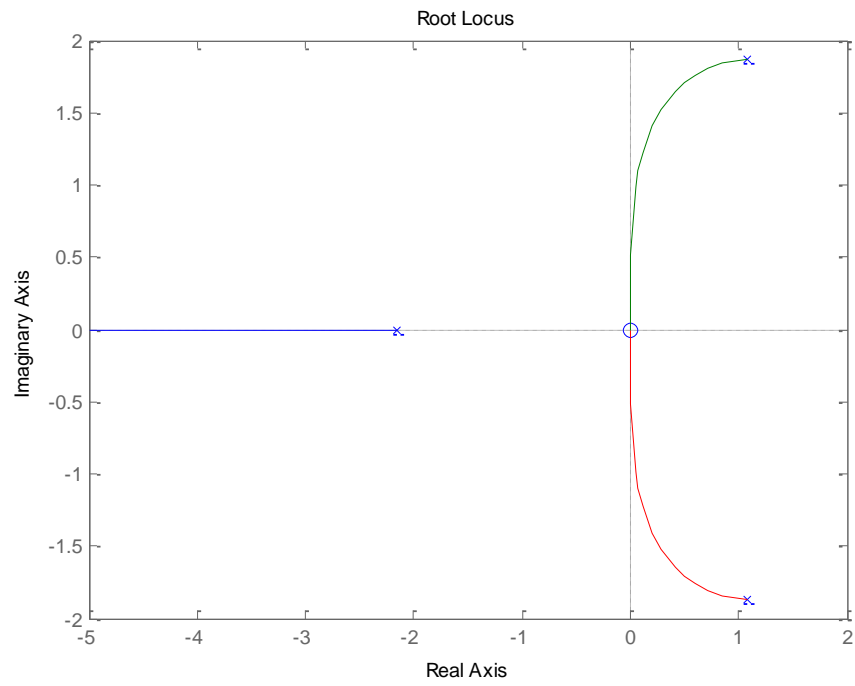
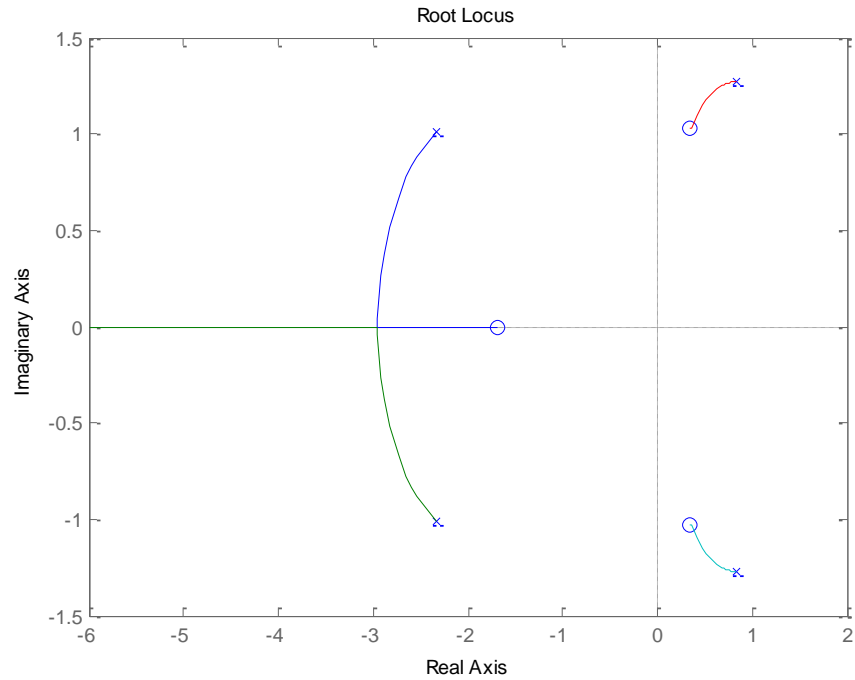
%k)
num_GH_k=1;
den_GH_k=(s^5+2*s^4+3*s^3+2*s^2+s);
GH_k=num_GH_k/den_GH_k;
figure(11);
rlocus(GH_k)

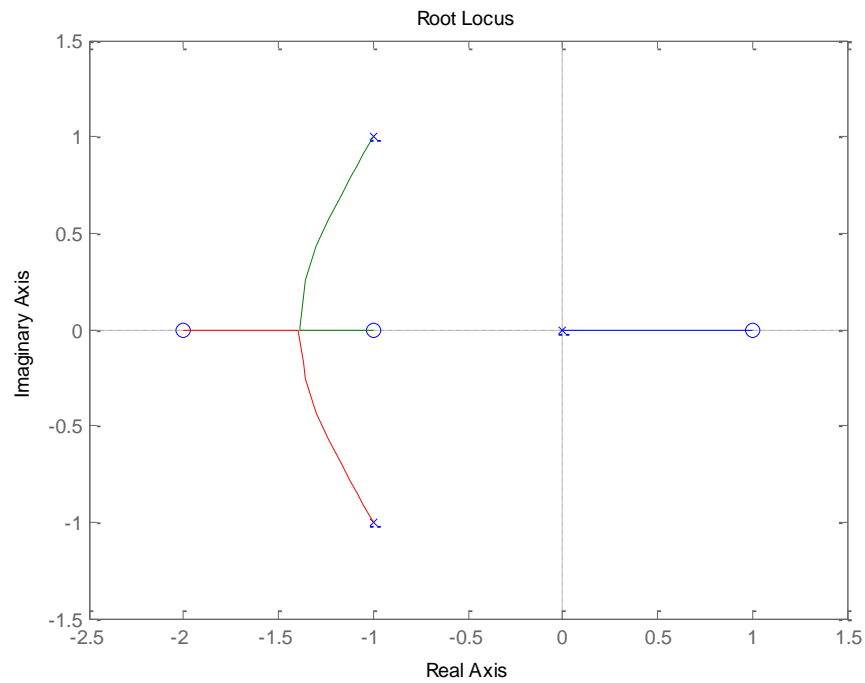
```

Root Locus diagram – 9-15(a):



Root Locus diagram – 9-15(b):**Root Locus diagram – 9-15(c):**

**Root Locus diagram – 9-15(d):****Root Locus diagram – 9-15(e):**

**Root Locus diagram – 9-15(f):**

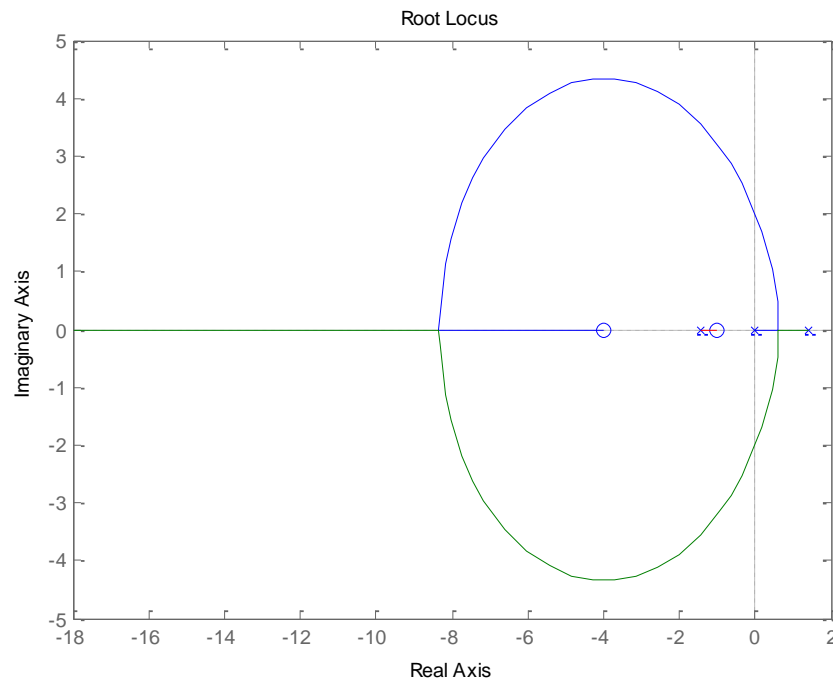
poles: $s = -1, -2 + j, -2 - j$

$$\sigma_1 = \frac{-1-2-j-2+j}{3} = -1.67$$

$$\text{Asymptotes angle: } \theta_i = \frac{2i+1}{|n-m|} \times 180 = \frac{2i+1}{3} \times 180$$

Therefore, $\theta_i = 60, 180, 300$

$$\text{Departure angle from: } \begin{cases} s = -2 - j & : \theta = 45 \\ s = -2 + j & : \theta = -45 \end{cases}$$

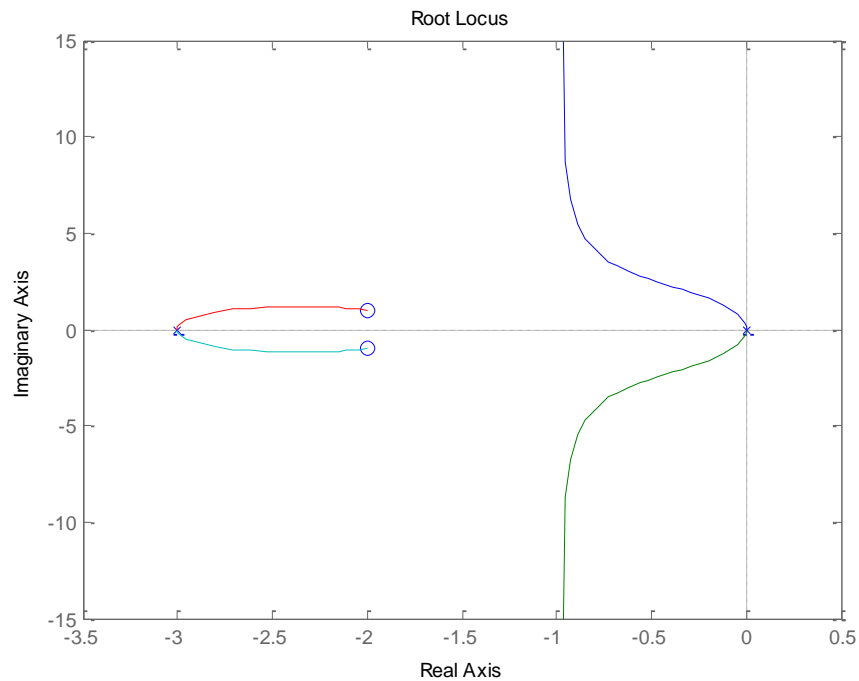
**Root Locus diagram – 9-15(g):**

Poles: $s = -1, -5 - j, 3 + j$ and zeroes: $s = -2$

$$\sigma_1 = \frac{-1-3-j-3+j+2}{2} = -2.5$$

$$\text{Asymptotes angles: } \begin{cases} \theta_i = \frac{2i+1}{n-m} 180 = \frac{2i-1}{3-1} 180 \\ \theta_i = 90, 270 \end{cases}$$

$$\text{Departure angles from: } \begin{cases} s = -3 - j & : \quad \theta = -72^\circ \\ s = -2 + j & : \quad \theta = 72^\circ \end{cases}$$

**Root Locus diagram – 9-15(h):**

Poles: $s = 0, -1$ and zeros: $s = -2, -3$

The break away points:

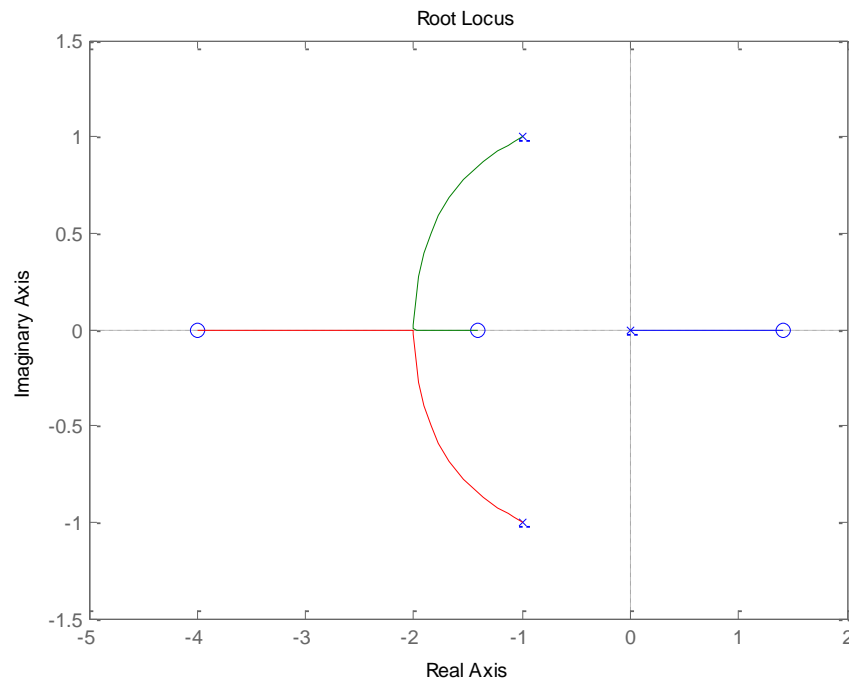
$$-\frac{d}{ds} \left[\frac{P(s)}{Q(s)} \right] = 0$$

which means:

$$-\frac{d}{ds} \left[\frac{s(s+1)}{(s+2)(s+3)} \right] = 0$$

or

$$\begin{aligned} \frac{1}{s+1} + \frac{1}{s} &= \frac{1}{s+2} + \frac{1}{s+3} \\ (2s+1)(s^2+5s+6) - (2s+5)(s^2+s) &= 0 \\ 4s^2 + 12s + 6 &= 0 \\ \begin{cases} s = -0.634 \\ s = -2.366 \end{cases} \end{aligned}$$

**Root Locus diagram – 9-15(i):**

Poles: $s = 0, -2 - j, -2 + j$

breaking points: $-\frac{d}{ds}(s^3 + 4s^2 + 5s) = 0$

which means :

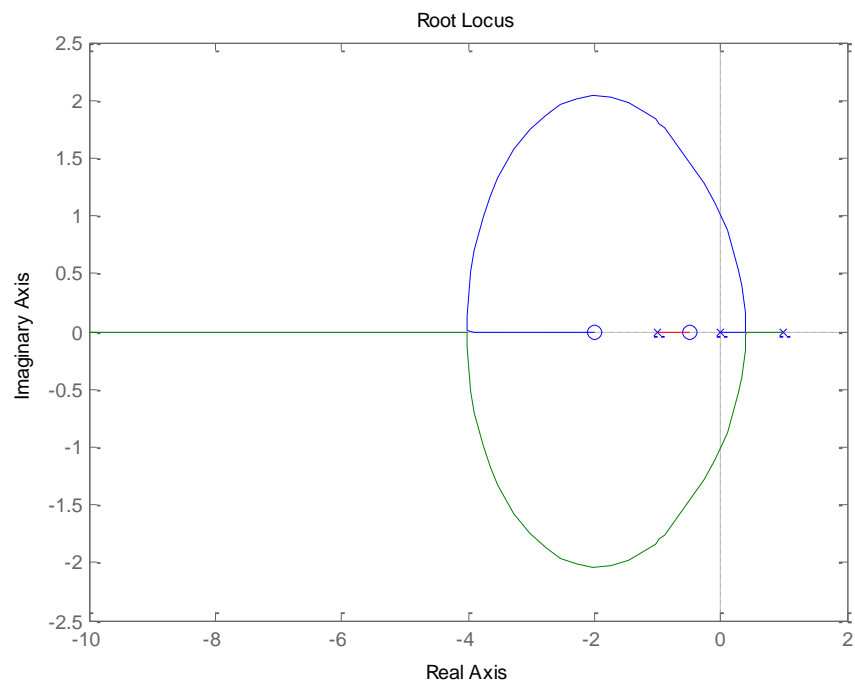
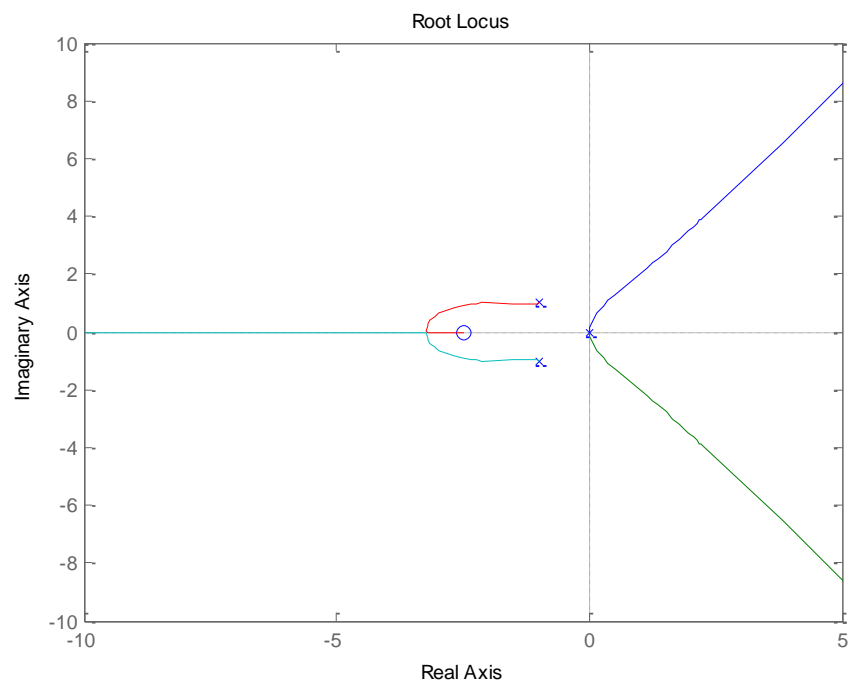
$$\begin{cases} s = -1 \\ s = -1.67 \end{cases}$$

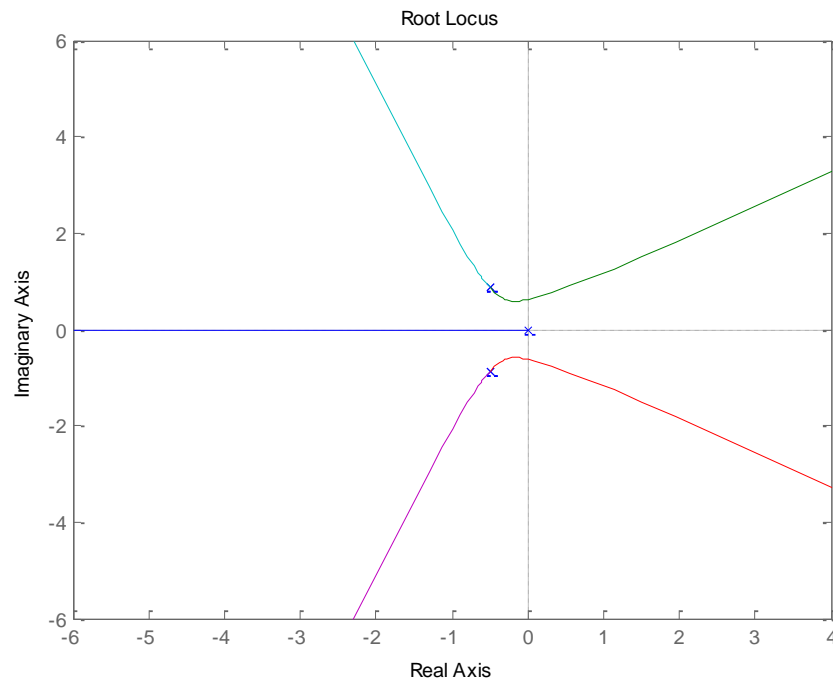
Departure angles from: $\begin{cases} s = -2j & : \quad \theta = -63.43 \\ s = -2 + j & : \quad \theta = 63.43 \end{cases}$

Asymptotes angles: $\theta_i = \frac{2i+1}{n-m} \times 180 = \frac{2i+1}{3} \times 180$

or $\theta = 60^\circ, 180^\circ, 300^\circ$

$$v_1 = \frac{-2 - j - 2 + j}{3} = -\frac{4}{3}$$

**Root Locus diagram – 9-15(j):****Root Locus diagram – 9-15(k):**



9-16) (a) Asymptotes: $K > 0$: 45° , 135° , 225° , 315°

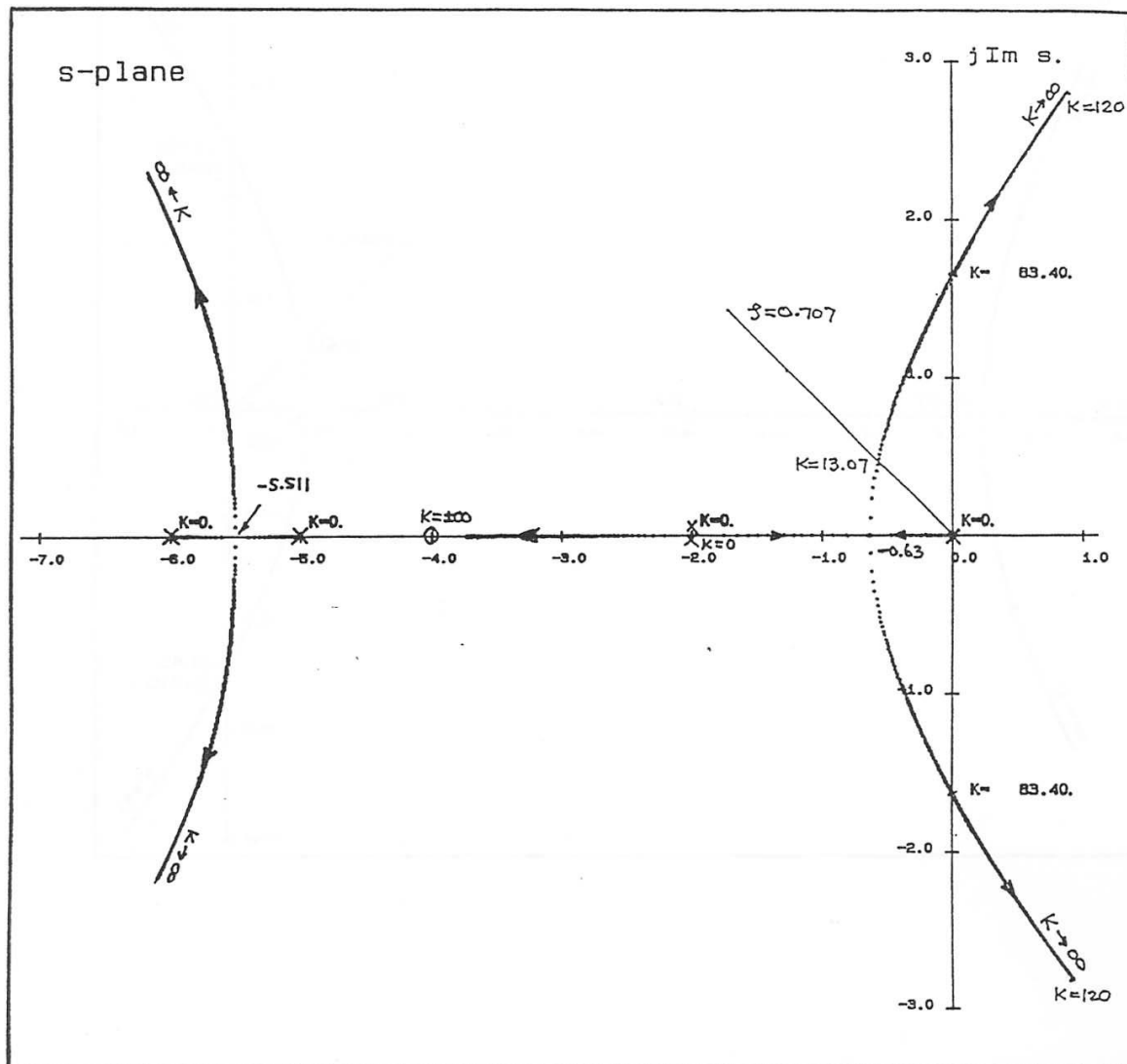
Intersect of Asymptotes:

$$\sigma_1 = \frac{-2 - 2 - 5 - 6 - (-4)}{5 - 1} = -2.75$$

Breakaway-point Equation: $4s^5 + 65s^4 + 396s^3 + 1100s^2 + 1312s + 480 = 0$

Breakaway Points: -0.6325 , -5.511 (on the RL)

When $\zeta = 0.707$, **$K = 13.07$**



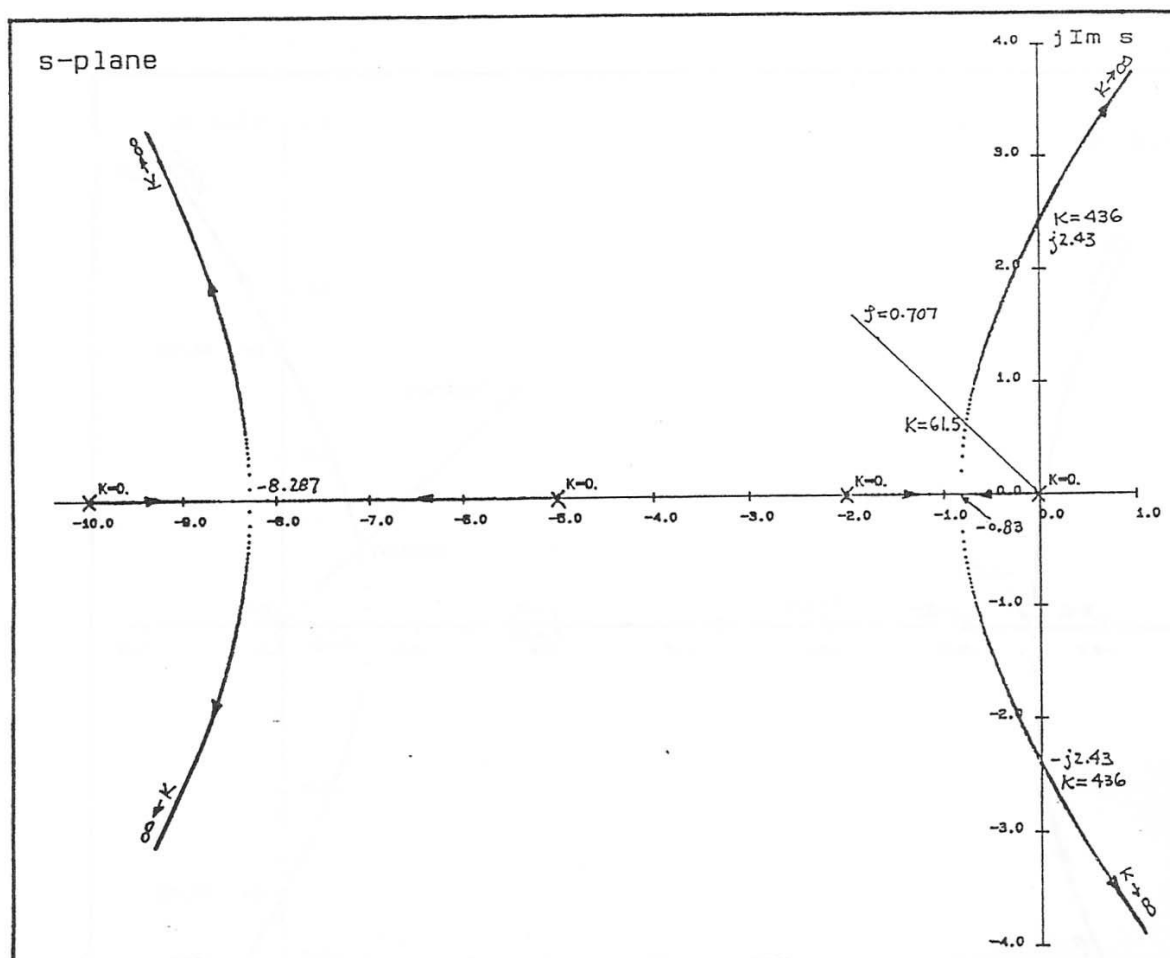
9-16 (b) Asymptotes: $K > 0$: 45° , 135° , 225° , 315°

Intersect of Asymptotes:

$$\sigma_1 = \frac{0-2-5-10}{4} = -4.25$$

Breakaway-point Equation: $4s^3 + 51s^2 + 160s + 100 = 0$

When $\zeta = 0.707$, $K = 61.5$

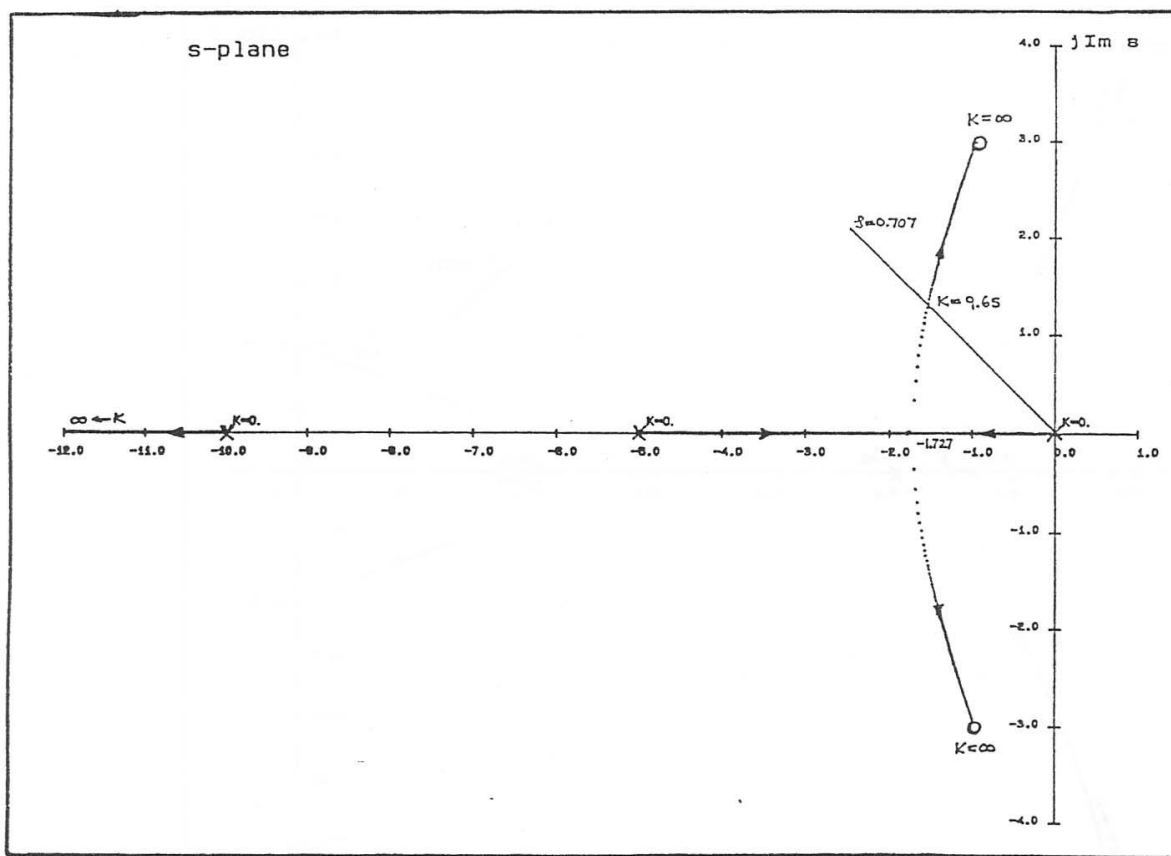


9-16 (c) Asymptotes: $K > 0$: 180°

Breakaway-point Equation: $s^4 + 4s^3 + 10s^2 + 300s + 500 = 0$

Breakaway Points: -1.727 (on the RL)

When $\zeta = 0.707$, **$K = 9.65$**

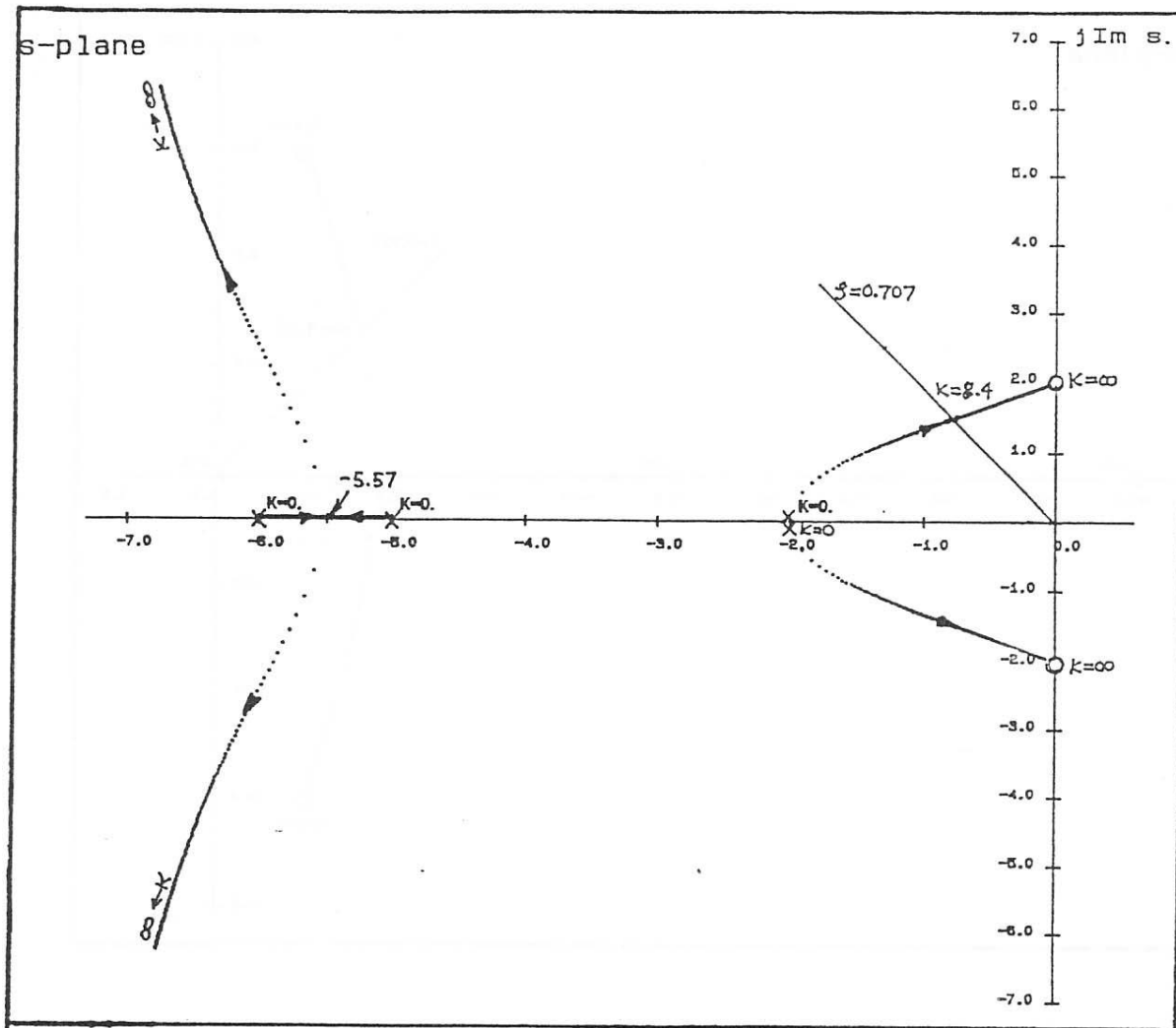


9-16 (d) $K > 0$: $90^\circ, 270^\circ$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-2-2-5-6}{4-2} = -7.5$$

When $\zeta = 0.707$, **$K = 8.4$**



9-17) MATLAB code:

```
clear all;
close all;
s = tf('s')

%a)
num_G_a=(s+3);
den_G_a=s*(s^2+4*s+4)*(s+5)*(s+6);
G_a=num_G_a/den_G_a;
figure(1);
rlocus(G_a)
```

```
%b)
```

```
num_G_b= 1;
den_G_b=s*(s+2)*(s+4)*(s+10);
G_b=num_G_b/den_G_b;
figure(2);
rlocus(G_b)

%c)
num_G_c=(s^2+2*s+8);
den_G_c=s*(s+5)*(s+10);
G_c=num_G_c/den_G_c;
figure(3);
rlocus(G_c)

%d)
num_G_d=(s^2+4);
den_G_d=(s+2)^2*(s+5)*(s+6);
G_d=num_G_d/den_G_d;
figure(4);
rlocus(G_d)

%e)
num_G_e=(s+10);
den_G_e=s^2*(s+2.5)*(s^2+2*s+2);
G_e=num_G_e/den_G_e;
figure(5);
rlocus(G_e)

%f)
num_G_f=1;
den_G_f=(s+1)*(s^2+4*s+5);
G_f=num_G_f/den_G_f;
figure(6);
rlocus(G_f)

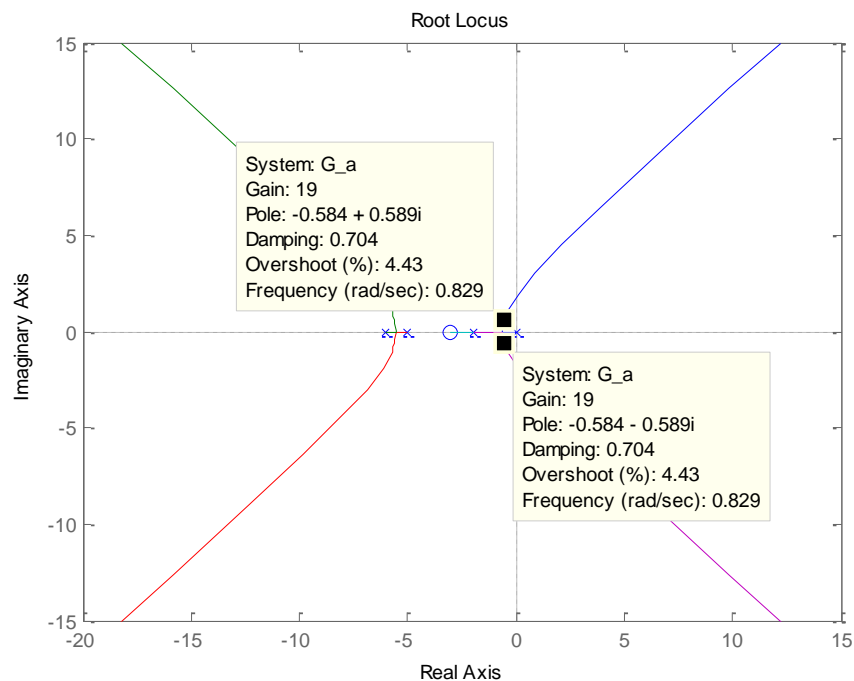
%g)
num_G_g=(s+2);
den_G_g=(s+1)*(s^2+6*s+10);
G_g=num_G_g/den_G_g;
figure(7);
rlocus(G_g)

%h)
num_G_h=(s+3)*(s+2);
den_G_h=s*(s+1);
G_h=num_G_h/den_G_h;
figure(8);
rlocus(G_h)
```

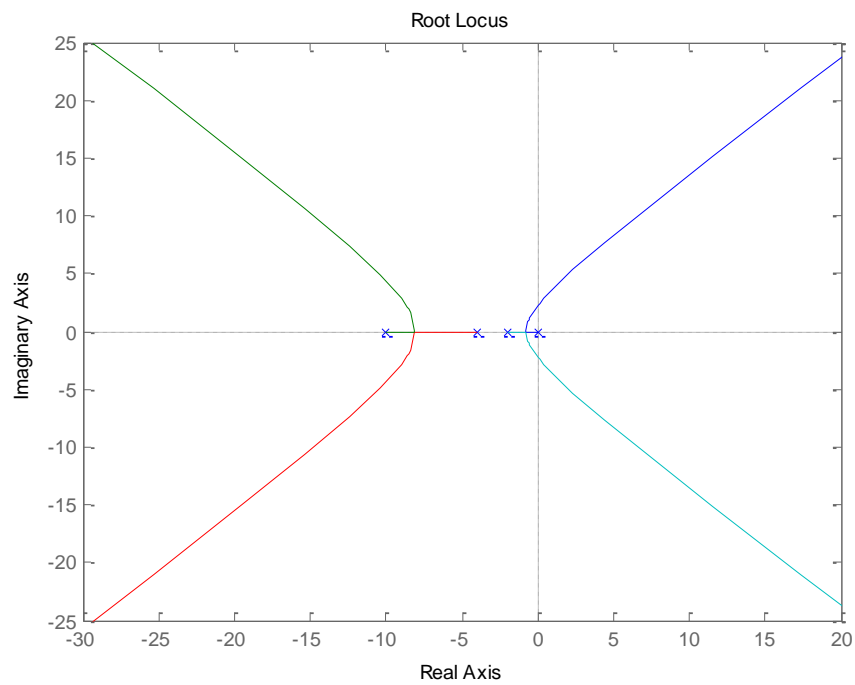
```
%i)
num_G_i=1;
den_G_i=s*(s^2+4*s+5);
G_i=num_G_i/den_G_i;
figure(9);
rlocus(G_i)
```

Root Locus diagram – 9-17(a):

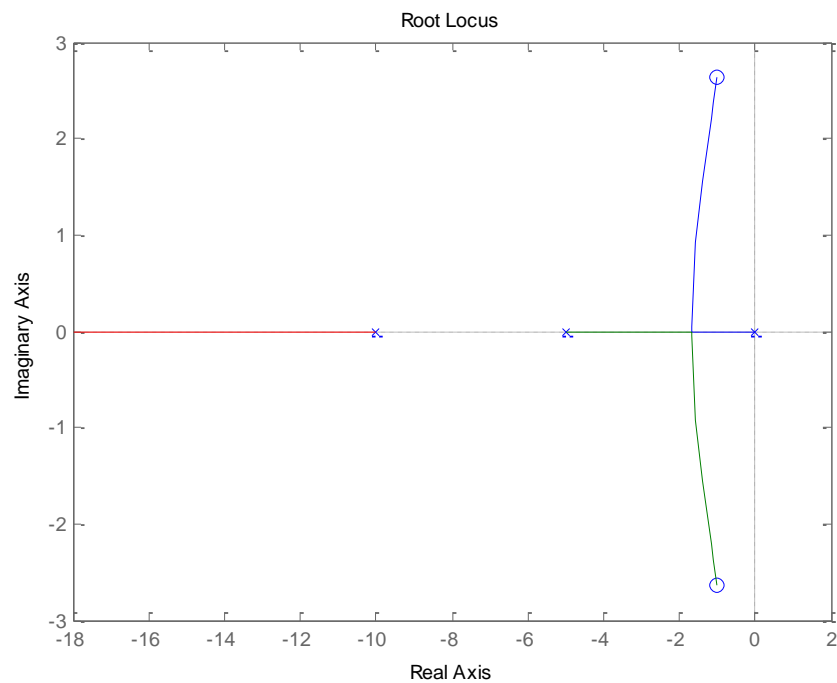
By using “Data Cursor” tab on the figure window and clicking on the root locus diagram, gain and damping values can be observed. Damping of ~ 0.707 can be observed on intersection of the root locus diagram with two lines originating from (0,0) by angles of $\text{ArcCos}(0.707)$ from the real axis. These intersection points are shown for part (a) where the corresponding gain is 19. In the other figures for section (b) to (i), similar points have been picked by the “Data Cursor”, and the gains are reported here.



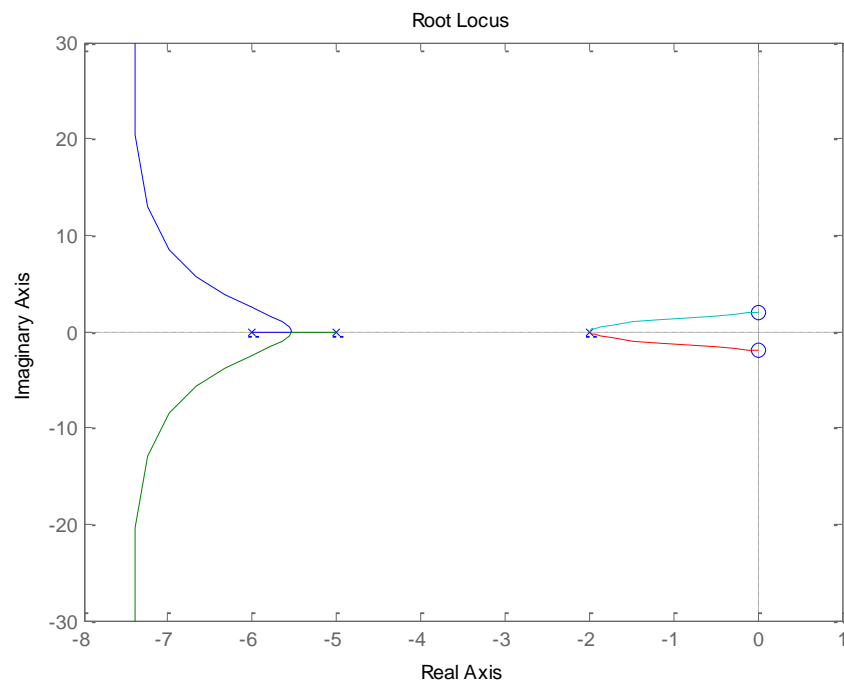
Root Locus diagram – 9-17(b): ($K = 45.5$ @ damping = ~ 0.707)



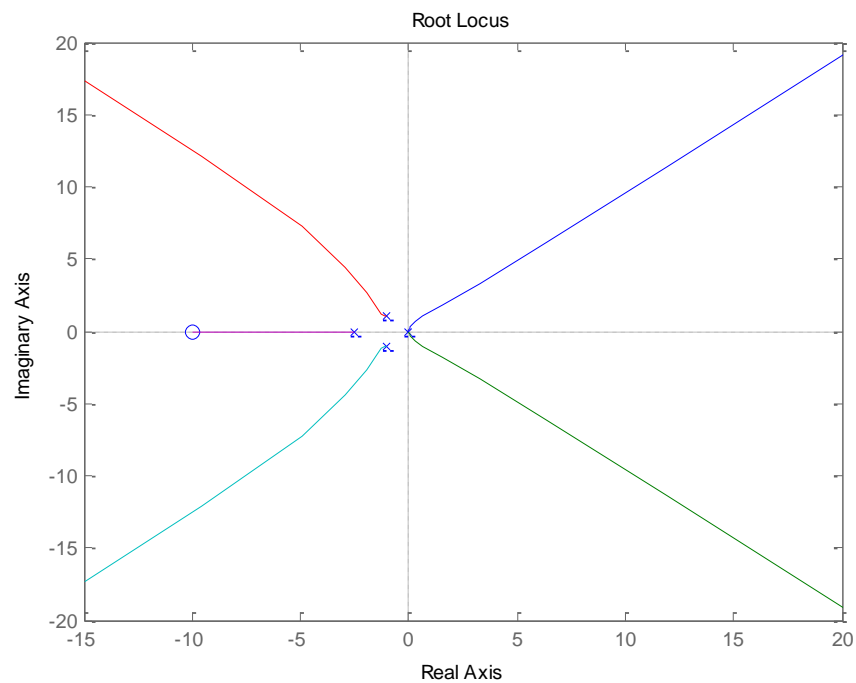
Root Locus diagram – 9-17(c): ($K = 12.8$ @ damping = ~ 0.0707)



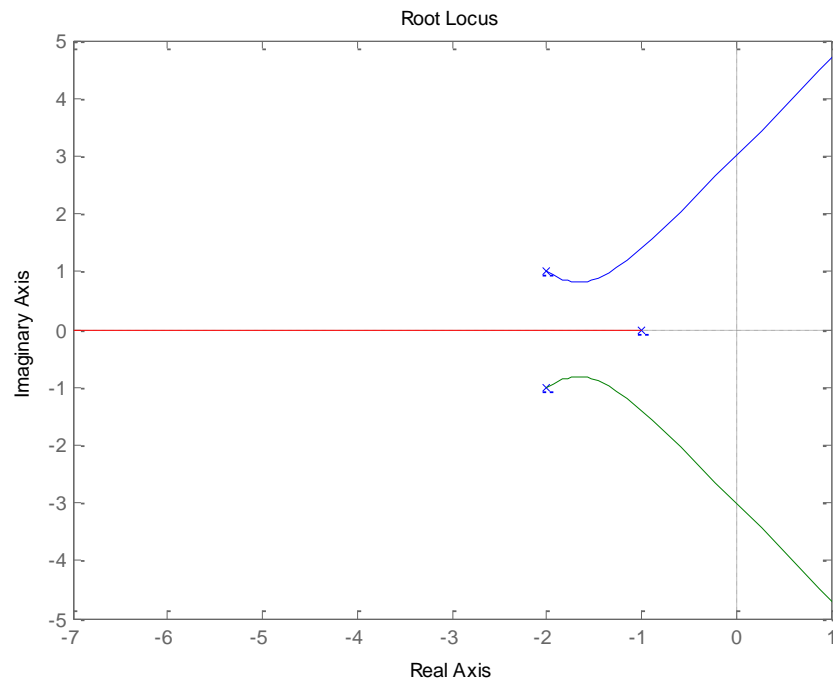
Root Locus diagram – 9-17(d): ($K = 8.3$ @ damping = ~ 0.0707)



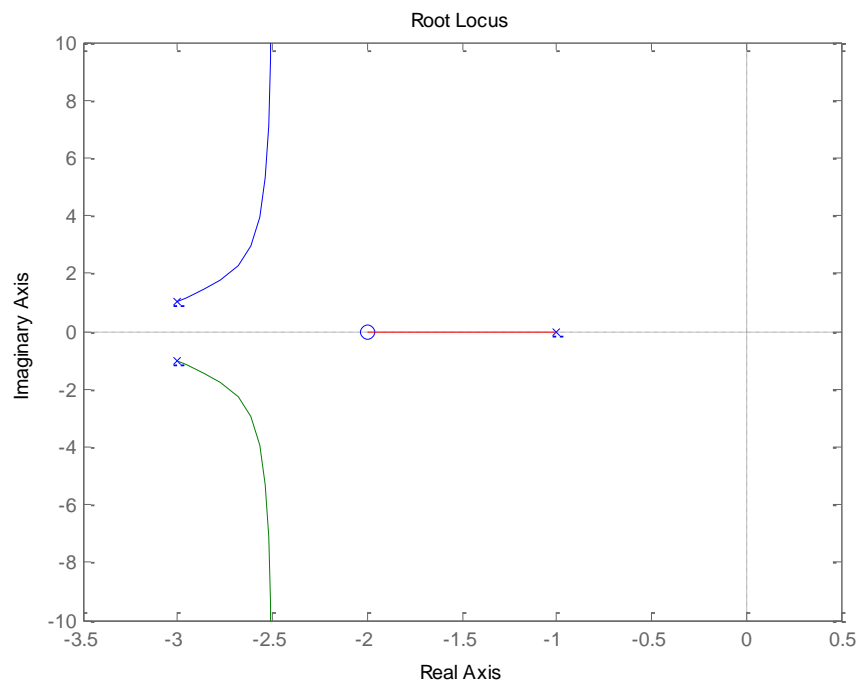
Root Locus diagram – 9-17(e): ($K = 0$ @ damping = 0.0707)



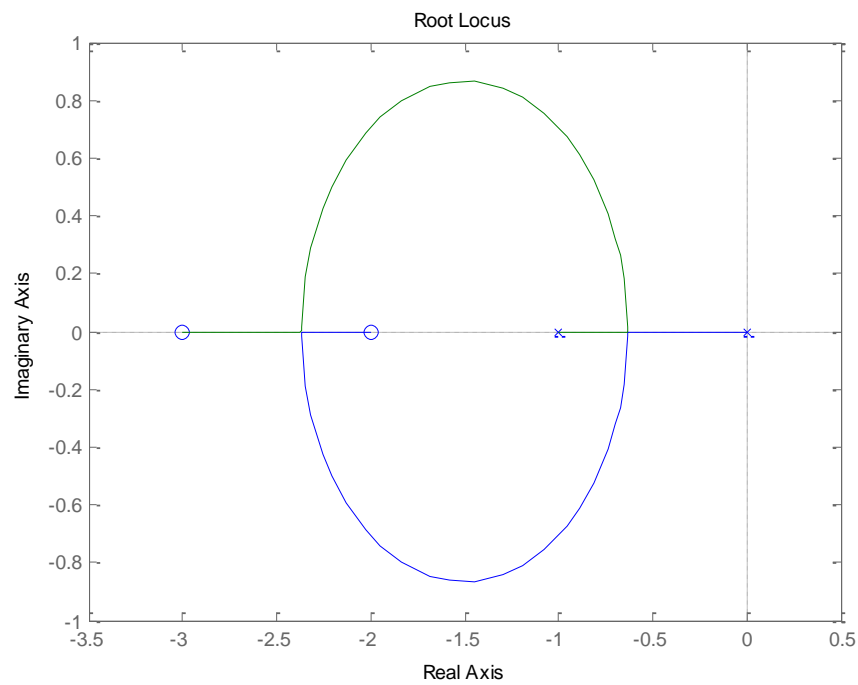
Root Locus diagram – 9-17(f): ($K = 2.33$ @ damping = ~ 0.0707)



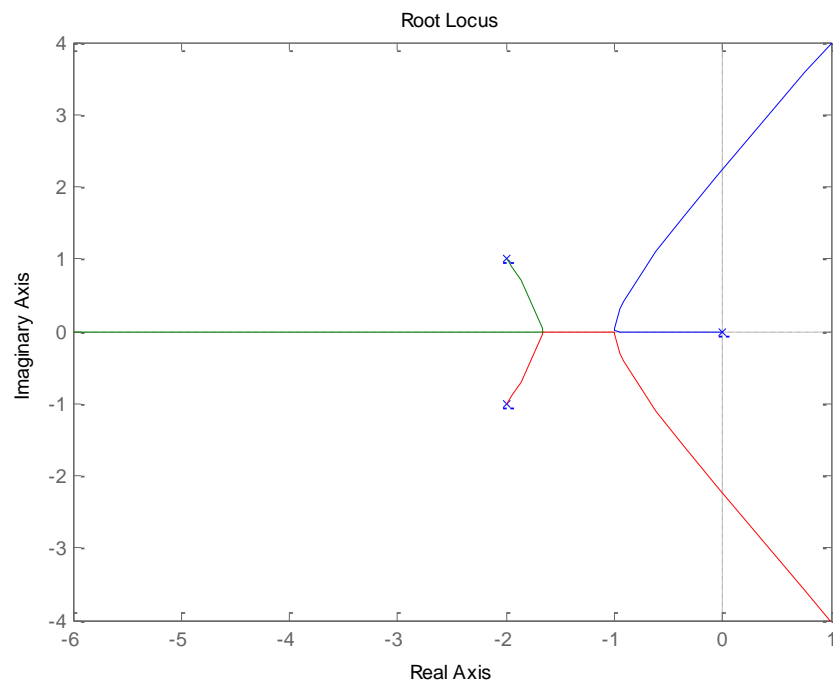
Root Locus diagram – 9-17(g): ($K = 7.03$ @ damping = ~ 0.0707)



Root Locus diagram – 9-17(h): (no solution exists for damping = 0.0707)



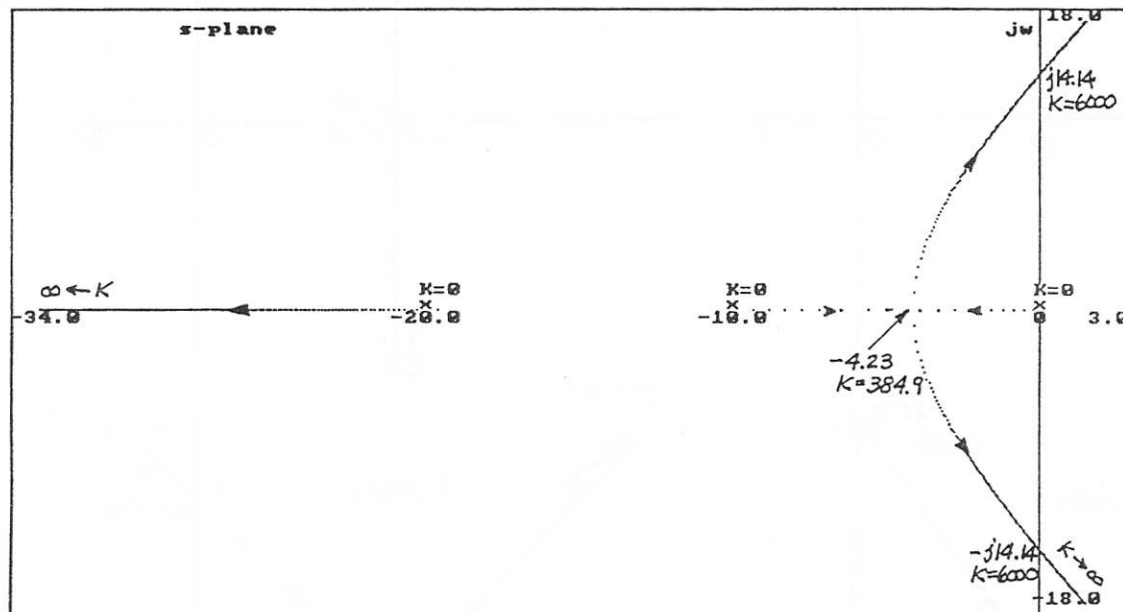
Root Locus diagram – 9-17(i): ($K = 2.93$ @ damping = ~ 0.0707)



9-18) (a) Asymptotes: $K > 0$: 60° , 180° , 300°

Intersect of Asymptotes:

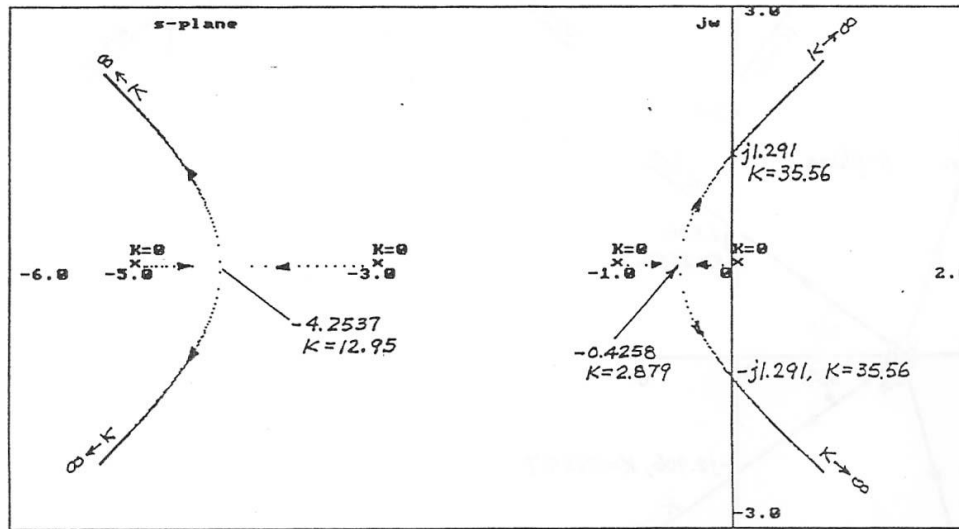
$$\sigma_1 = \frac{0-10-20}{3} = -10$$

Breakaway-point Equation: $3s^2 + 60s + 200 = 0$ Breakaway Point: (RL) -4.2265 , $K = 384.9$ (b) Asymptotes: $K > 0$: 45° , 135° , 225° , 315°

Intersect of Asymptotes:

$$\sigma_1 = \frac{0-1-3-5}{4} = -2.25$$

Breakaway-point Equation: $4s^3 + 27s^2 + 46s + 15 = 0$ Breakaway Points: (RL) -0.4258 $K = 2.879$, -4.2537 $K = 12.95$



- c) Zeros: $s = 0.5$ and poles: $s = 1$

Angle of asymptotes: $\theta = (2i + 1)180 = 180$

The breakaway points: $\frac{1}{(s+1)^2} = \frac{1}{s+0.5} \Rightarrow s^2 + s + 0.5 = 0$

Then $s = -0.5 - 0.5j, -0.5 + 0.5j$ and $\sigma_1 = \frac{+1-0.5}{1} = 0.5$

- d) Poles: $s = -0.5, 4.5$

Angle of asymptotes: $\theta_i = \frac{2i+1}{2} \times 180 = 90, 270$

breakaway points:

$s^2 + s + 0.75 = 0 \Rightarrow s = -1 - \sqrt{2}j, -1 + \sqrt{2}j$

$\sigma_1 = \frac{-0.5+1.5}{2} = 0.5$

- e) Zeros: $s = -\frac{1}{3}, -1$ and poles: $s = 0, 0.5, 1$

Angle of asymptotes: $\theta_i = \frac{2i+1}{3-2} 180 = 180$

breakaway points: $\frac{1}{s} + \frac{1}{s+\frac{1}{2}} + \frac{1}{s-1} = \frac{1}{s+\frac{1}{3}} + \frac{1}{s+1} \Rightarrow s = 0.383, -2.22$

$$\sigma = -\frac{1-0.5+\frac{1}{3}+1}{1} = -\frac{11}{6}$$

f) Poles: $s = 0, -3 + 4j, -3 - 4j$

Angles of asymptotes: $\theta_i = \frac{2i+1}{3} \times 180 = 60, 180, 300$

$$\sigma_1 = -\frac{0+3-4j+3+4j}{3} = 2$$

breakaway point: $-\frac{d}{ds}[s(s^2 + 6s + 25)] = 0$

$$3s^2 + 12s + 25 = 0 \rightarrow s \approx -2 + 2.1j, -2 - 2.1j$$

9-19) MATLAB code:

```
clear all;
close all;
s = tf('s')
```

```
%a)
```

```
num_G_a=1;
den_G_a=s*(s+10)*(s+20);
G_a=num_G_a/den_G_a;
figure(1);
rlocus(G_a)
```

```
%b)
```

```
num_G_b= 1;
den_G_b=s*(s+1)*(s+3)*(s+5);
G_b=num_G_b/den_G_b;
figure(2);
rlocus(G_b)
```

```
%c)
```

```
num_G_c=(s-0.5);
den_G_c=(s-1)^2;
G_c=num_G_c/den_G_c;
figure(3);
rlocus(G_c)
```

```
%d)
```

```
num_G_d=1;
den_G_d=(s+0.5)*(s-1.5);
G_d=num_G_d/den_G_d;
figure(4);
rlocus(G_d)
```

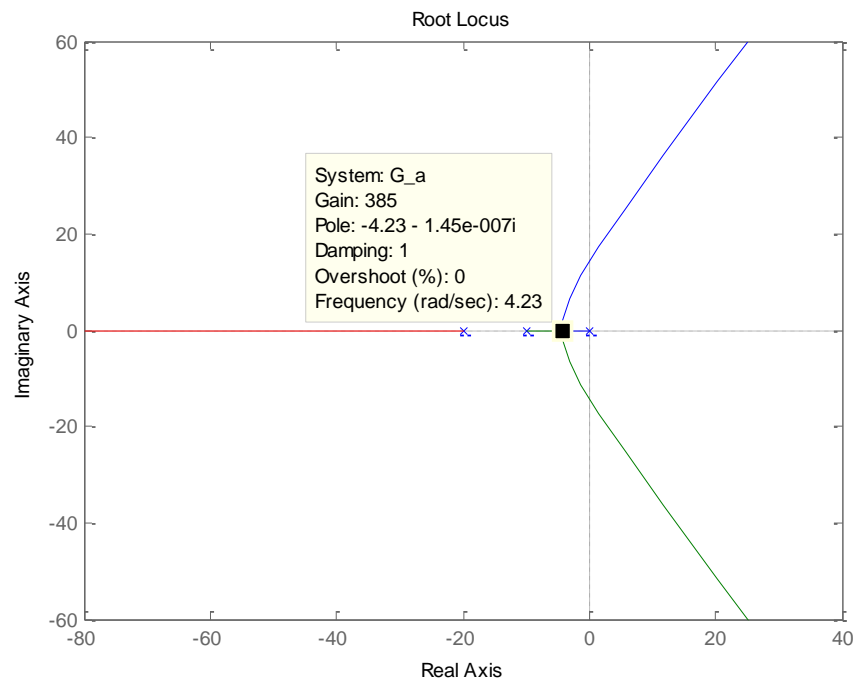
```

%e)
num_G_e=(s+1/3)*(s+1);
den_G_e=s*(s+1/2)*(s-1);
G_e=num_G_e/den_G_e;
figure(5);
rlocus(G_e)

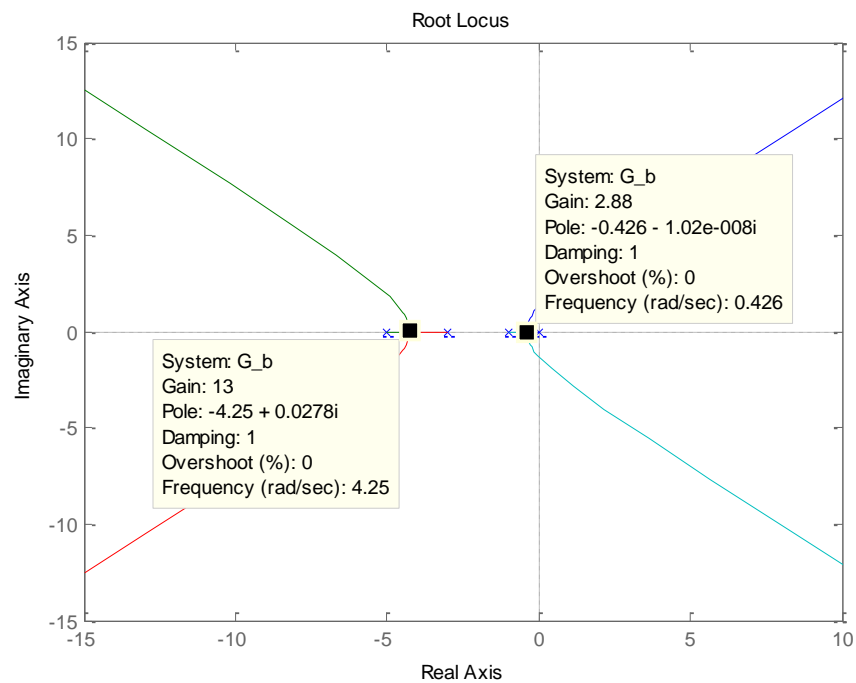
%f)
num_G_f=1;
den_G_f=s*(s^2+6*s+25);
G_f=num_G_f/den_G_f;
figure(6);
rlocus(G_f)

```

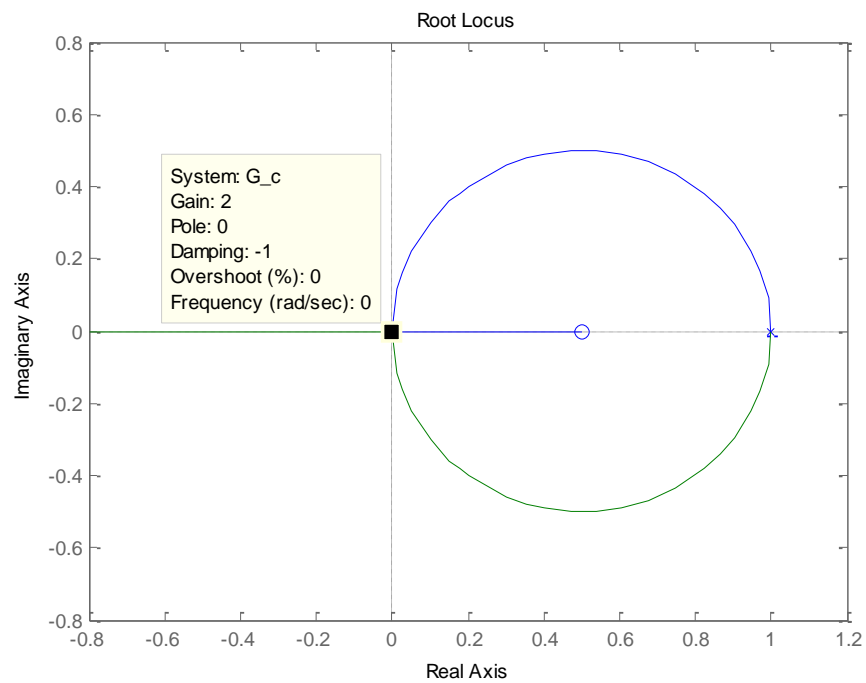
Root Locus diagram – 9-19(a):



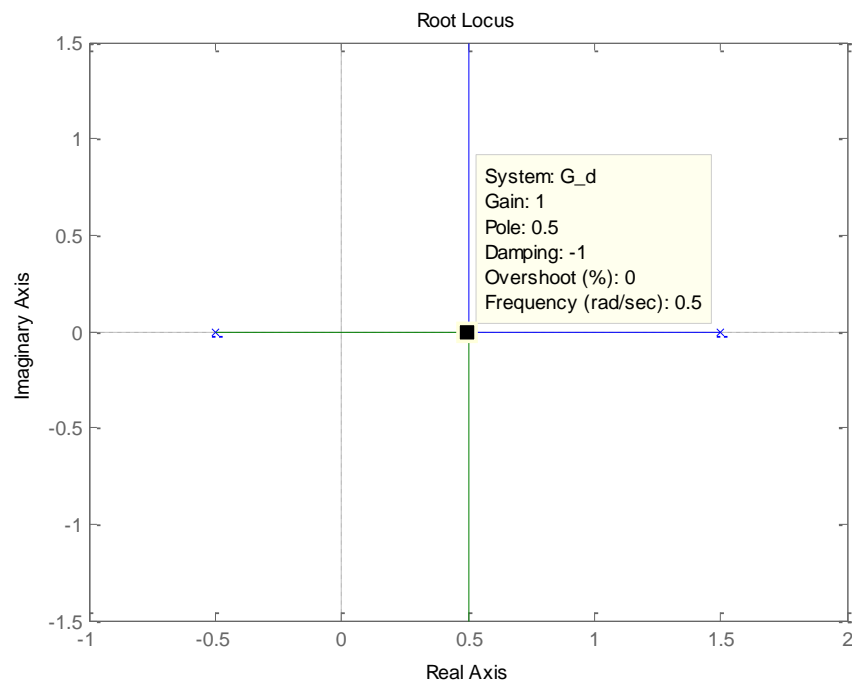
Root Locus diagram – 9-19(b):



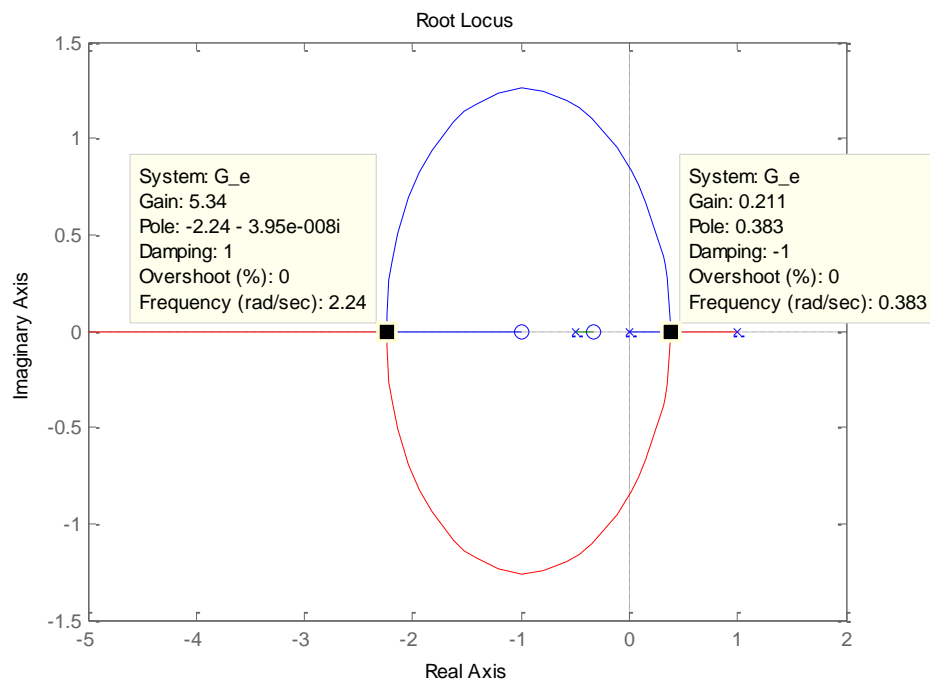
Root Locus diagram – 9-19(c):



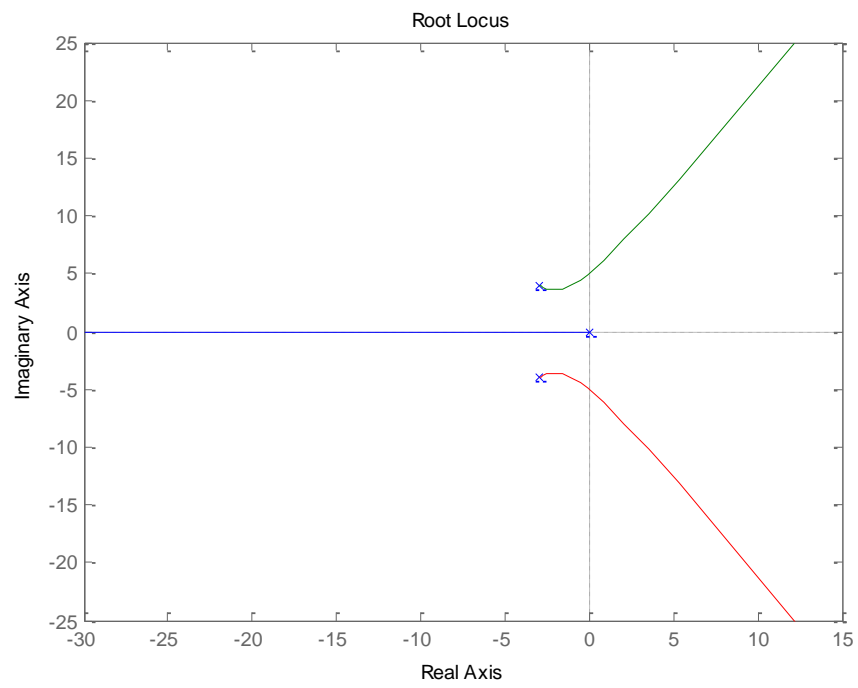
Root Locus diagram – 9-19(d):



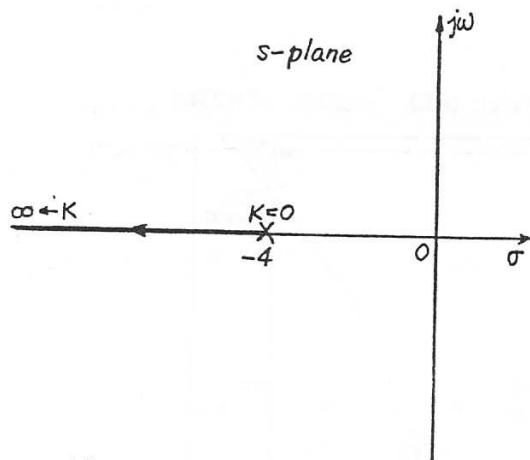
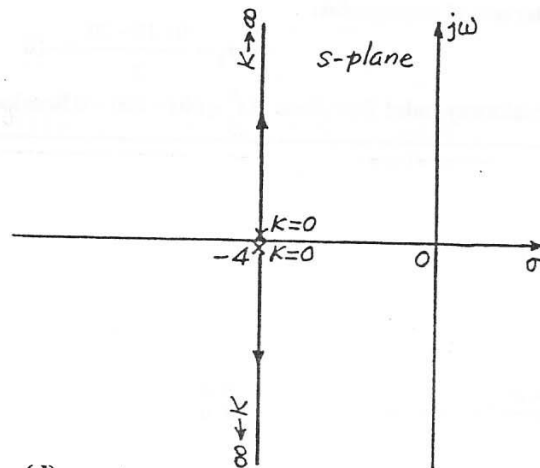
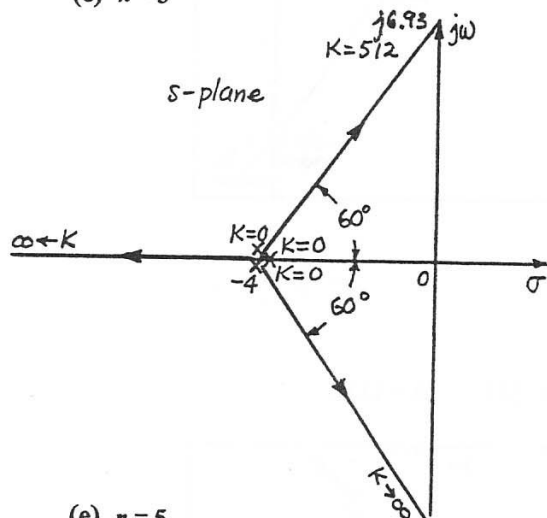
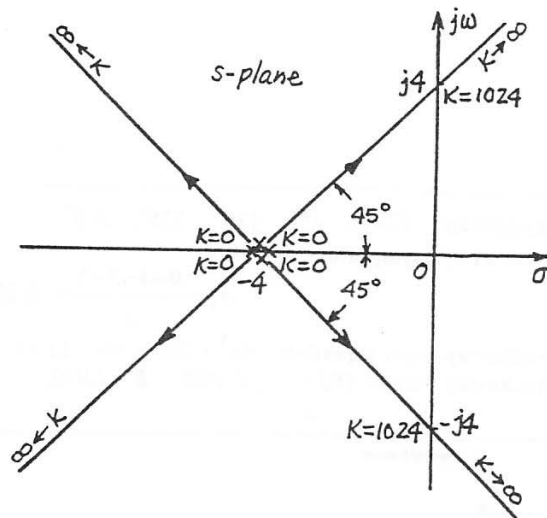
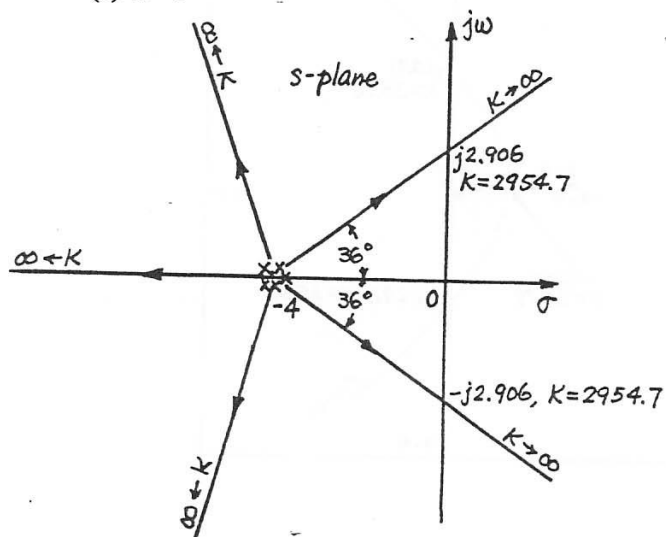
Root Locus diagram – 9-19(e):



Root Locus diagram – 9-19(f): (No breakaway points)



9-20)

8-9 (a) $n=1$ (b) $n=2$ (c) $n=3$ (d) $n=4$ (e) $n=5$ 

9-21) MATLAB code:

```
clear all;
close all;
s = tf('s')

%a)
n=1;
num_G_a= 1;
den_G_a=(s+4)^n;
G_a=num_G_a/den_G_a;
figure(n);
rlocus(G_a)

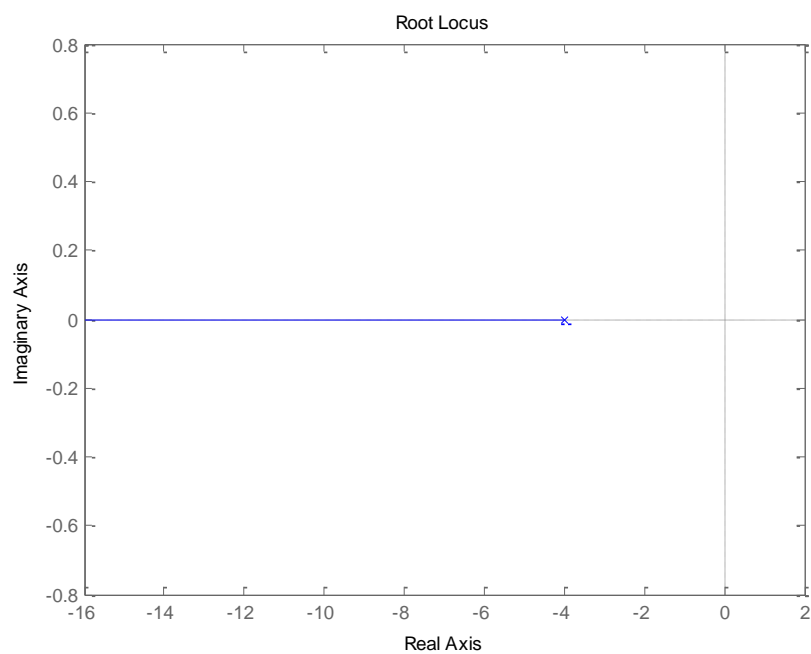
%b)
n=2;
num_G_b= 1;
den_G_b=(s+4)^n;
G_b=num_G_b/den_G_b;
figure(n);
rlocus(G_b)

%c)
n=3;
num_G_c= 1;
den_G_c=(s+4)^n;
G_c=num_G_c/den_G_c;
figure(n);
rlocus(G_c)

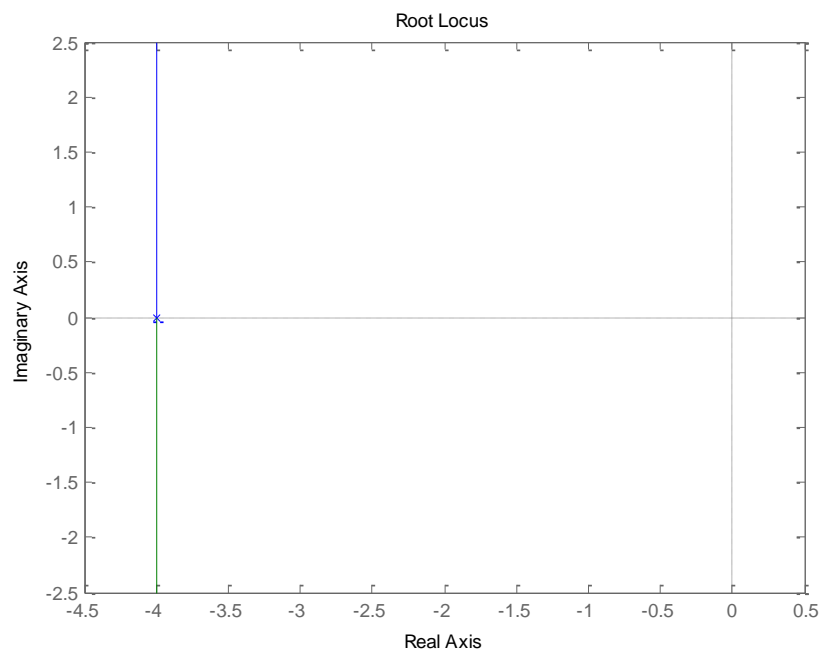
%d)
n=4;
num_G_d= 1;
den_G_d=(s+4)^n;
G_d=num_G_d/den_G_d;
figure(n);
rlocus(G_d)

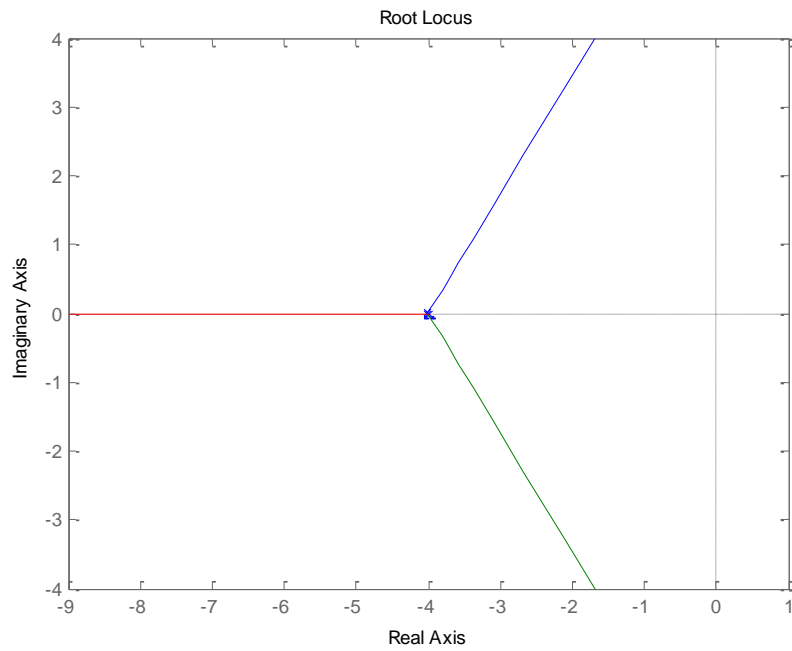
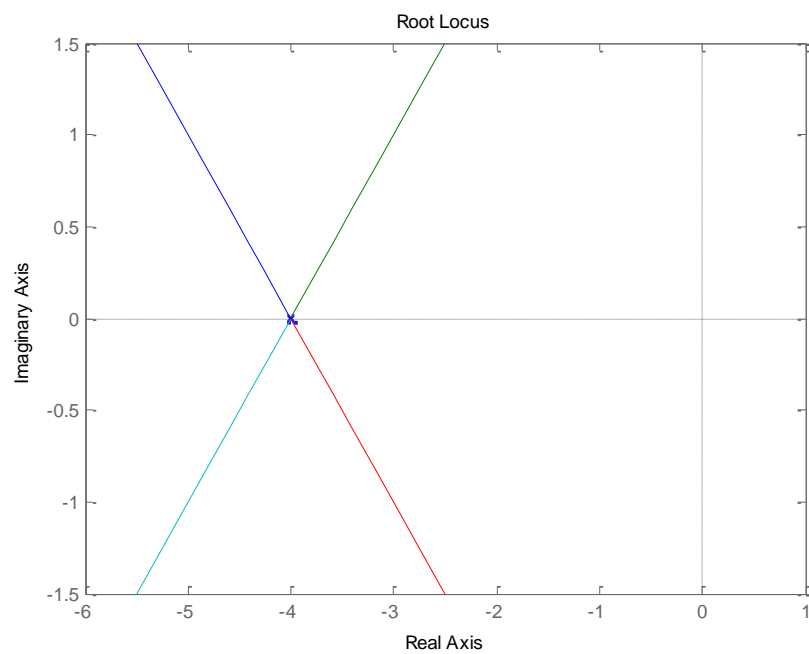
%e)
n=5;
num_G_e= 1;
den_G_e=(s+4)^n;
G_e=num_G_e/den_G_e;
figure(n);
rlocus(G_e)
```

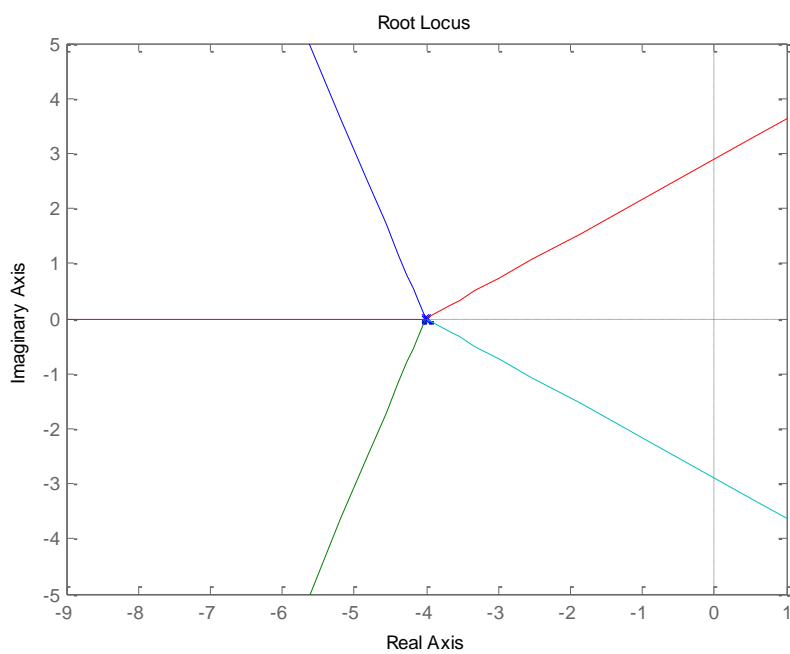
Root Locus diagram – 9-21(a):



Root Locus diagram – 9-21(b):



Root Locus diagram – 9-21(c):**Root Locus diagram – 9-21(d):**

Root Locus diagram – 9-21(e):

9-22) $P(s) = s^3 + 25s^2 + 2s + 100$ $Q(s) = 100s$

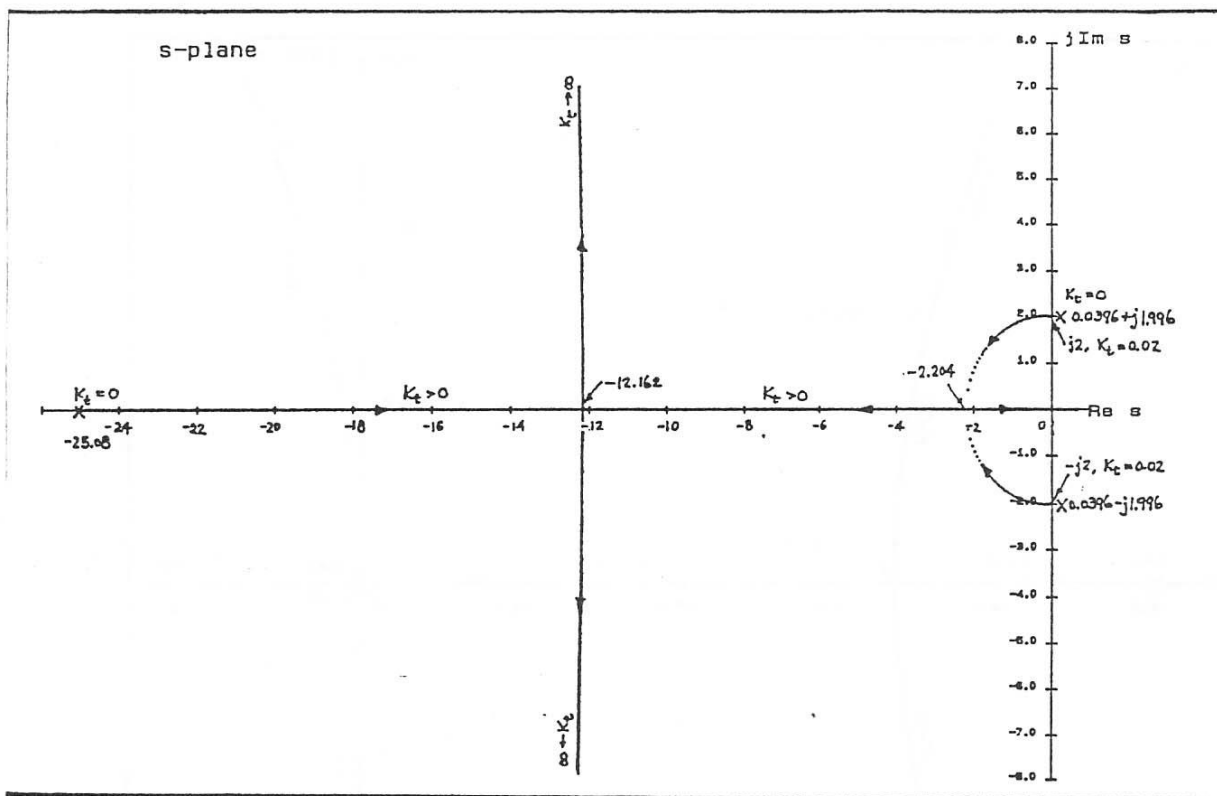
Asymptotes: $K_t > 0$: 90° , 270°

Intersect of Asymptotes:

$$\sigma_1 = \frac{-25-0}{3-1} = -12.5$$

Breakaway-point Equation: $s^3 + 12.5s^2 - 50 = 0$

Breakaway Points: (RL) -2.2037 , -12.162



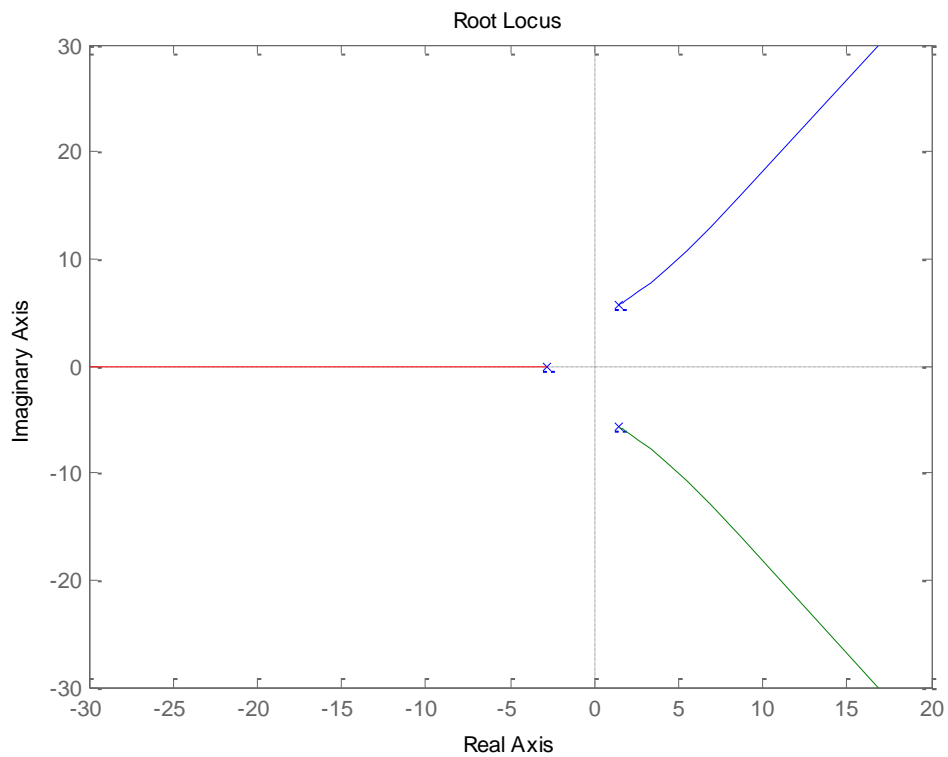
9-23) MATLAB code:

```

s = tf('s')
num_G= 100;
den_G=s^3+25*s+2*s+100;
G=num_G/den_G;
figure(1);
rlocus(G)

```

Root Locus diagram – 9-23:



9-24) Characteristic equation: $s^3 + 5s^2 + K_t s + K = 0$

(a) $K_t = 0$: $P(s) = s^2(s + 5)$ $Q(s) = 1$

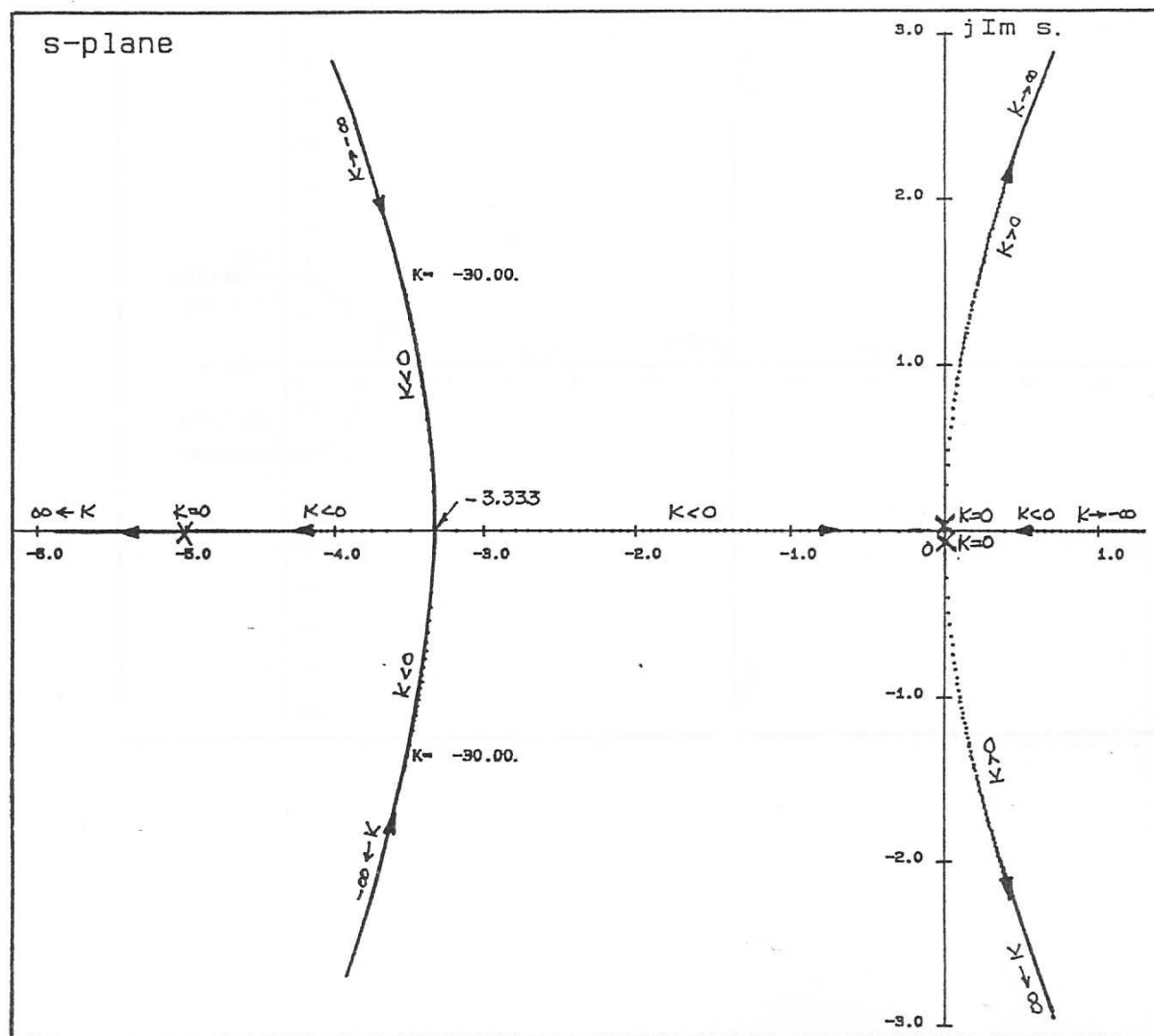
Asymptotes: $K > 0$: $60^\circ, 180^\circ, 300^\circ$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-5-0}{3} = -1.667$$

Breakaway-point Equation: $3s^2 + 10s = 0$

Breakaway Points: 0, -3.333



9-24 (b) $P(s) = s^3 + 5s^2 + 10 = 0$ $Q(s) = s$

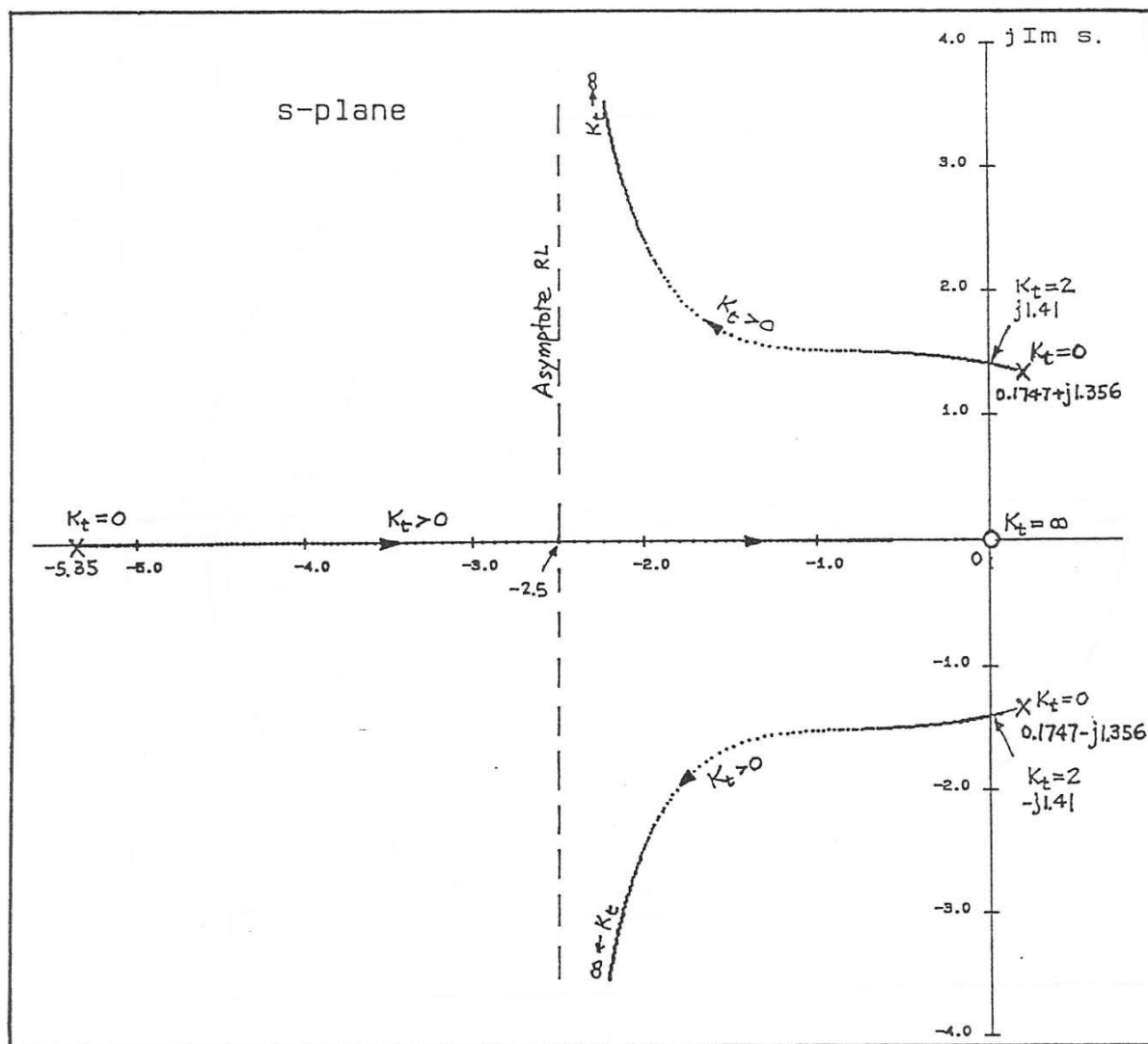
Asymptotes: $K > 0$: $90^\circ, 270^\circ$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-5-0}{2-1} = 0$$

Breakaway-point Equation: $2s^3 + 5s - 10 = 0$

There are no breakaway points on RL.



9-25)

By collapsing the two loops, and finding the overall close loop transfer function, the characteristic equation (denominator of closed loop transfer function) can be found as:

$$1 + GH = \frac{s^3 + 5s^2 + K_t s + K}{s^2(s + 5) + K_t s}$$

For part (a):

$K_t = 0$. Therefore, assuming

$\text{Den}(GH) = s^3 + 5s^2$ and

$\text{Num}(GH) = 1$, we can use rlocus command to construct the root locus diagram.

For part (b):

$K = 10$. Therefore, assuming

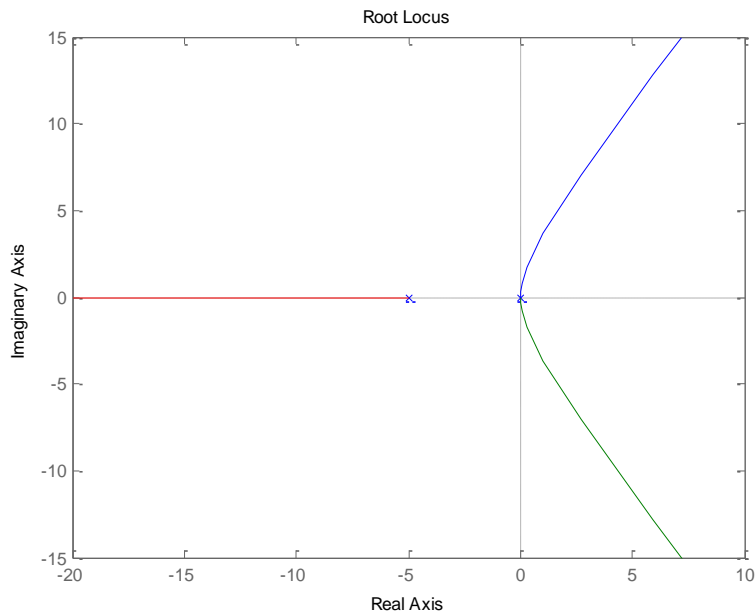
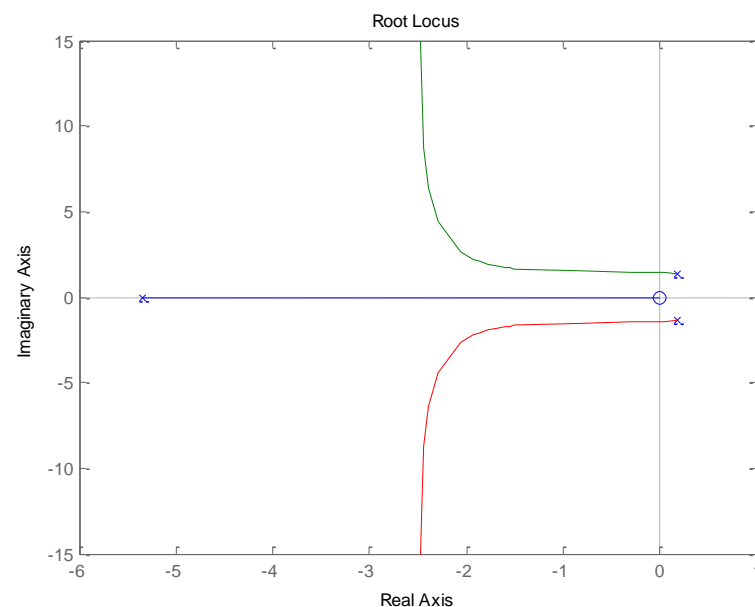
$\text{Den}(GH) = s^3 + 5s^2 + 10$ and

$\text{Num}(GH) = s$, we can use rlocus command to construct the root locus diagram.

MATLAB code (9-25):

```
s = tf('s')
%a)
num_G_a = 1;
den_G_a = s^3 + 5*s^2;
GH_a = num_G_a/den_G_a;
figure(1);
rlocus(GH_a)

%b)
num_G_b = s;
den_G_b = s^3 + 5*s^2 + 10;
GH_b = num_G_b/den_G_b;
figure(2);
rlocus(GH_b)
```

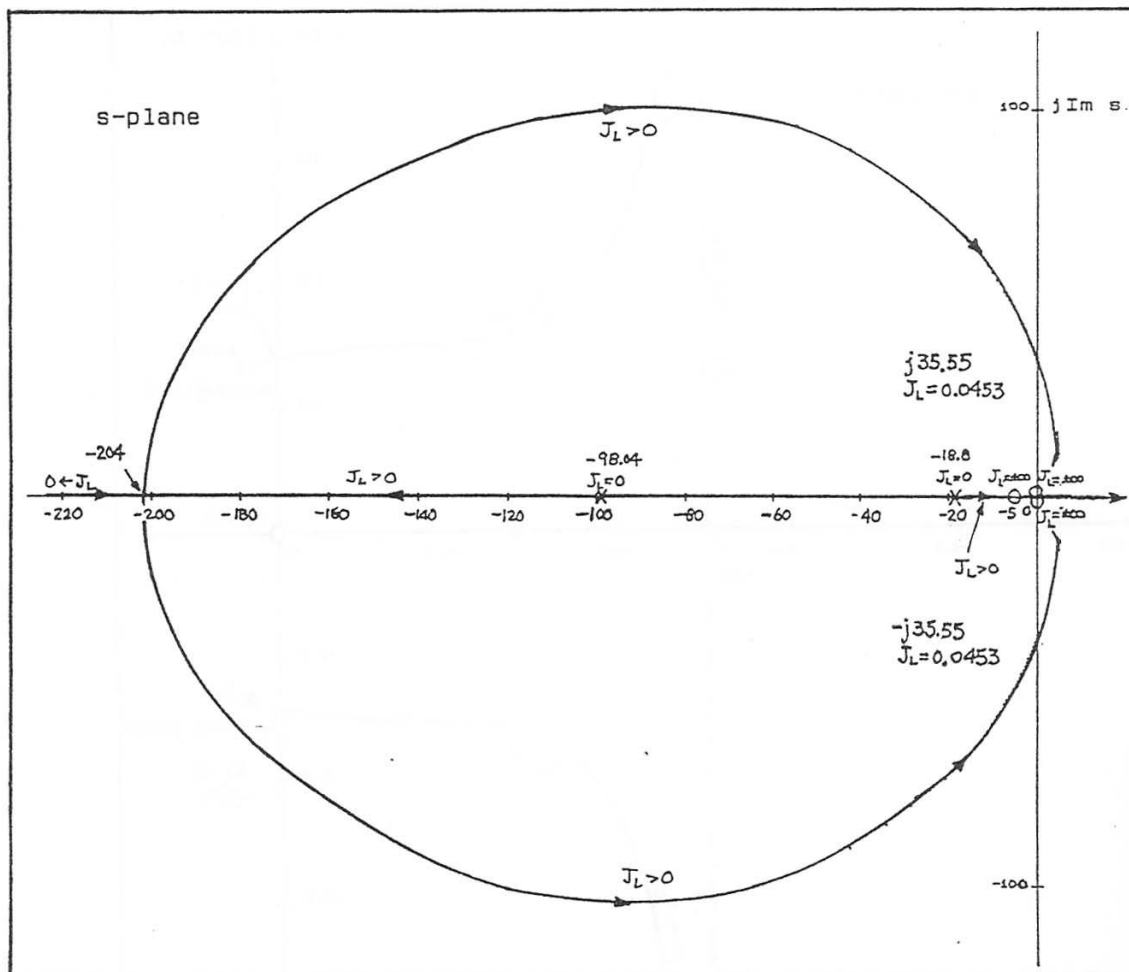
Root locus diagram, part (a):**Root locus diagram, part (b):**

9-26) $P(s) = s^2 + 116.84s + 1843$ $Q(s) = 2.05s^2(s + 5)$

Asymptotes: $J_L = 0: 180^\circ$

Breakaway-point Equation: $-2.05s^4 - 479s^3 - 12532s^2 - 37782s = 0$

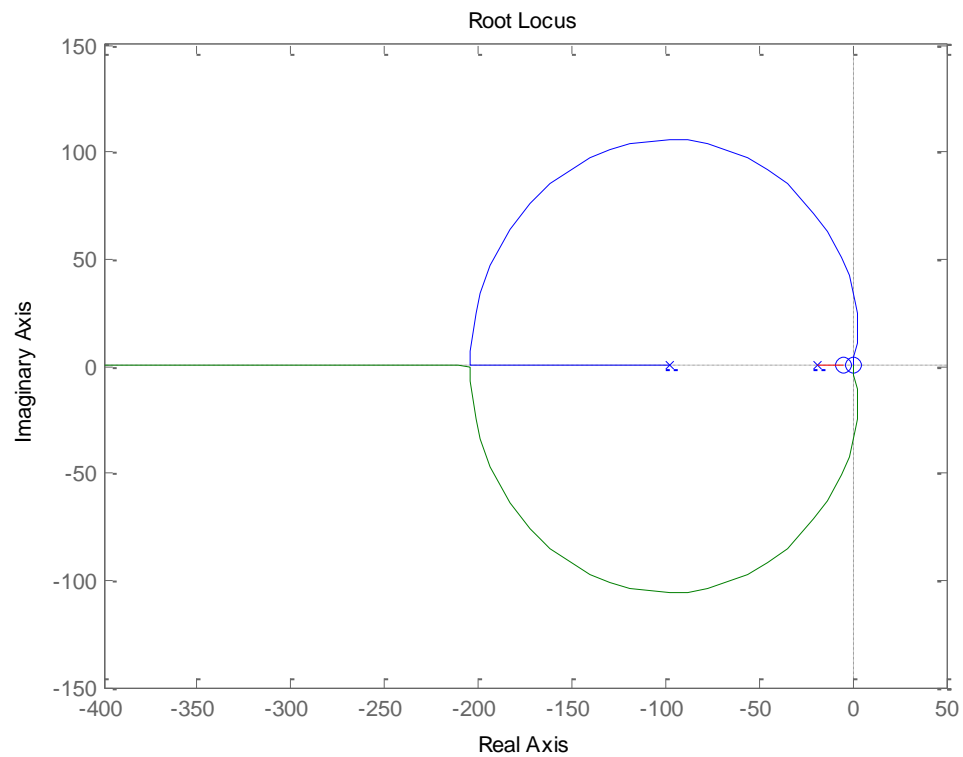
Breakaway Points: **(RL)** 0, -204.18



9-27) MATLAB code:

```
s = tf('s')
num_G = (2.05*s^3 + 10.25*s^2);
den_G = (s^2 + 116.84*s + 1843);
G = num_G/den_G;
figure(1);
rlocus(G)
```

Root locus diagram:

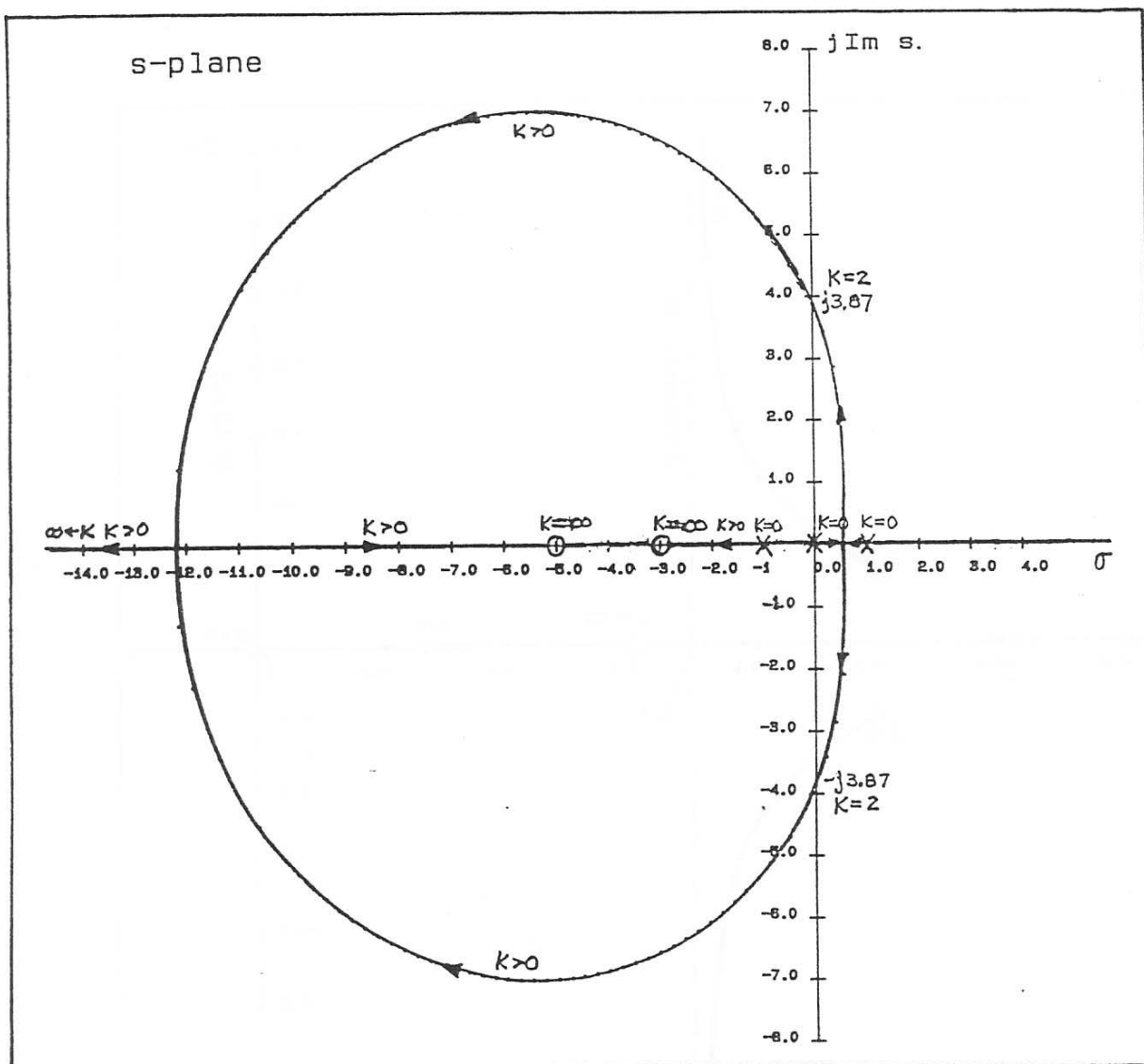


9-28) (a) $P(s) = s(s^2 - 1)$ $Q(s) = (s+5)(s+3)$

Asymptotes: $K > 0$: 180°

Breakaway-point Equation: $s^4 + 16s^3 + 46s^2 - 15 = 0$

Breakaway Points: (RL) 0.5239, -12.254



9-28 (b) $P(s) = s(s^2 + 10s + 29)$ $Q(s) = 10(s + 3)$

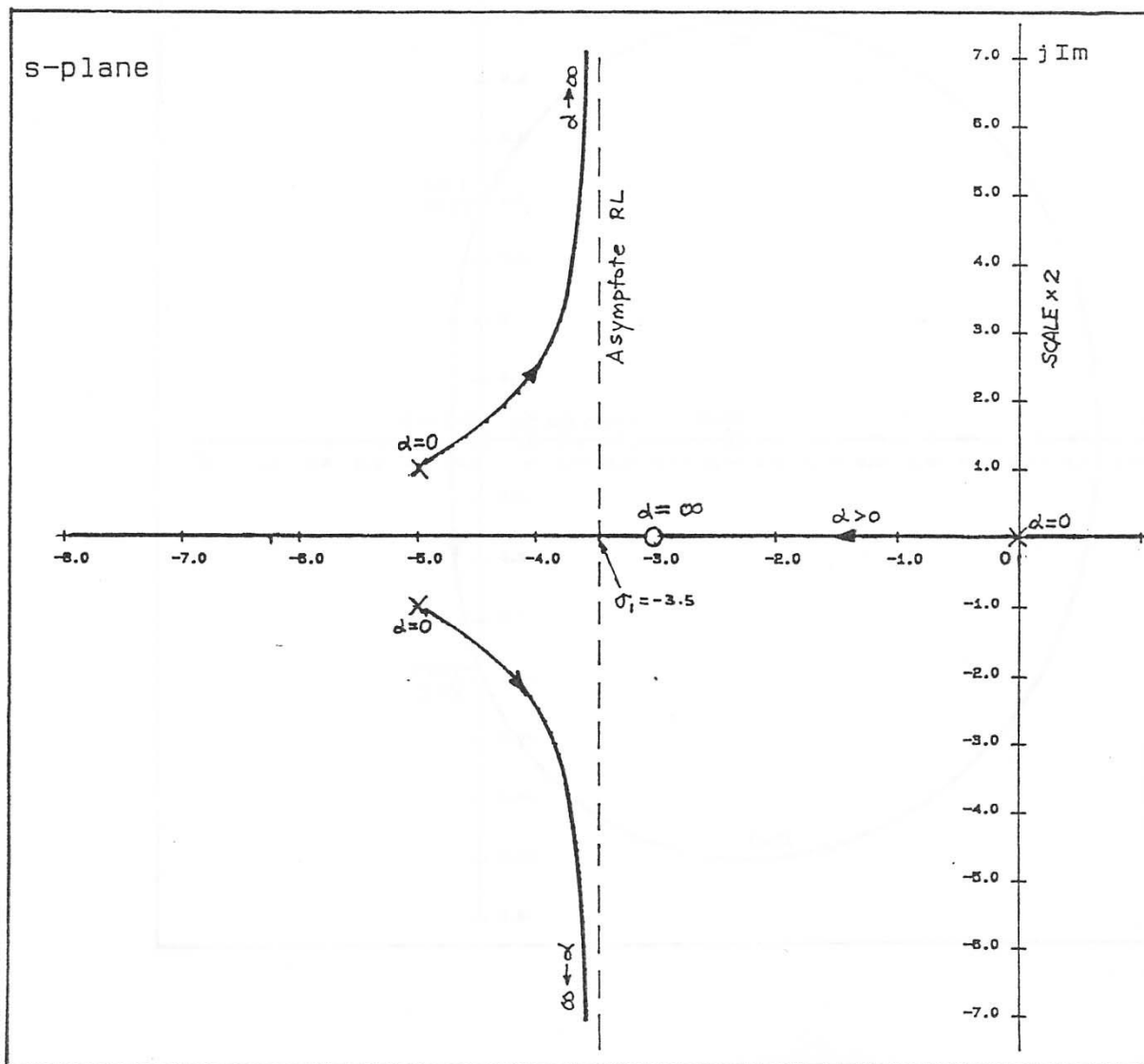
Asymptotes: $K > 0$: 90° , 270°

Intersect of Asymptotes:

$$\sigma_1 = \frac{0 - 10 - (-3)}{3 - 1} = -3.5$$

Breakaway-point Equation: $20s^3 + 190s^2 + 600s + 870 = 0$

There are no breakaway points on the RL.



9-29)

MATLAB code (9-29):

```

s = tf('s')
%a)
num_G_a = (s+5)*(s+3);
den_G_a = s*(s^2 - 1);
G_a = num_G_a/den_G_a;
figure(1);
rlocus(G_a)

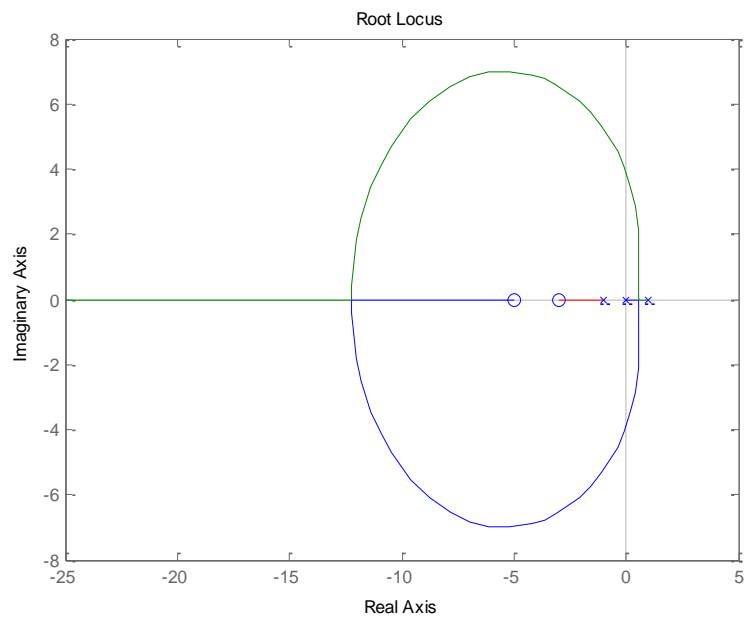
```

Root locus diagram, part (a):

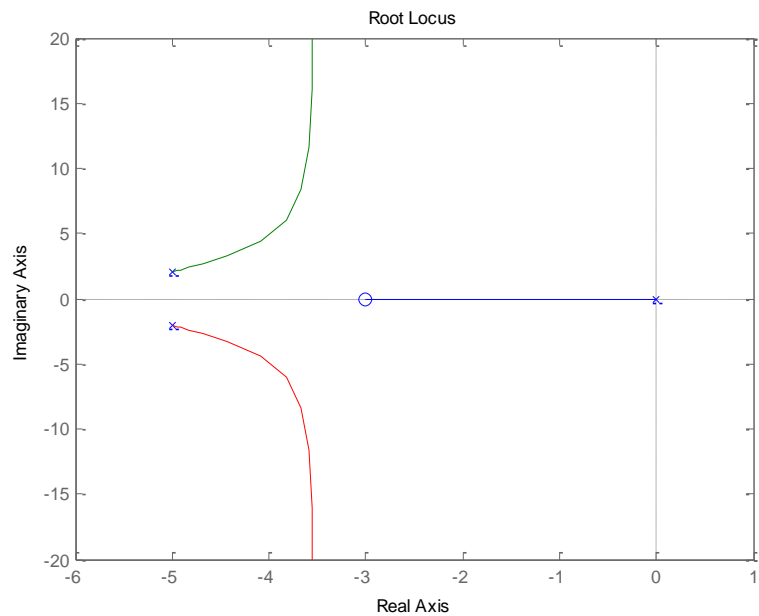
```

K=10;
%b)
num_G_b = (3*K+K*s);
den_G_b = (s^3+K*s^2+K*3*s-s);
G_b = num_G_b/den_G_b;
figure(2);
rlocus(G_b)

```



Root locus diagram, part (b):



9-30) Poles: $s = 0, -3.6$ zeros: $s = -0.4$

Angles of asymptotes: $\theta_i = \frac{2i+1}{3-1} \times 180 = 90^\circ, 270^\circ$

$$\sigma = -\frac{3.6-0.4}{3-1} = -1.6$$

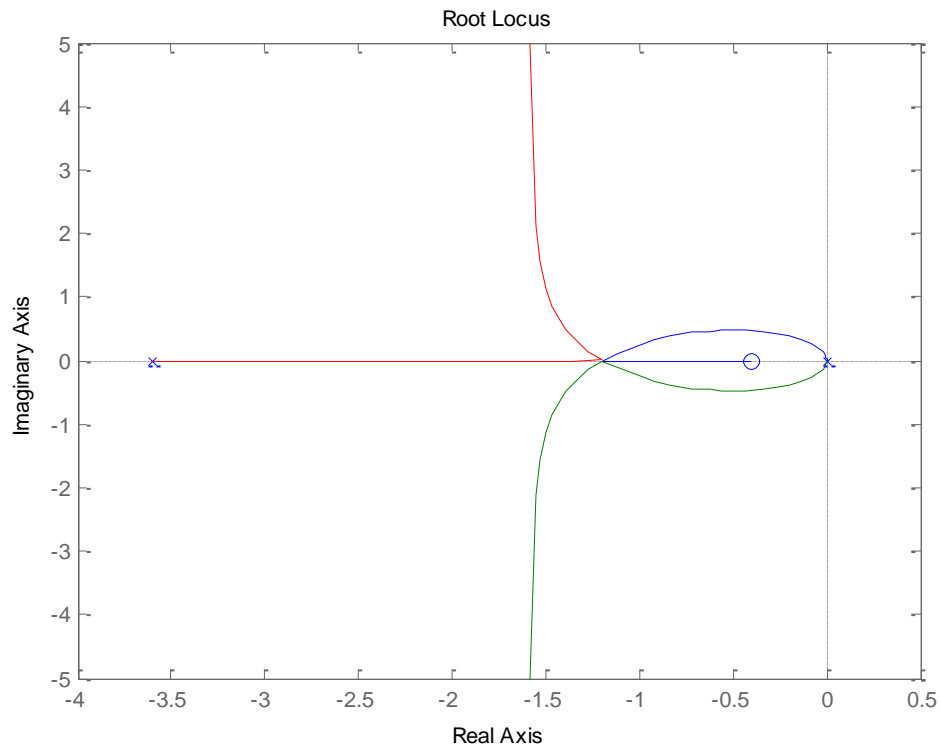
breakaway points: $\frac{1}{s^2} + \frac{1}{s+3.6} = \frac{1}{s+0.4}$

$$\Rightarrow s^3 + 2.4s^2 + 1.44s = 0 \rightarrow s = 0, -1.2$$

MATLAB code:

```
s = tf('s')
num_G=(s+0.4);
den_G=s^2*(s+3.6);
G=num_G/den_G;
figure(1);
rlocus(G)
```

Root locus diagram:



9-31 (a) $P(s) = s(s+12.5)(s+1)$ $Q(s) = 83.333$

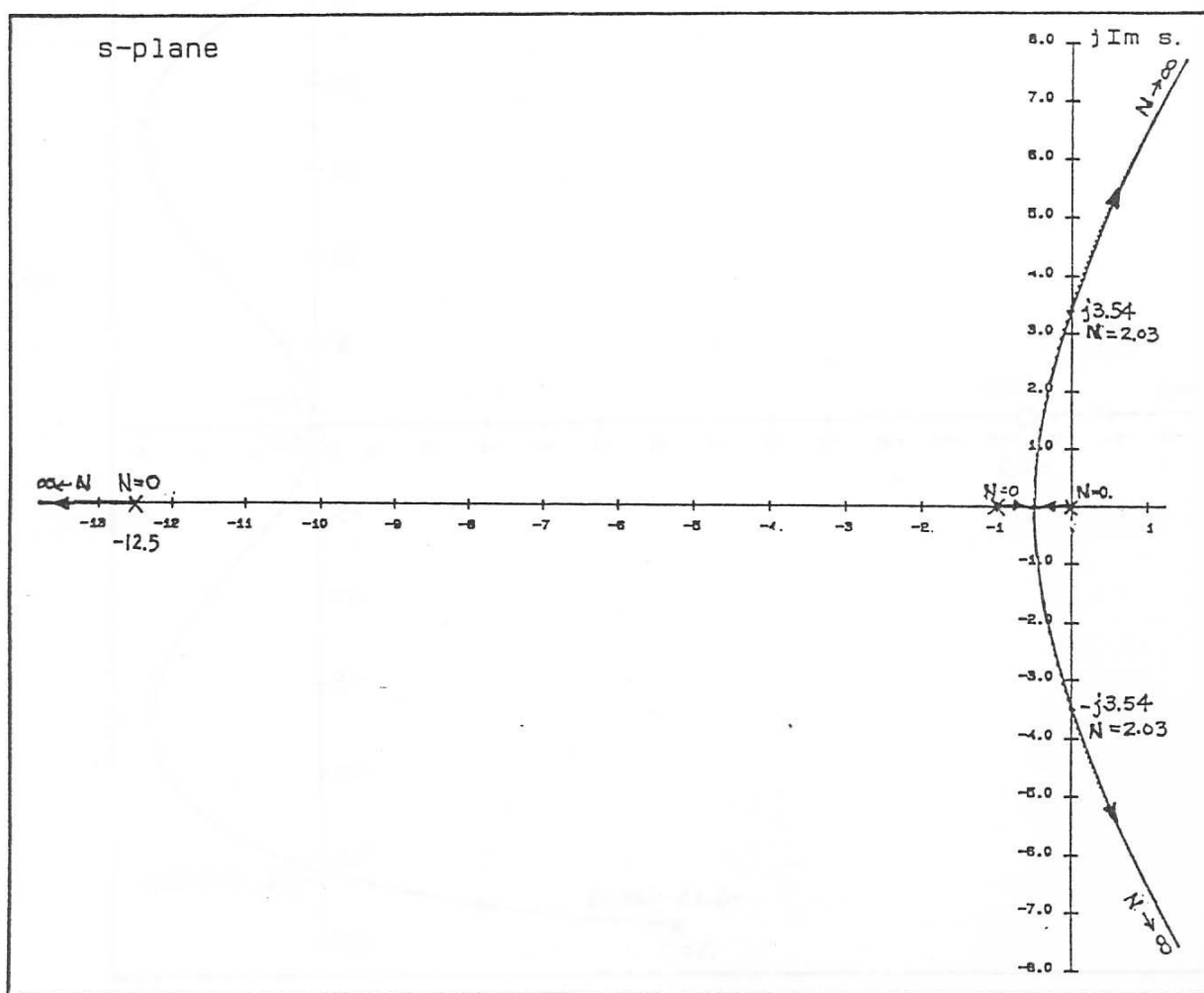
Asymptotes: $N > 0$: 60° , 180° , 300°

Intersect of Asymptotes:

$$\sigma_1 = \frac{0 - 12.5 - 1}{3} = -4.5$$

Breakaway-point Equation: $3s^2 + 27s - 12.5 = 0$

Breakaway Point: (RL) -0.4896

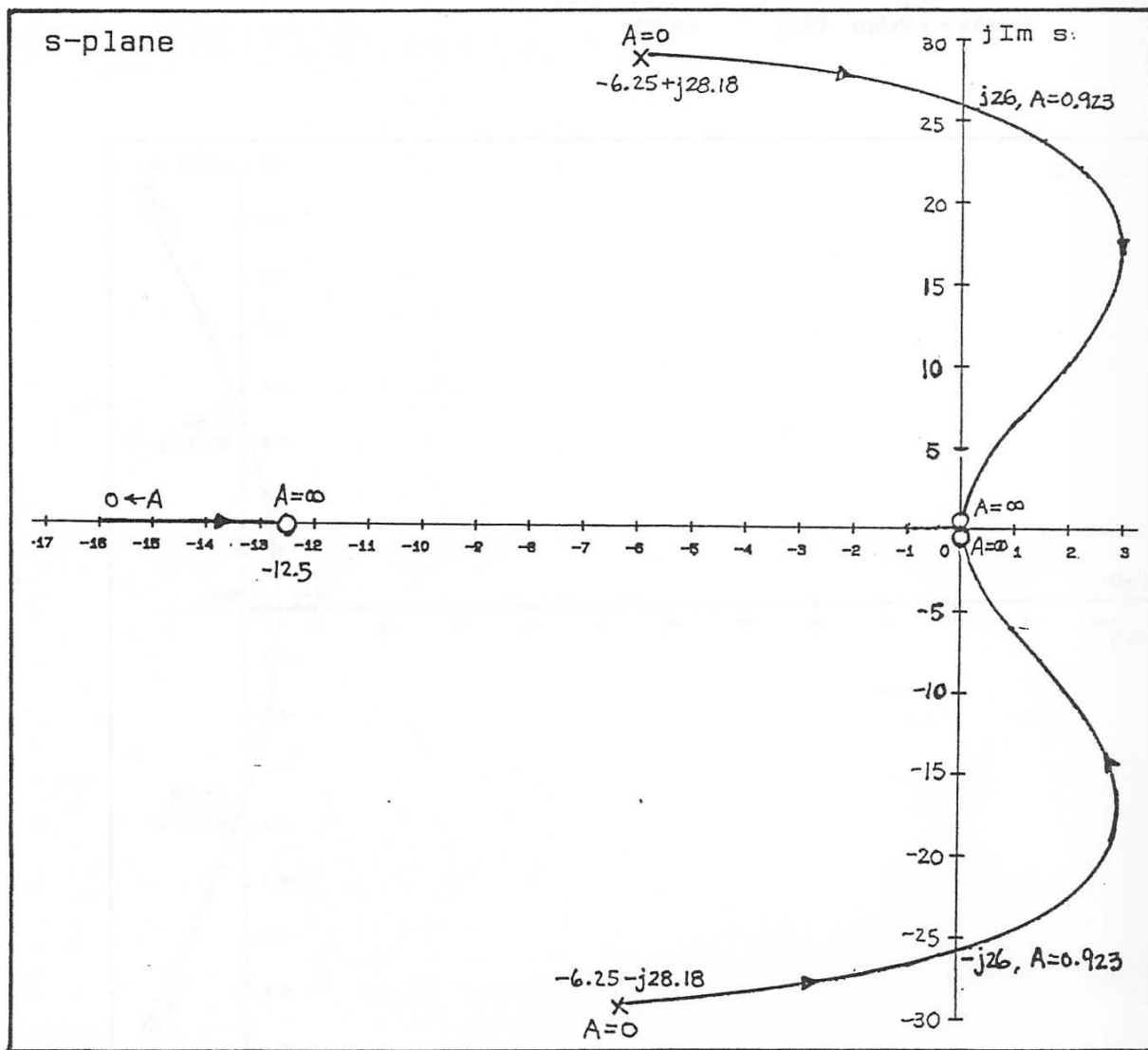


9-31 (b) $P(s) = s^2 + 12.5s + 833.333$ $Q(s) = 0.02s^2(s + 12.5)$

$$A > 0: \quad 180^\circ$$

$$\text{Breakaway-point Equation:} \quad 0.02s^4 + 0.5s^3 + 53.125s^2 + 416.67s = 0$$

$$\text{Breakaway Points:} \quad (RL) 0$$



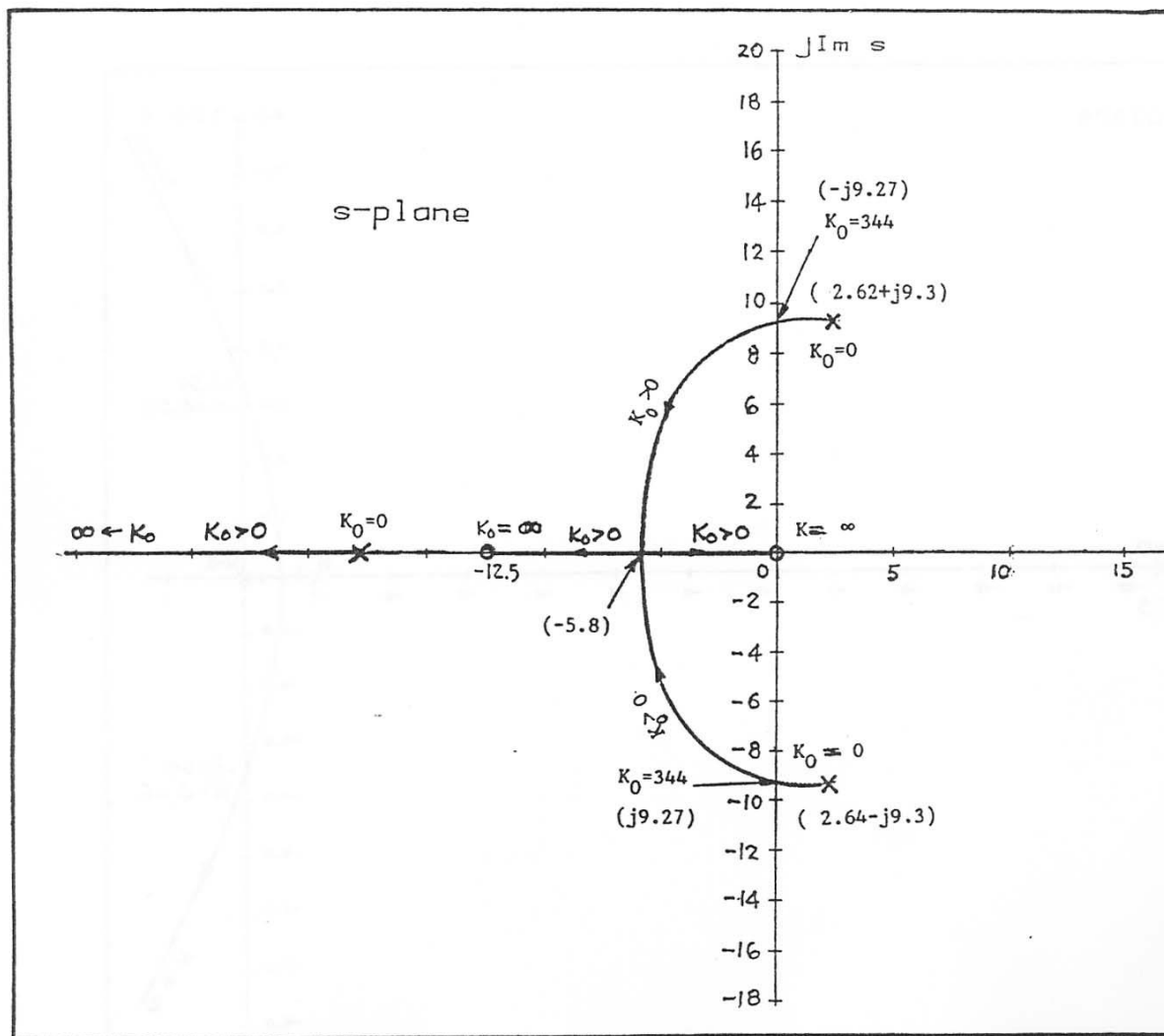
$$9-31 \text{ c) } P(s) = s^3 + 12.5s^2 + 1666.67 = (s + 17.78)(s - 2.64 + j9.3)(s - 2.64 - j9.3)$$

$$Q(s) = 0.02s(s + 12.5)$$

Asymptotes: $K_o > 0$: 180°

Breakaway-point Equation: $0.02s^4 + 0.5s^3 + 3.125s^2 - 66.67s - 416.67 = 0$

Breakaway Point: (RL) -5.797



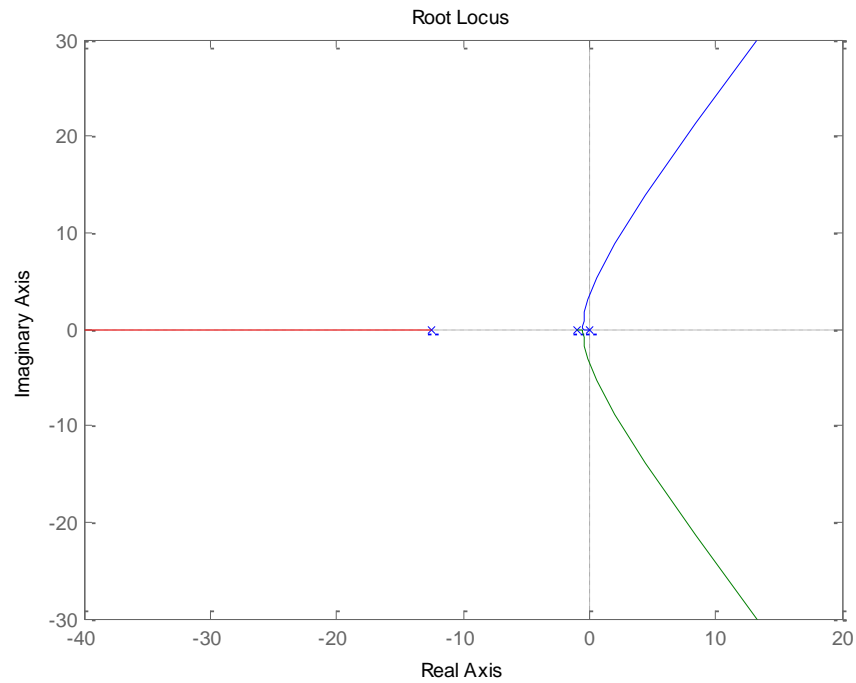
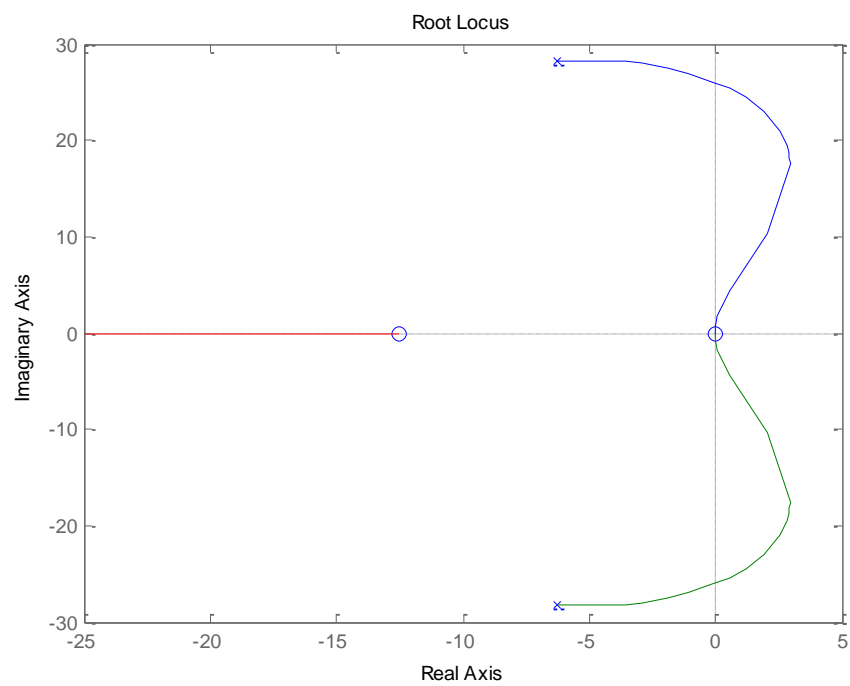
9-32) MATLAB code:

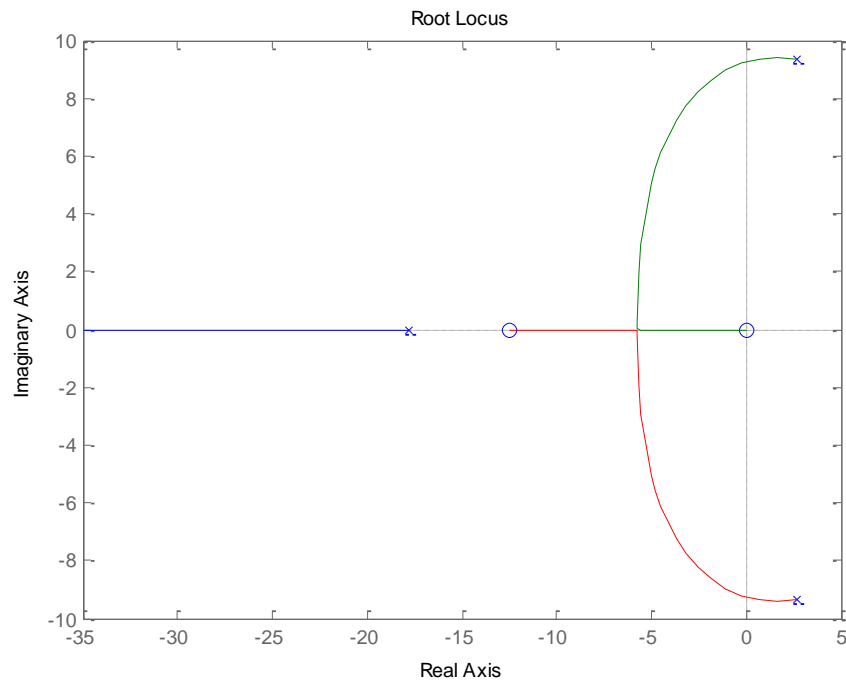
```
s = tf('s')
%a)
A=50;
K0=50;
num_G_a = 250;
den_G_a = 0.06*s*(s + 12.5)*(A*s+K0);
G_a = num_G_a/den_G_a;
figure(1);
rlocus(G_a)

%b)
N=10;
K0=50;
num_G_b = 0.06*s*(s+12.5)*s
den_G_b = K0*(0.06*s*(s+12.5))+250*N;
G_b = num_G_b/den_G_b;
figure(2);
rlocus(G_b)

%c)
A=50;
N=20;
num_G_c = 0.06*s*(s+12.5);
den_G_c = 0.06*s*(s+12.5)*A*s+250*N;
G_c = num_G_c/den_G_c;
figure(3);
rlocus(G_c)
```

Root locus diagram, part (a):

**Root locus diagram, part (b):****Root locus diagram, part (c):**



9-33) (a) $A = K_o = 100$: $P(s) = s(s+12.5)(s+1)$ $Q(s) = 41.67$

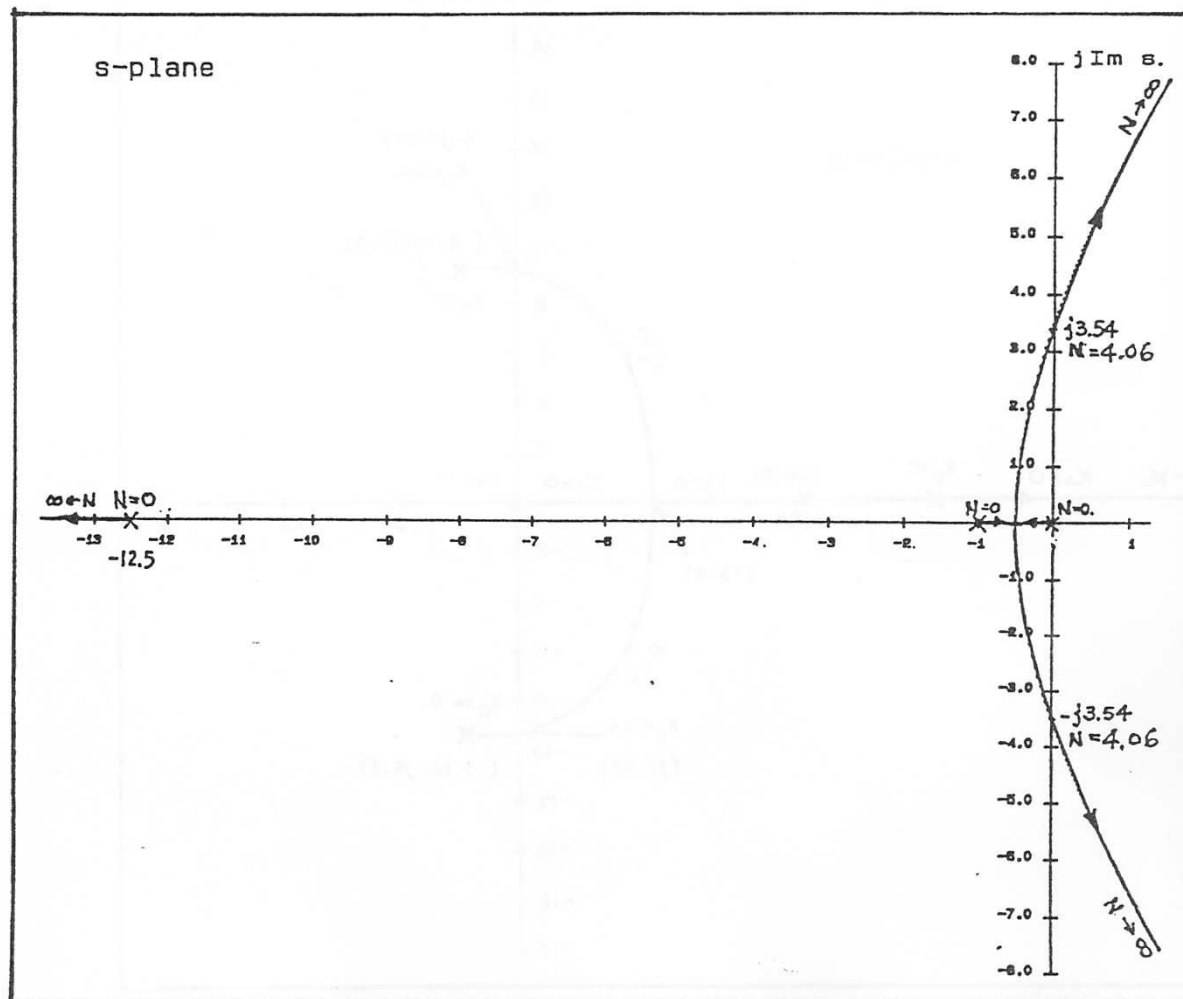
Asymptotes: $N > 0$: 60° 180° 300°

Intersect of Asymptotes:

$$\sigma_1 = \frac{0 - 1 - 12.5}{3} = -4.5$$

Breakaway-point Equation: $3s^2 + 27s + 12.5 = 0$

Breakaway Points: (RL) -0.4896



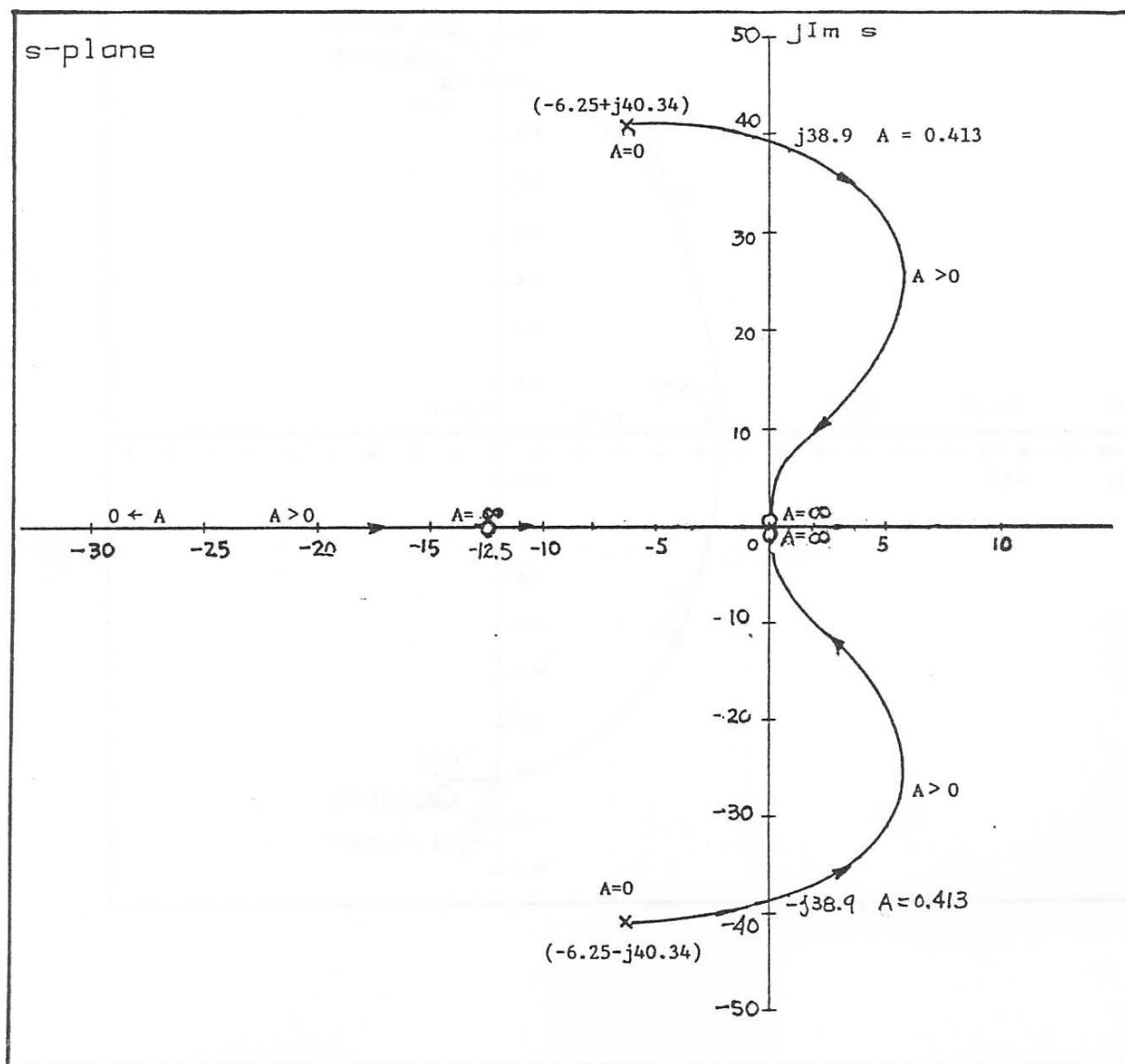
9-33 (b) $P(s) = s^2 + 12.5 + 1666.67 = (s + 6.25 + j40.34)(s + 6.25 - j40.34)$

$$Q(s) = 0.02s^2(s + 12.5)$$

Asymptotes: $A > 0$: 180°

Breakaway-point Equation: $0.02s^4 + 0.5s^3 + 103.13s^2 + 833.33s = 0$

Breakaway Points: (RL) 0



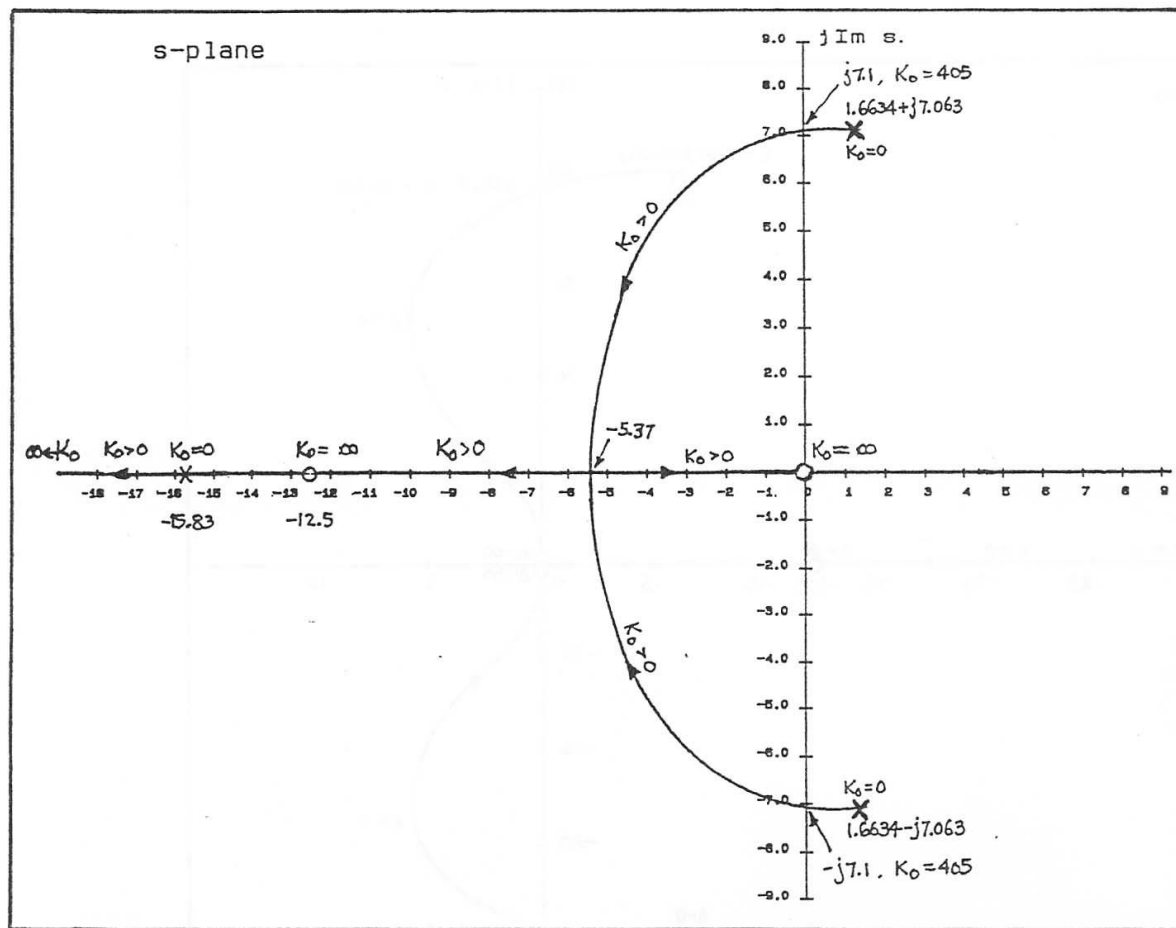
9-33 (c) $P(s) = s^3 + 12.5s^2 + 833.33 = (s + 15.83)(s - 1.663 + j7.063)(s - 1.663 - j7.063)$

$$Q(s) = 0.01s(s + 12.5)$$

Asymptotes: $K_o > 0$: 180°

Breakaway-point Equation: $0.01s^4 + 0.15s^3 + 1.5625s^2 - 16.67s - 104.17 = 0$

Breakaway Point: (RL) -5.37



9-34) MATLAB code:

```
s = tf('s')
%a)
A=100;
K0=100;
num_G_a = 250;
den_G_a = 0.06*s*(s + 12.5)*(A*s+K0);
G_a = num_G_a/den_G_a;
figure(1);
rlocus(G_a)
```

```
%b)
N=20;
K0=50;
```

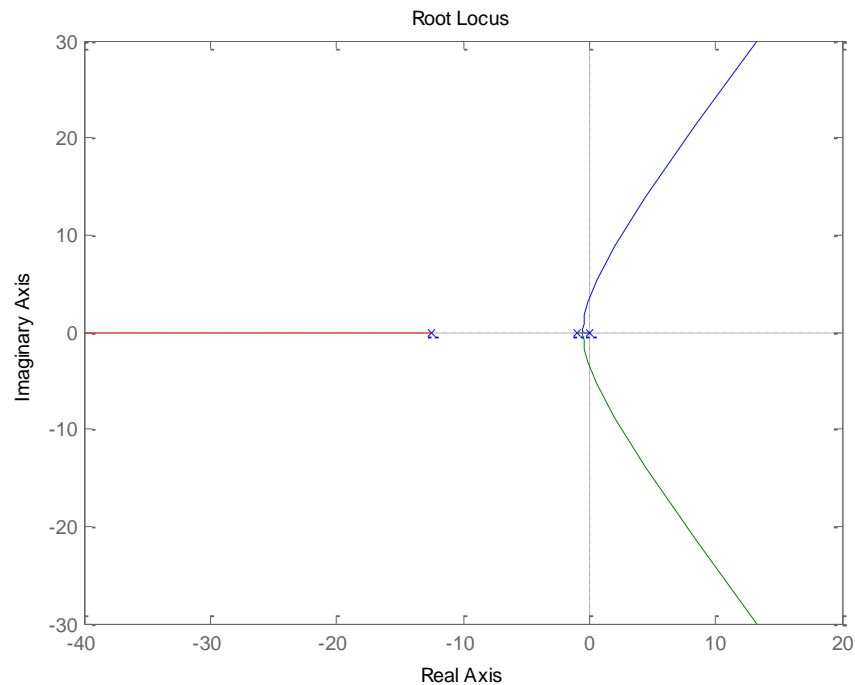
```

num_G_b = 0.06*s*(s+12.5)*s
den_G_b = K0*(0.06*s*(s+12.5))+250*N;
G_b = num_G_b/den_G_b;
figure(2);
rlocus(G_b)

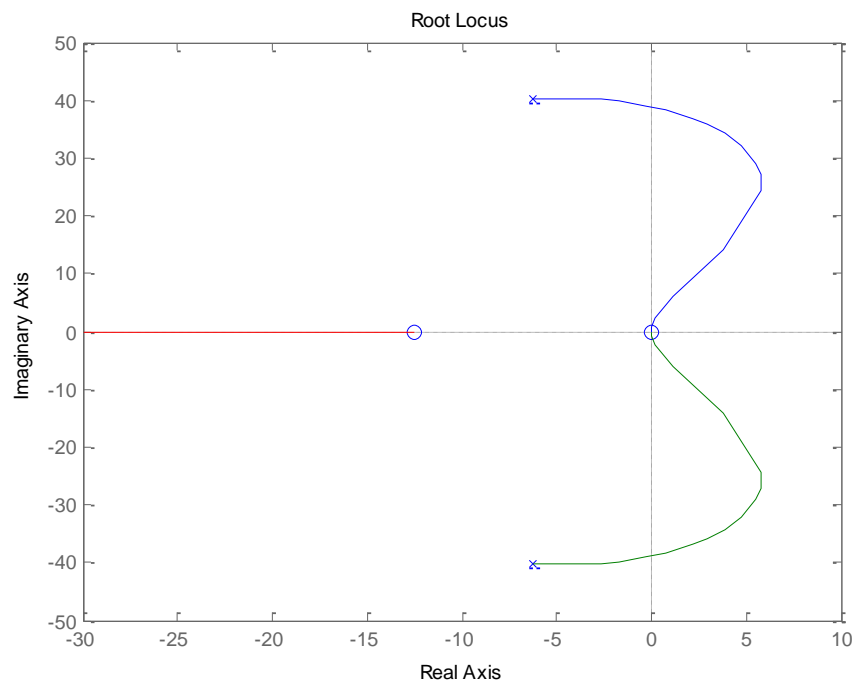
%c)
A=100;
N=20;
num_G_c = 0.06*s*(s+12.5);
den_G_c = 0.06*s*(s+12.5)*A*s+250*N;
G_c = num_G_c/den_G_c;
figure(3);
rlocus(G_c)

```

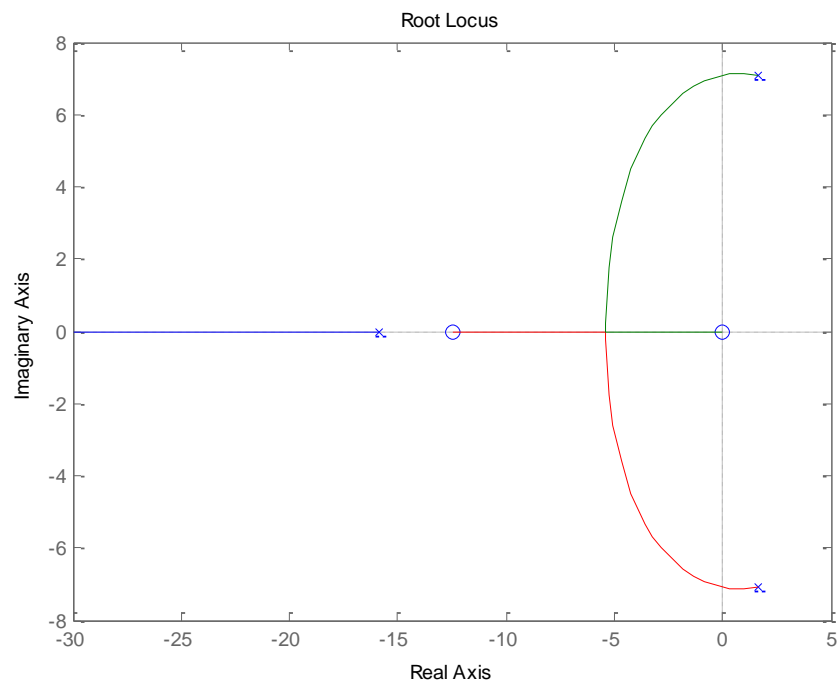
Root locus diagram, part (a):



Root locus diagram, part (b):



Root locus diagram, part (c):



9-35) a) zeros: $s = -2$, poles: $s = -2j, +2j, -5$

$$\text{Angle of asymptotes: } \theta_l = \frac{2l+1}{4-2} \times 180 = 90, 270$$

$$\sigma = -3$$

$$\text{Breakaway points: } \frac{1}{s^2+4} + \frac{1}{(s+5)^2} = \frac{1}{(s+2)^2}$$

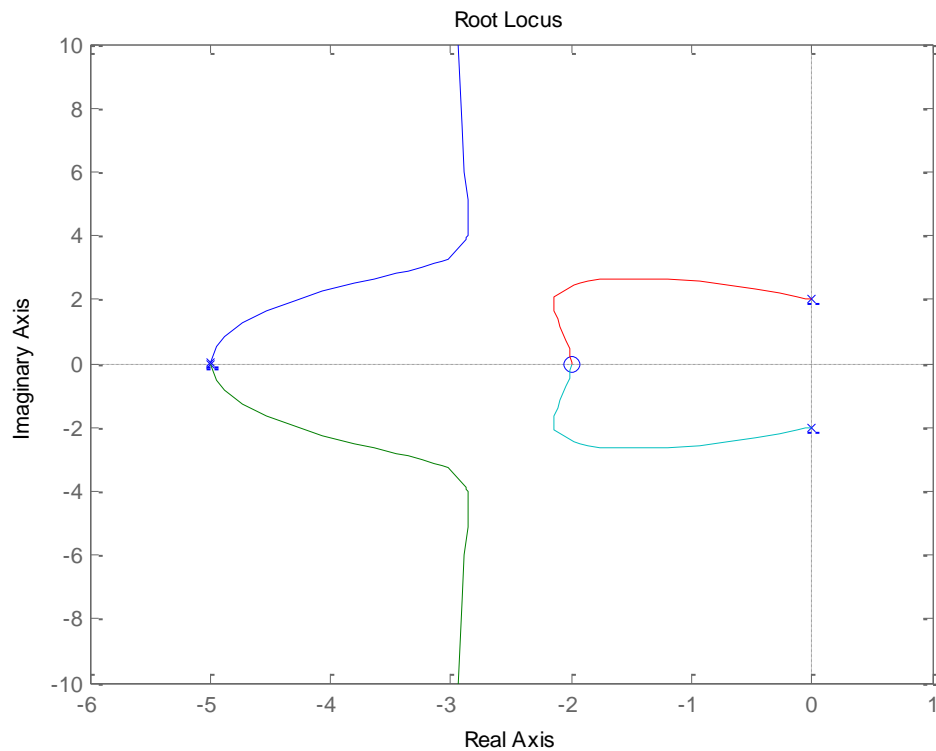
$$\Rightarrow s^2 + 6s + 25 = 0 \rightarrow s = -1.5 + 2j, -1.5 - 2j$$

b) There is no closed loop pole in the right half s-plane; therefore the system is stable for all $K > 0$

c) MATLAB code:

```
num_G=25*(s+2)^2;
den_G=(s^2+4)*(s+5)^2;
G_a=num_G/den_G;
figure(1);
rlocus(G_a)
```

Root locus diagram:



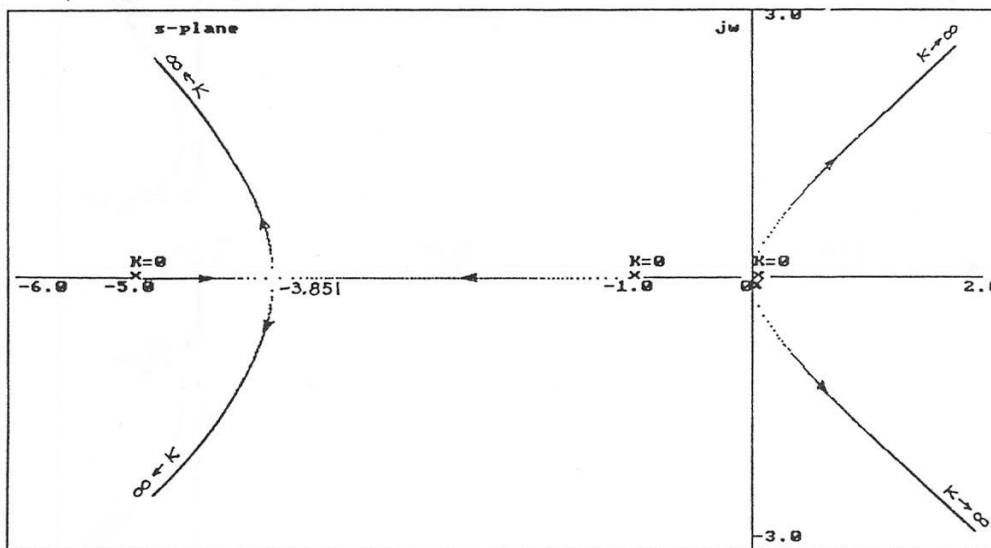
9-36) (a) $P(s) = s^2(s+1)(s+5)$ $Q(s) = 1$

Asymptotes: $K > 0$: 45° , 135° , 225° , 315°

Intersect of Asymptotes:

$$\sigma_1 = \frac{0+0-1-5}{4} = -1.5$$

Breakaway-point Equation: $4s^3 + 18s^2 + 10s = 0$ **Breakaway point: (RL)** $0, -3.851$



(b) $P(s) = s^2(s+1)(s+5)$ $Q(s) = 5s+1$

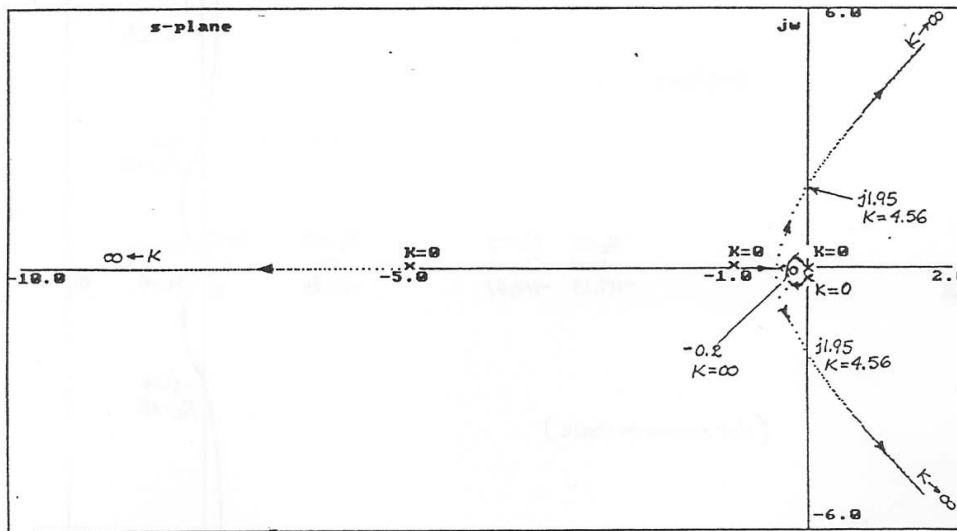
Asymptotes: $K > 0$: 60° , 180° , 300°

Intersect of Asymptotes:

$$\sigma_1 = \frac{0+0-1-5-(-0.2)}{4-1} = -\frac{5.8}{3} = -1.93$$

Breakaway-point Equation: $15s^4 + 64s^3 + 43s^2 + 10s = 0$

Breakaway Points: (RL) -3.5026



9-37)

MATLAB code (9-37):

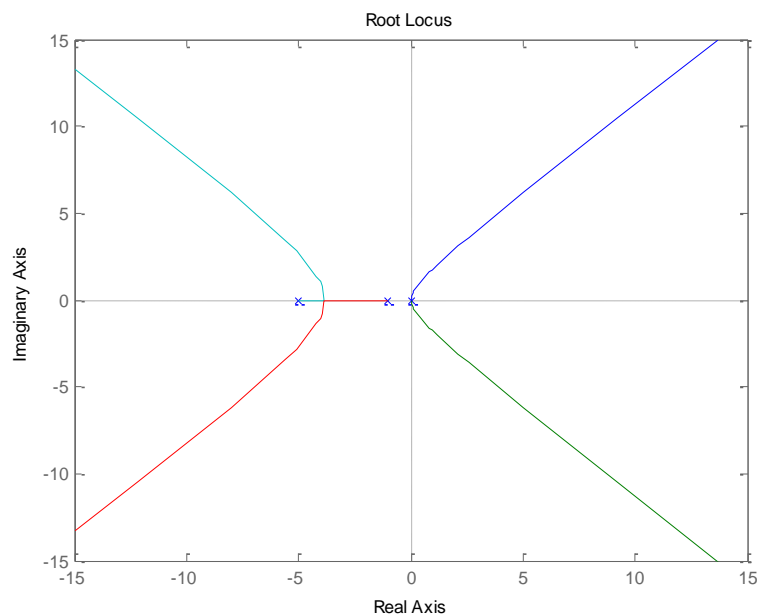
```

s = tf('s')
%a)
num_GH_a= 1;
den_GH_a=s^2*(s+1)*(s+
5);
GH_a=num_GH_a/den_GH_a
;
figure(1);
rlocus(GH_a)

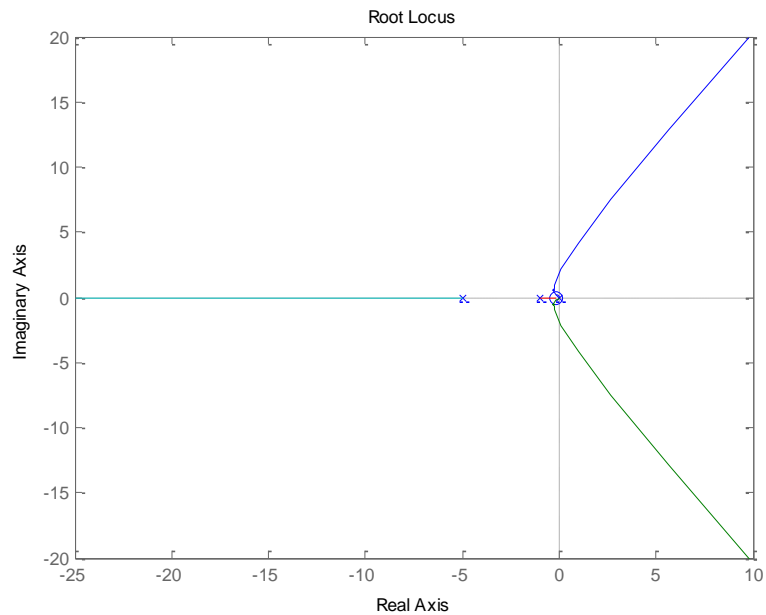
%b)
num_GH_b= (5*s+1);
den_GH_b=s^2*(s+1)*(s+
5);
GH_b=num_GH_b/den_GH_b
;
figure(2);
rlocus(GH_b)

```

Root locus diagram, part (a):



Root locus diagram, part (b):



9-38) a) e^{-s} can be approximated by (easy way to verify is to compare both funtions' Taylor series expansions)

$$e^{-s} \approx \frac{2-s}{2+s}$$

Therefore:

$$G(s) = -\frac{K(s-2)}{(s+1)(s+2)}$$

Zeros: $s = 2$ and poles: $s = -1, -2$

Angle of asymptotes : $\theta_i = (2i + 1)180 = 180$

$$\sigma_c = -(1 + 2 - 2) = -1$$

Breakaway points: $\frac{1}{s+1} + \frac{1}{s+2} = \frac{1}{2-s}$

Which means: $s^2 + 4s = 0 \rightarrow s = 0, s = \pm 2$

b) $s + 1 + K \frac{2-s}{s+2} = 0 \rightarrow s^2 + 3s + 2 - Ks + 2K = 0$

s^2	1	$2+2k$
s	$3-k$	0

$$s^0 \quad | \quad (3-k)(2+2k)$$

As a result:

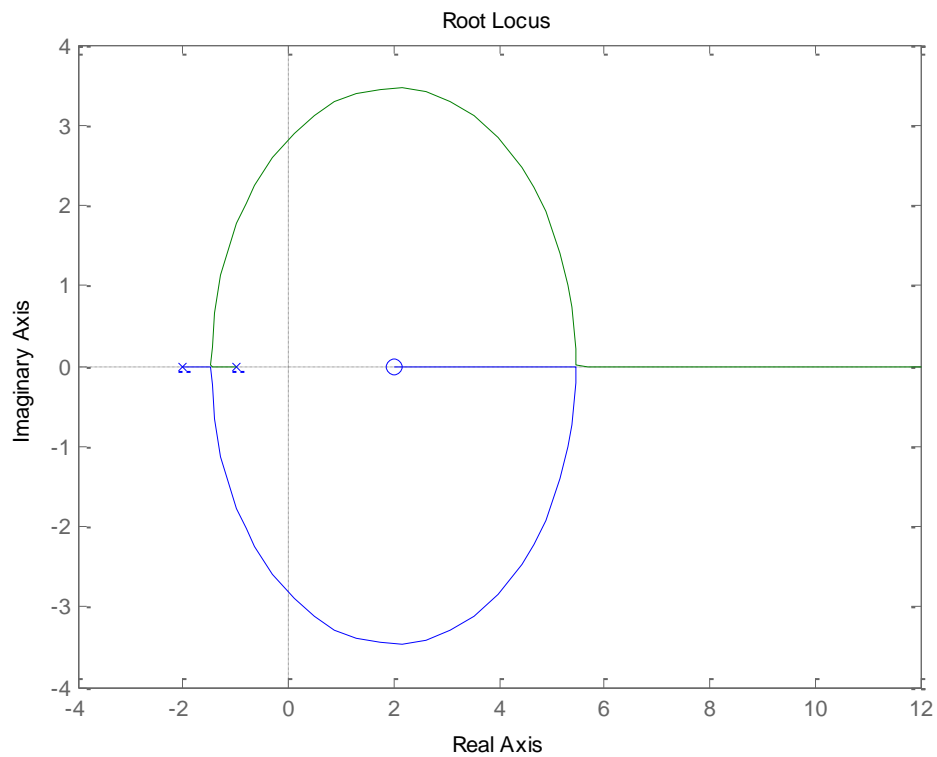
$$\begin{cases} 3 - K > 0 \rightarrow K < 3 \\ (3 - K)(2 + 2K) > 0 \rightarrow 2 + 2k > 0 \rightarrow K > -1 \end{cases}$$

Since K must be positive, the range of stability is then $0 < k < 3$

c) In this problem, e^{-Ts} term is a time delay. Therefore, MATLAB PADE command is used for pade approximation, where brings e^{-Ts} term to the polynomial form of degree N.

```
s = tf('s')
T=1
N=1;
num_GH= pade(exp(-1*T*s),N);
den_GH=(s+1);
GH=num_GH/den_GH;
figure(5);
rlocus(GH)
```

Root locus diagram:



9-39)

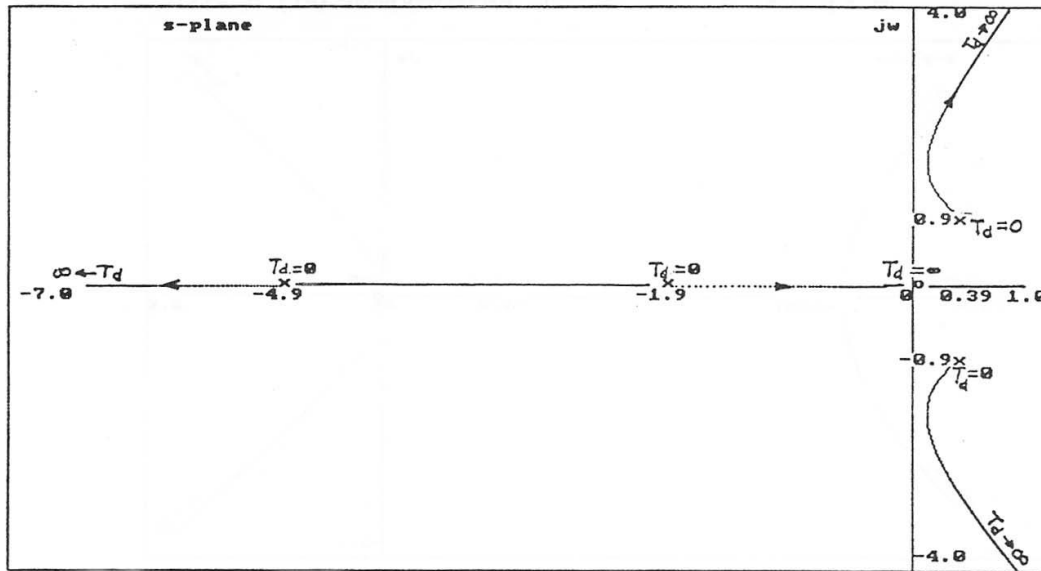
(a) $P(s) = s^2(s+1)(s+5)+10 = (s+4.893)(s+1.896)(s-0.394+j0.96)(s-0.394-j0.96)$

$$Q(s) = 10s$$

Asymptotes: $T_d > 0$: 60° , 180° , 300°

Intersection of Asymptotes:
$$\sigma_1 = \frac{-4.893 - 1.896 + 0.3944 + 0.3944}{4 - 1} = -2$$

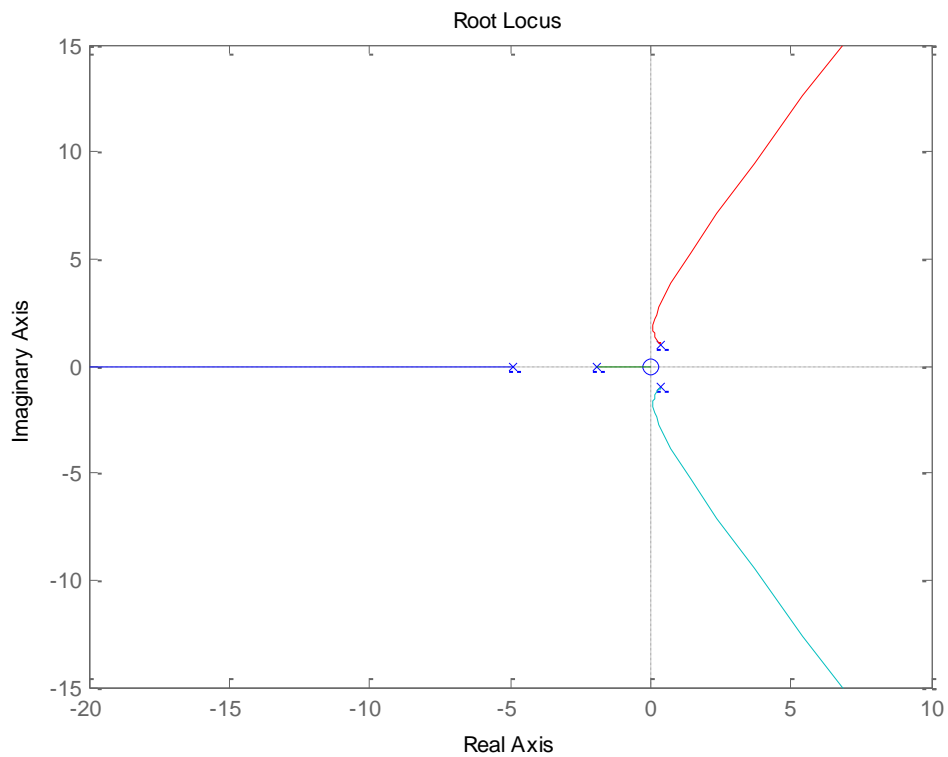
There are no breakaway points on the RL.



(b) MATLAB code:

```
s = tf('s')
num_GH= 10*s;
den_GH=s^2*(s+1)*(s+5)+10;
GH=num_GH/den_GH;
figure(1);
rlocus(GH)
```

Root locus diagram:



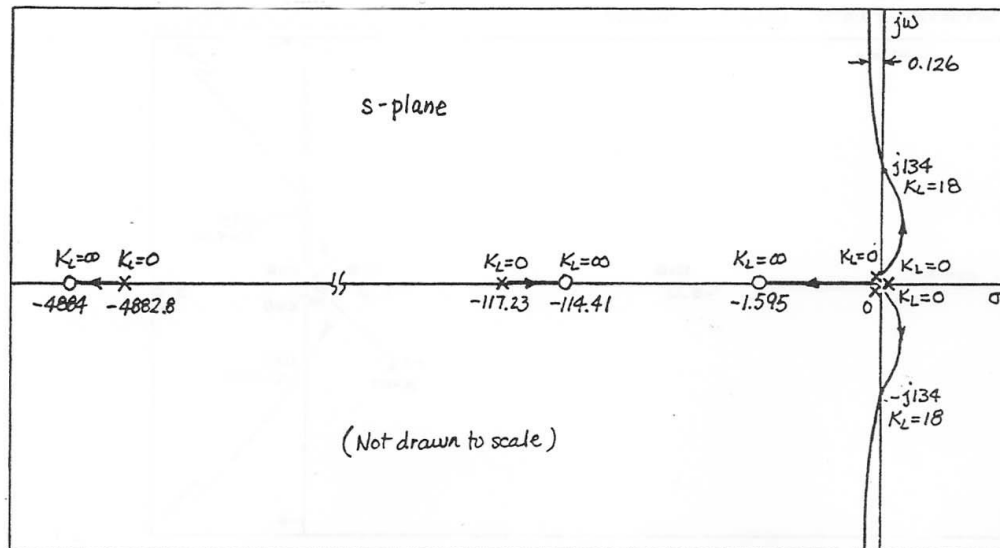
9-40) (a) $\kappa = 1$: $P(s) = s^3(s+117.23)(s+4882.8)$ $Q(s) = 1010(s+1.5948)(s+114.41)(s+4884)$

Asymptotes: $K_L > 0$: $90^\circ, 270^\circ$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-117.23 - 4882.8 + 1.5948 + 114.41 + 4884}{5 - 3} = -0.126$$

Breakaway Point: (RL) 0



9-40 (b) $K = 1000$: $P(s) = s^3(s + 117.23)(s + 4882.8)$

$$Q(s) = 1010(s^3 + 5000s^2 + 5.6673 \times 10^5 s + 891089110)$$

$$= 1010(s + 4921.6)(s + 39.18 + j423.7)(s + 39.18 - j423.7)$$

Asymptotes: $K_L > 0$: 90° , 270°

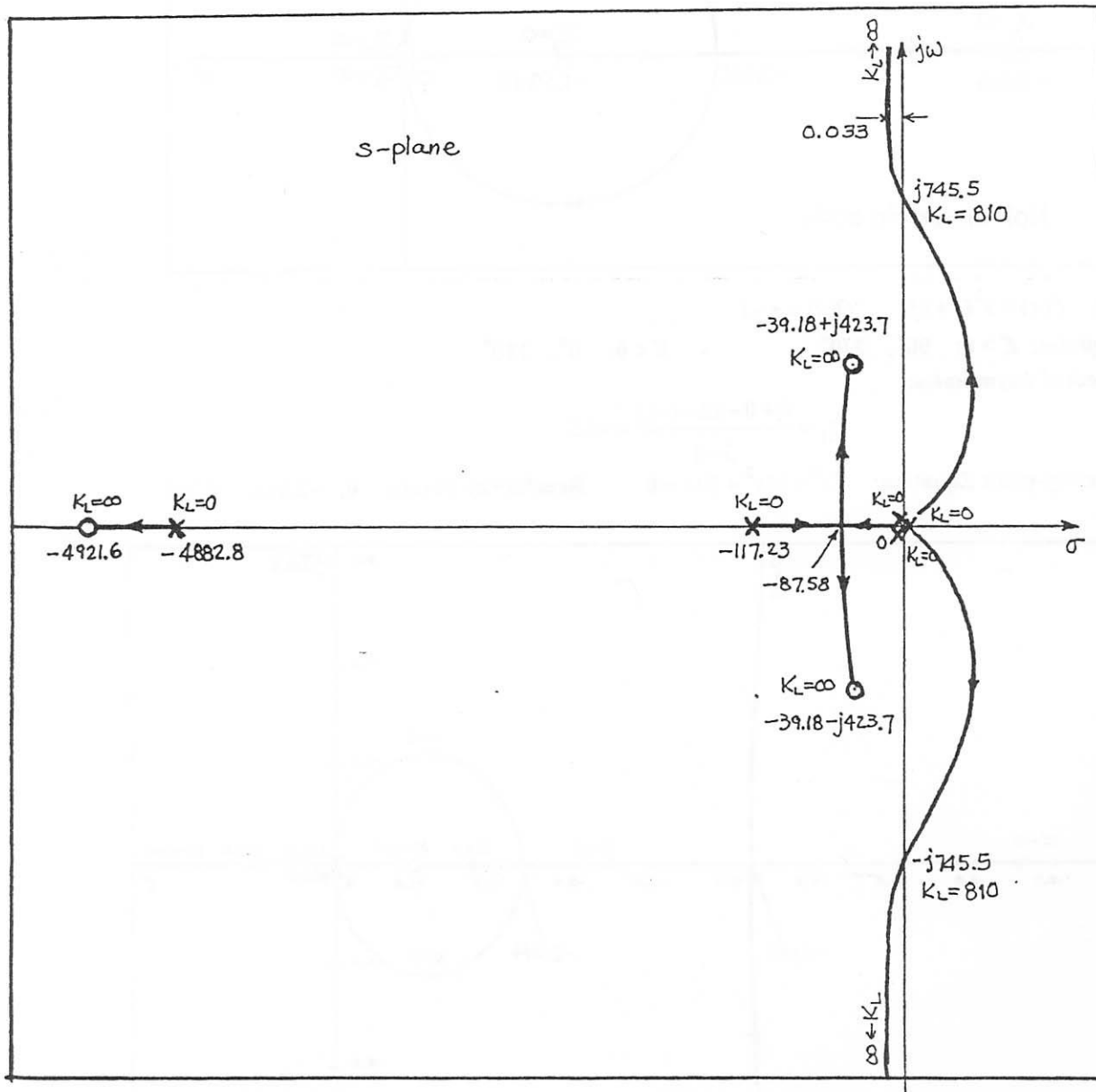
Intersect of Asymptotes:

$$\sigma_1 = \frac{-117.23 - 4882.8 + 4921.6 + 39.18 + 39.18}{5 - 3} = -0.033$$

Breakaway-point Equation:

$$2020s^7 + 2.02 \times 10^7 s^6 + 5.279 \times 10^{10} s^5 + 1.5977 \times 10^{13} s^4 + 1.8655 \times 10^{16} s^3 + 1.54455 \times 10^{18} s^2 = 0$$

Breakaway points: (RL) $0, -87.576$



9-41) MATLAB code:

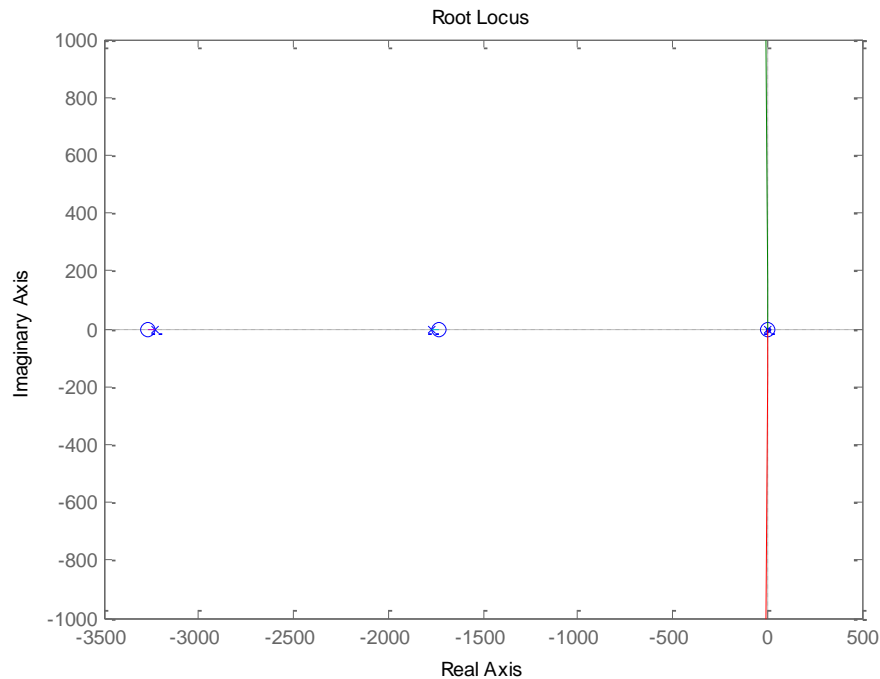
```
s = tf('s')
Ki=9;
Kb=0.636;
Ra=5;
La=.001;
Ks=1;
n=.1;
Jm=0.001;
Jl=0.001;
Bm=0;
%a)
```

```

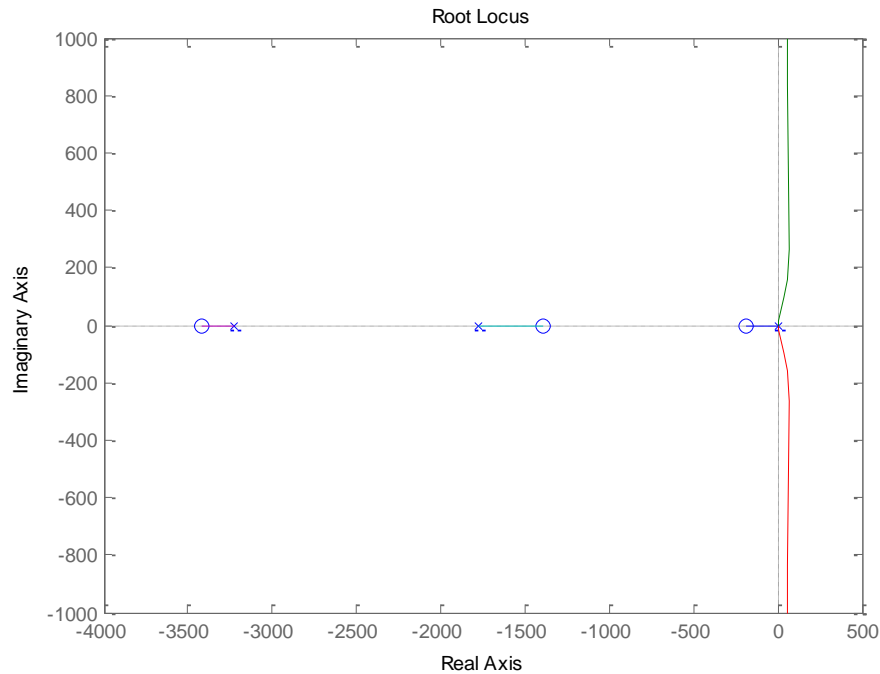
K=1;
num_G_a=( (n^2*La*Jl+La*Jm)*s^3+(n^2*Ra*Jl+Ra*Jm+Bm*La)*s^2+Ra*Bm*s+Ki
*Kb*s+n*Ks*K*Ki);
den_G_a=( (La*Jm*Jl)*s^5+(Jl*Ra*Jm+Jl*Bm*La)*s^4+(Ki*Kb*Jl+Ra*Bm*Jl)*s
^3);
G_a=num_G_a/den_G_a;
figure(1);
rlocus(G_a)
%b)
K=1000;
num_G_b=( (n^2*La*Jl+La*Jm)*s^3+(n^2*Ra*Jl+Ra*Jm+Bm*La)*s^2+Ra*Bm*s+Ki
*Kb*s+n*Ks*K*Ki);
den_G_b=( (La*Jm*Jl)*s^5+(Jl*Ra*Jm+Jl*Bm*La)*s^4+(Ki*Kb*Jl+Ra*Bm*Jl)*s
^3);
G_b=num_G_b/den_G_b;
figure(2);
rlocus(G_b)

```

Root locus diagram, part (a):



Root locus diagram, part (b):



9-42

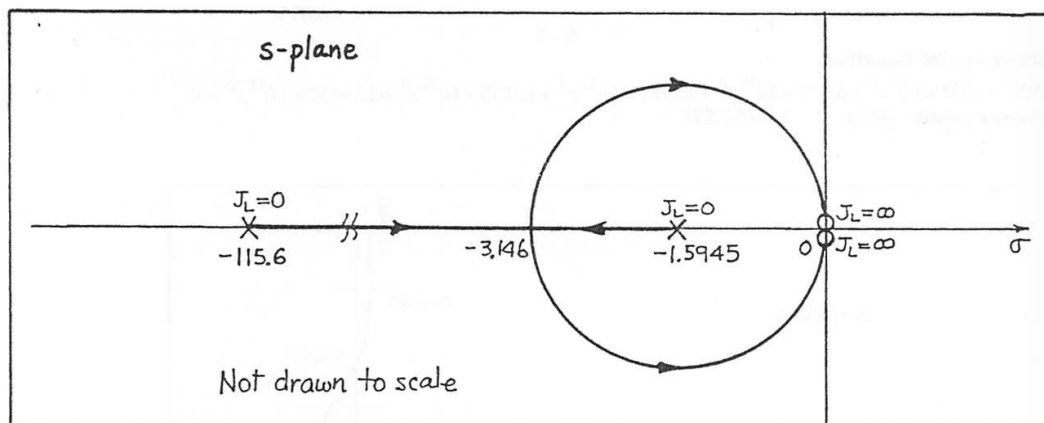
(a) Characteristic Equation: $s^3 + 5000s^2 + 572,400s + 900,000 + J_L(10s^3 + 50,000s^2) = 0$

$$P(s) = s^3 + 5000s^2 + 572,400s + 900,000 = (s + 1.5945)(s + 115.6)(s + 4882.8) \quad Q(s) = 10s^2(s + 5000)$$

Since the pole at -5000 is very close to the zero at -4882.8 , $P(s)$ and $Q(s)$ can be approximated as:

$$P(s) \cong (s + 1.5945)(s + 115.6) \quad Q(s) \cong 10.24s^2$$

Breakaway-point Equation: $1200s^2 + 3775s = 0$ Breakaway Points: (RL): $0, -3.146$



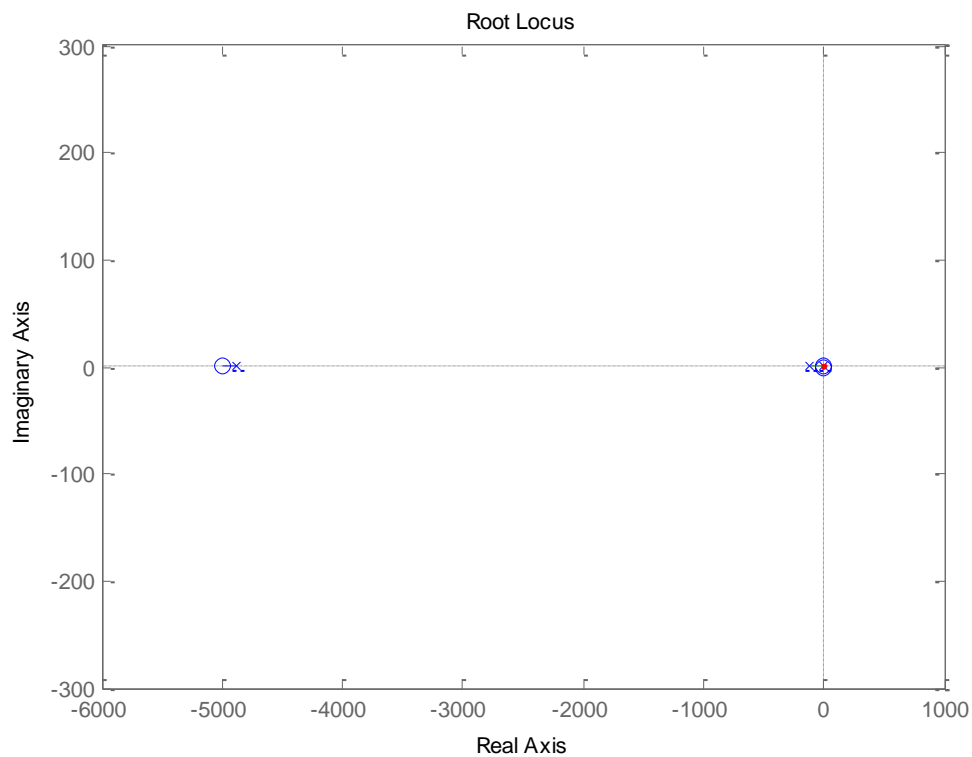
(b) MATLAB code:

```

s = tf('s')
K=1;
Jm=0.001;
La=0.001;
n=0.1;
Ra=5;
Ki=9;
Bm=0;
Kb=0.0636;
Ks=1;

num_G_a = (n^2*La*s^3+n^2*Ra*s^2);
den_G_a = (La*Jm*s^3+ (Ra*Jm+Bm*La) *s^2+ (Ra*Bm+Ki*Kb) *s+n*Ki*Ks*K);
G_a = num_G_a/den_G_a;
figure(1);
rlocus(G_a)

```

Root locus diagram:

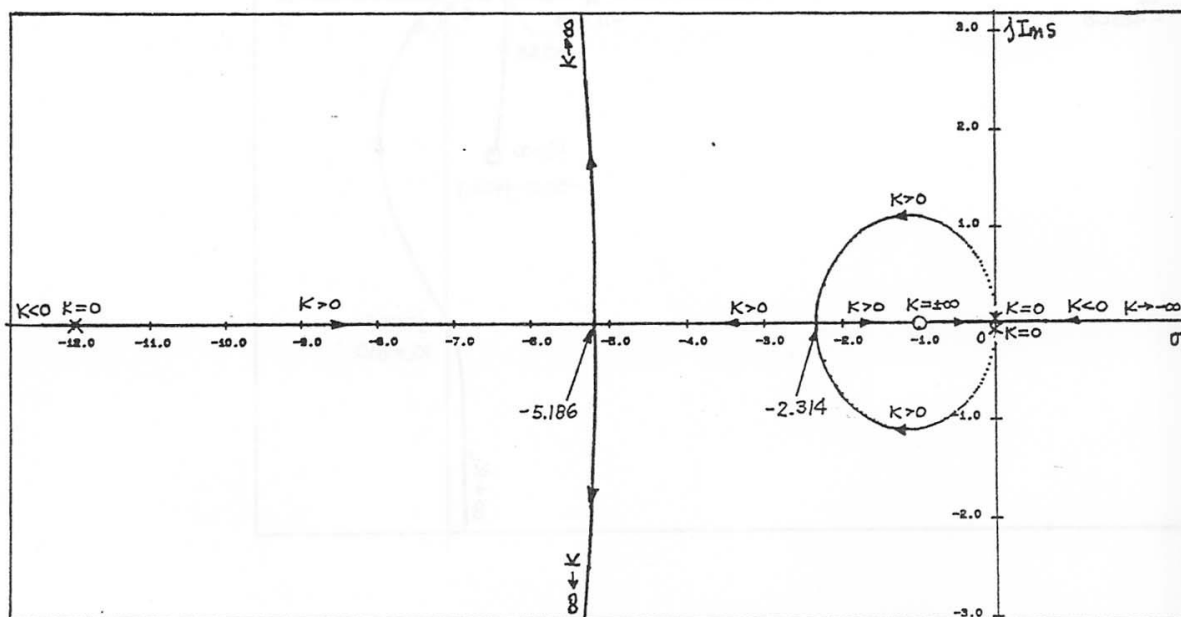
9-43) (a) $\alpha = 12$: $P(s) = s^2(s+12)$ $Q(s) = s+1$

Asymptotes: $K > 0$: $90^\circ, 270^\circ$ $K < 0$: $0^\circ, 180^\circ$

Intersect of Asymptotes:

$$\sigma_1 = \frac{0+0-12-(-1)}{3-1} = -5.5$$

Breakaway-point Equation: $2s^3 + 15s^2 + 24s = 0$ **Breakaway Points:** $0, -2.314, -5.186$



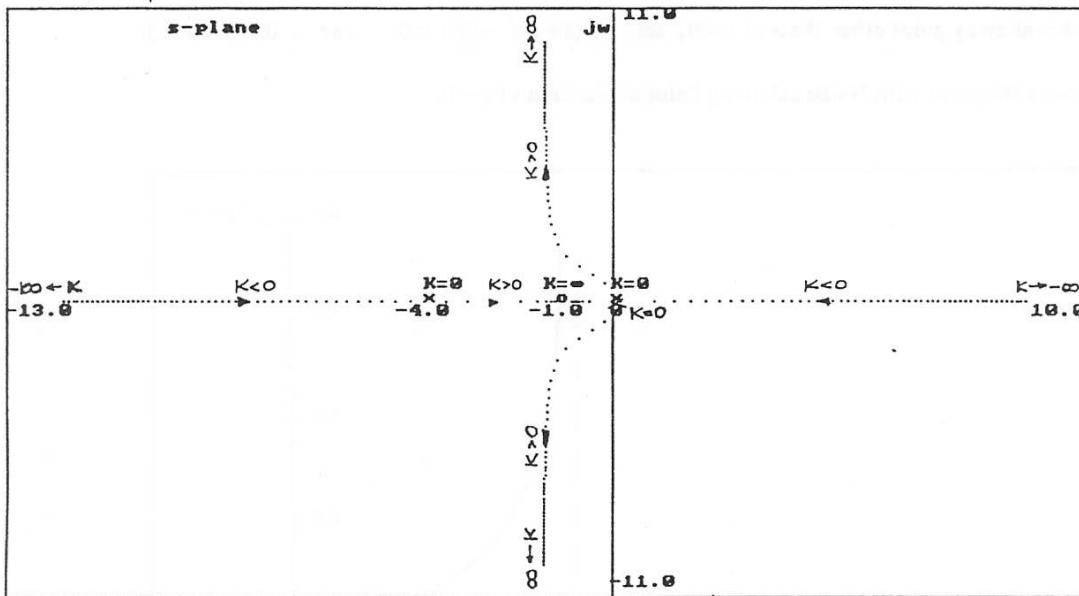
9-43 (b) $\alpha = 4$: $P(s) = s^2(s+4)$ $Q(s) = s+1$

Asymptotes: $K > 0$: $90^\circ, 270^\circ$ $K < 0$: $0^\circ, 180^\circ$

Intersect of Asymptotes:

$$\sigma_1 = \frac{0+0-4-(-1)}{3-1} = -1.5$$

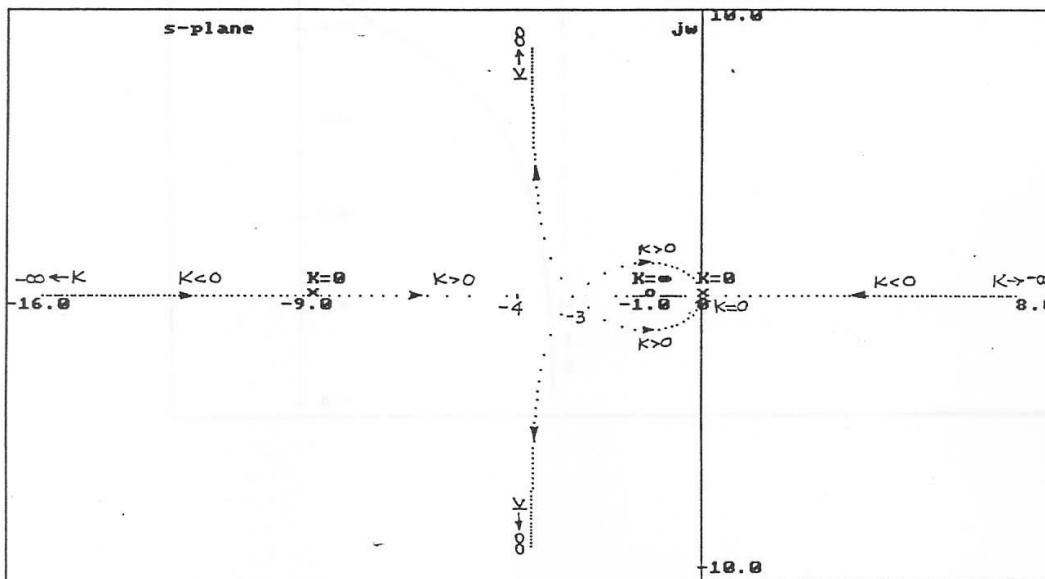
Breakaway-point Equation: $2s^3 + 7s^2 + 8s = 0$ **Breakaway Points:** $K > 0$ 0 . None for $K < 0$.



(c) Breakaway-point Equation: $2s^2 + (\alpha + 3)s + 2s = 0$ Solutions: $s = -\frac{\alpha + 3}{4} \pm \frac{\sqrt{(\alpha + 3)^2 - 16\alpha}}{4}$, $s = 0$

For one nonzero breakaway point, the quantity under the square-root sign must equal zero.

Thus, $\alpha^2 - 10\alpha + 9 = 0$, $\alpha = 1$ or $\alpha = 9$. The answer is $\alpha = 9$. The $\alpha = 1$ solution represents pole-zero cancellation in the equivalent $G(s)$. When $\alpha = 9$, the nonzero breakaway point is at $s = -3$. $\sigma_1 = -4$.



9-44)

For part (c), after finding the expression for:

$$\frac{dk}{ds} = \frac{-3 - \alpha \pm \sqrt{(\alpha - 1)(\alpha - 9)}}{4},$$

there is one acceptable value of alpha that makes the square root zero ($\alpha = 9$). Zero square root means one answer to the breakaway point instead of 2 answers as a result of \pm sign. $\alpha = 1$ is not acceptable

since it results in $s = -1 @ \frac{dk}{ds} = 0$ and then $k = \frac{0}{0}$.

MATLAB code:

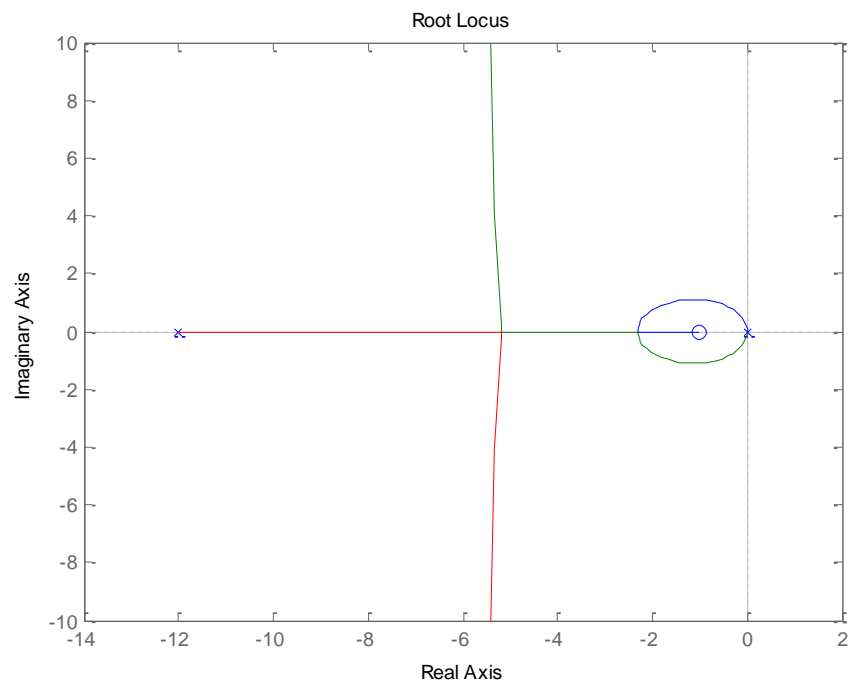
```
s = tf('s')

% (a)
alpha=12
num_GH= s+1;
den_GH=s^3+alpha*s^2;
GH=num_GH/den_GH;
figure(1);
rlocus(GH)

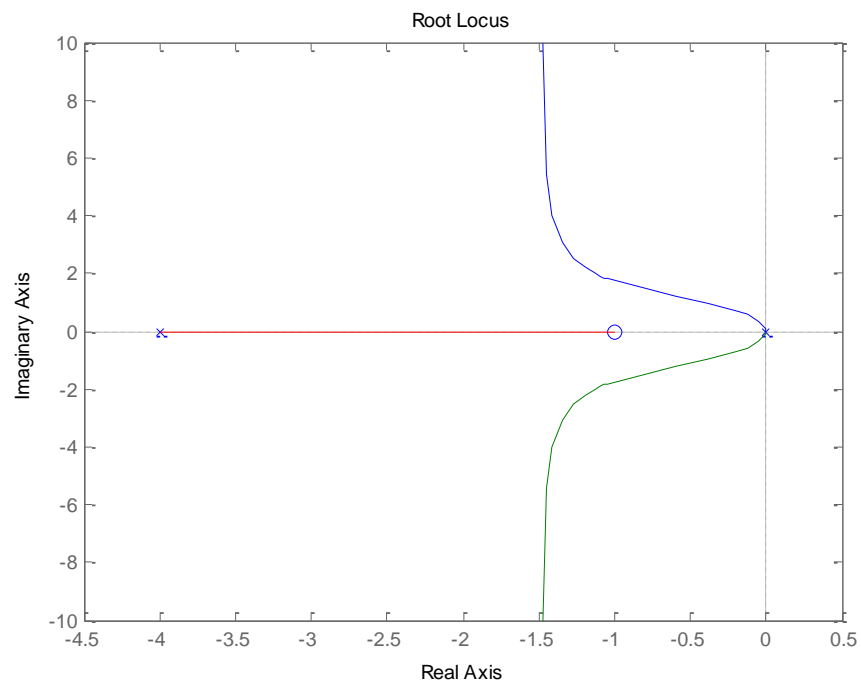
% (b)
alpha=4
num_GH= s+1;
den_GH=s^3+alpha*s^2;
GH=num_GH/den_GH;
figure(2);
rlocus(GH)

% (c)
alpha=9
num_GH= s+1;
den_GH=s^3+alpha*s^2;
GH=num_GH/den_GH;
figure(3);
rlocus(GH)
```

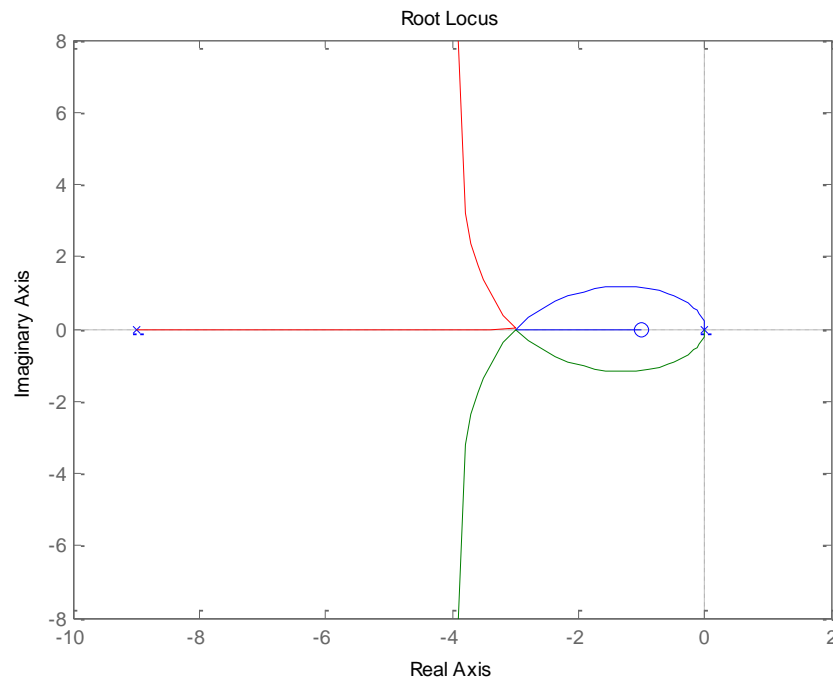
Root locus diagram, part (a):



Root locus diagram, part (b):



Root locus diagram, part (c): ($\alpha=9$ resulting in 1 breakaway point)



9-45) (a) $P(s) = s^2(s+3)$ $Q(s) = s + \alpha$

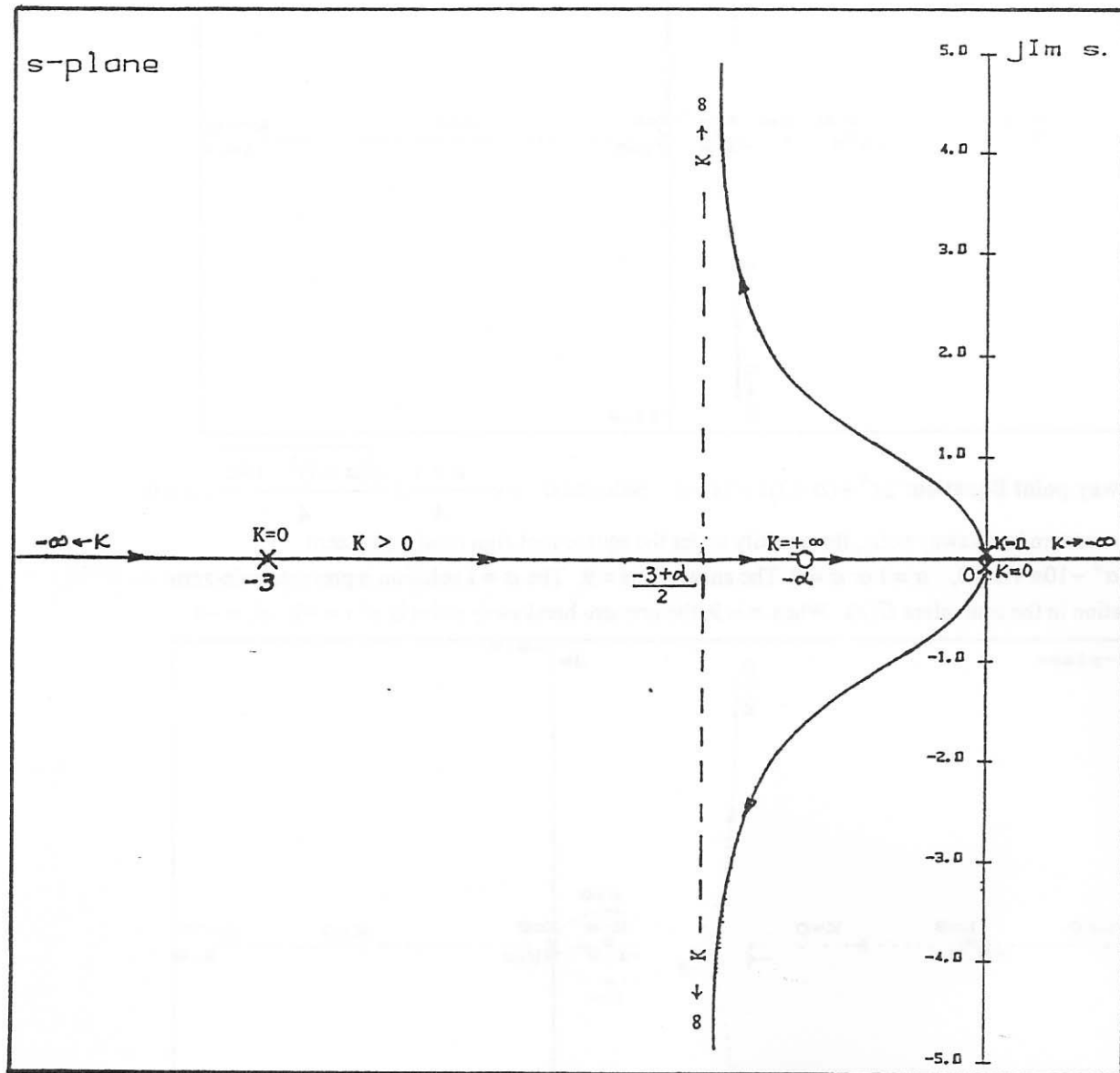
Breakaway-point Equation: $2s^3 + 3(1+\alpha)s + 6\alpha = 0$

The roots of the breakaway-point equation are:

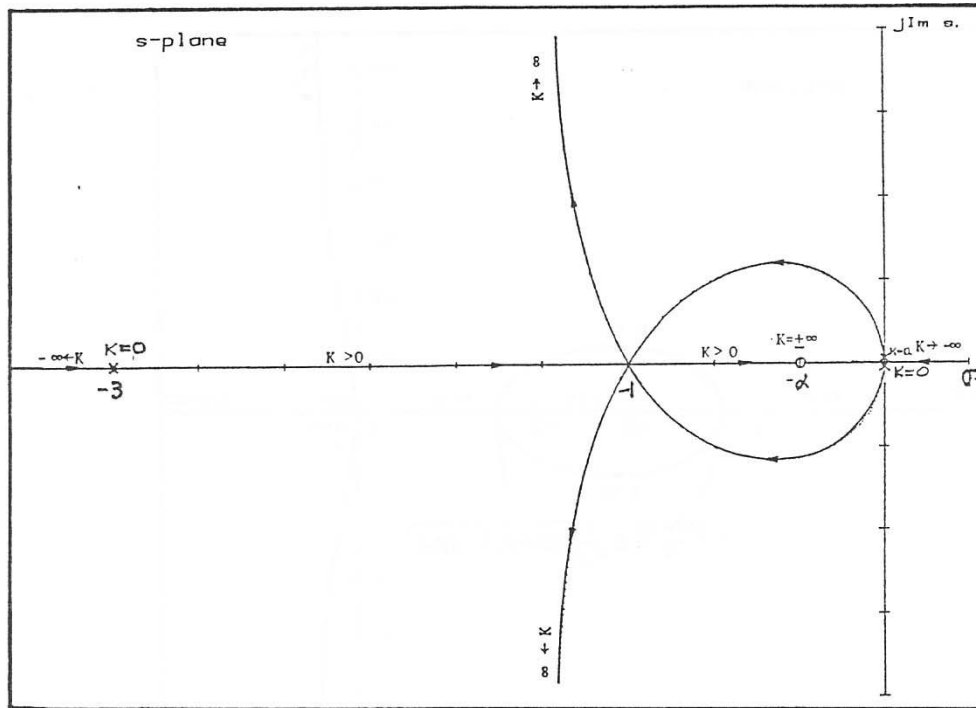
$$s = \frac{-3(1+\alpha)}{4} \pm \frac{\sqrt{9(1+\alpha)^2 - 48\alpha}}{4}$$

For no breakaway point other than at $s=0$, set $9(1+\alpha)^2 - 48\alpha < 0$ **or** $-0.333 < \alpha < 3$

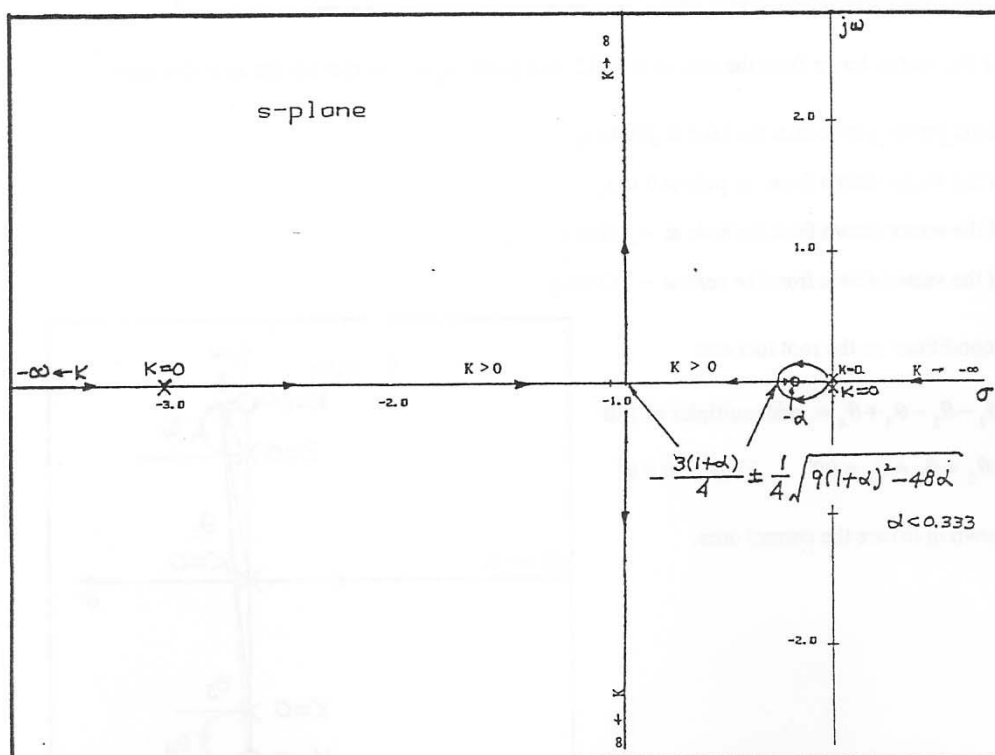
Root Locus Diagram with No Breakaway Point other than at $s=0$.



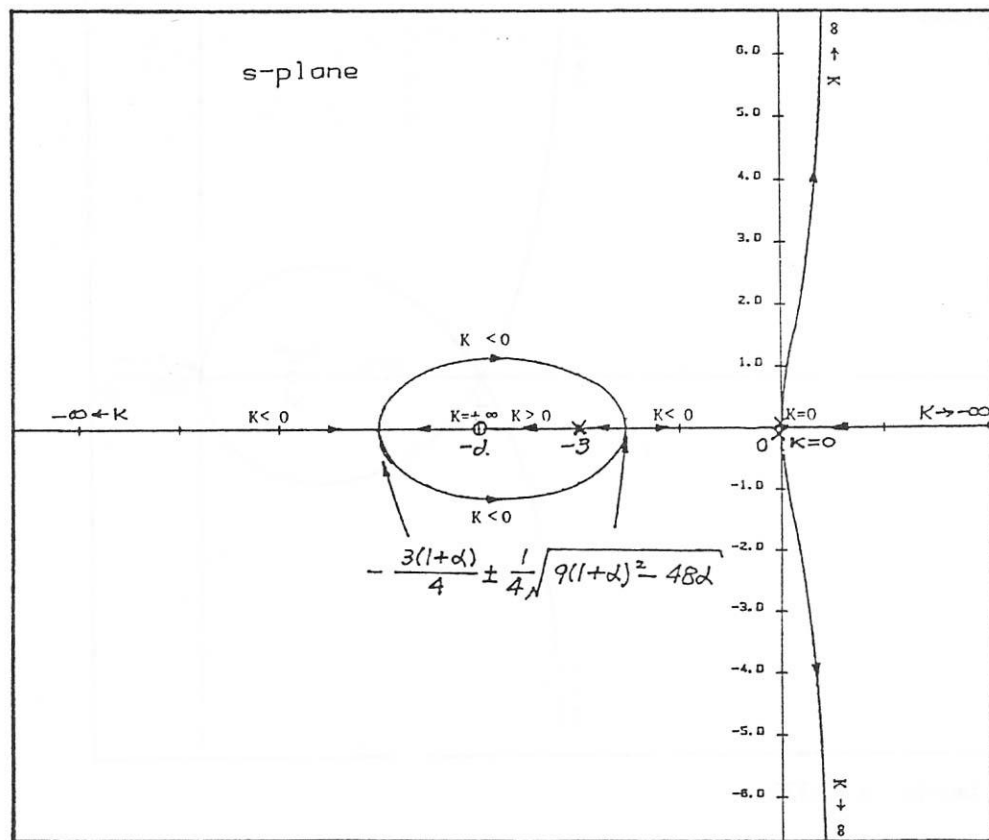
9-45 (b) One breakaway point other than at $s = 0$: $\alpha = 0.333$, Breakaway point at $s = -1$.



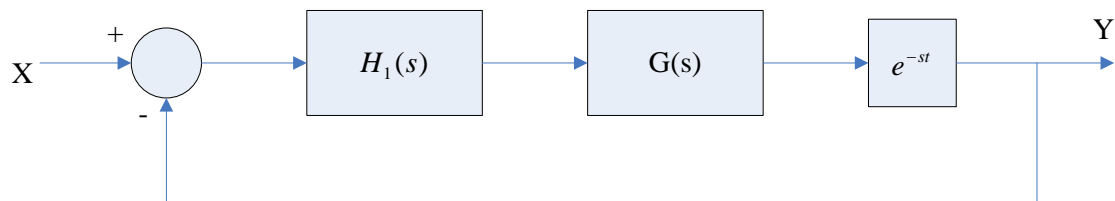
(c) Two breakaway points: $\alpha < 0.333$.



9-45 (d) Two breakaway points: $\alpha > 3$:



9-46) First we can rearrange the system as:



where

$$H_1(s) = \frac{H}{1 + (1 - e^{-st})GH}$$

Now designing a controller is similar to the designing a controller for any unity feedback system.

9-47) Let the angle of the vector drawn from the zero at $s = j12$ to a point s_1 on the root locuss near the zero

be θ . Let

θ_1 = angle of the vector drawn from the pole at $j10$ to s_1 .

θ_2 = angle of the vector drawn from the pole at 0 to s_1 .

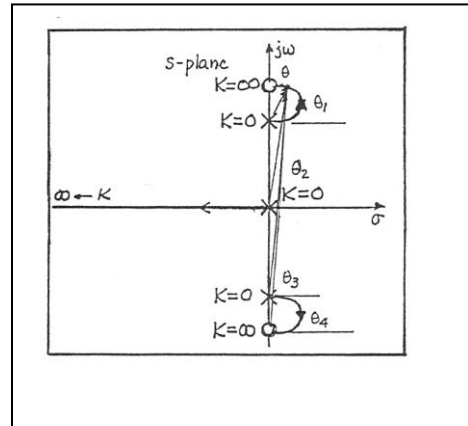
θ_3 = angle of the vector drawn from the pole at $-j10$ to s_1 .

θ_4 = angle of the vector drawn from the zero at $-j12$ to s_1 .

Then the angle conditions on the root loci are:

$$\theta = \theta_1 - \theta_2 - \theta_3 + \theta_4 = \text{odd multiples of } 180^\circ$$

$$\theta_1 = \theta_2 = \theta_3 = \theta_4 = 90^\circ \quad \text{Thus, } \theta = 0^\circ$$



The root loci shown in (b) are the correct ones.

Answers to True and False Review Questions:

6. (F) 7. (T) 8. (T) 9. (F) 10. (F) 11. (T) 12. (T) 13. (T) 14. (T)

Chapter 10

10-1 (a) $\kappa = 5$ $\omega_n = \sqrt{5} = 2.24 \text{ rad/sec}$ $\zeta = \frac{6.54}{4.48} = 1.46$ $M_r = 1$ $\omega_r = 0 \text{ rad/sec}$

(b) $\kappa = 21.39$ $\omega_n = \sqrt{21.39} = 4.62 \text{ rad/sec}$ $\zeta = \frac{6.54}{9.24} = 0.707$ $M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = 1$

$$\omega_r = \omega_n \sqrt{1-\zeta^2} = 3.27 \text{ rad/sec}$$

(c) $\kappa = 100$ $\omega_n = 10 \text{ rad/sec}$ $\zeta = \frac{6.54}{20} = 0.327$ $M_r = 1.618$ $\omega_r = 9.45 \text{ rad/sec}$

10-2

MATLAB code:

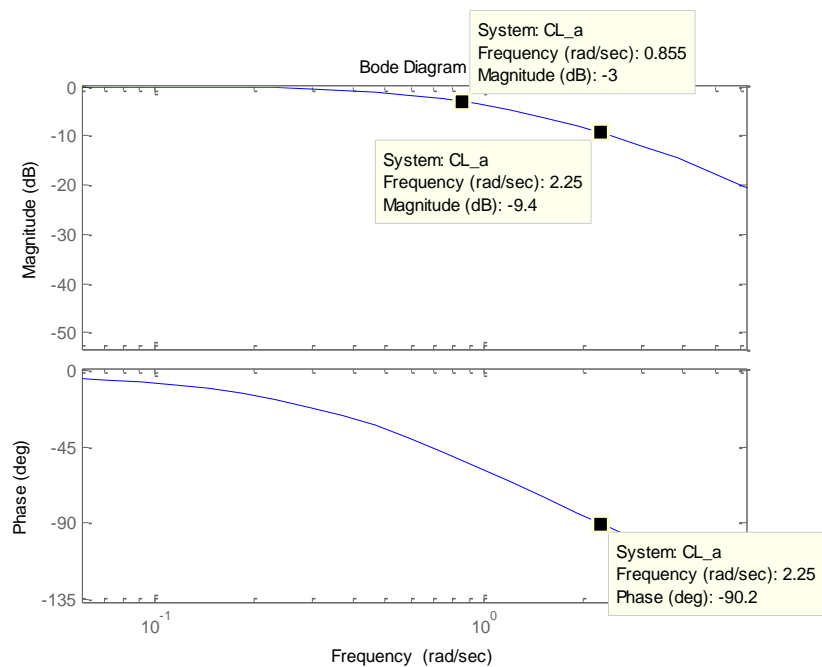
```
% Question 10-2,
clear all;
close all;
s = tf('s')

%a)
num_G_a = 5;
den_G_a = s*(s+6.54);
G_a = num_G_a/den_G_a;
CL_a = G_a/(1+G_a)
BW = bandwidth(CL_a)
bode(CL_a)

%b)
figure(2);
num_G_b = 21.38;
den_G_b = s*(s+6.54);
G_b = num_G_b/den_G_b;
CL_b = G_b/(1+G_b)
BW = bandwidth(CL_b)
bode(CL_b)

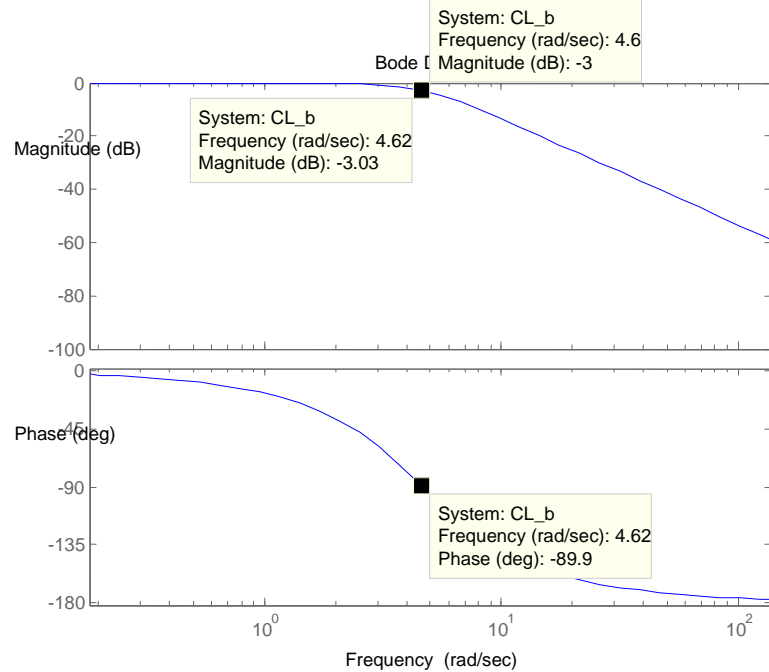
%c)
figure(3);
num_G_c = 100;
den_G_c = s*(s+6.54);
G_c = num_G_c/den_G_c;
```

Bode diagram (a) – $\kappa=5$: data points from top to bottom indicate bandwidth BW, resonance peak M_r , and resonant frequency ω_r .

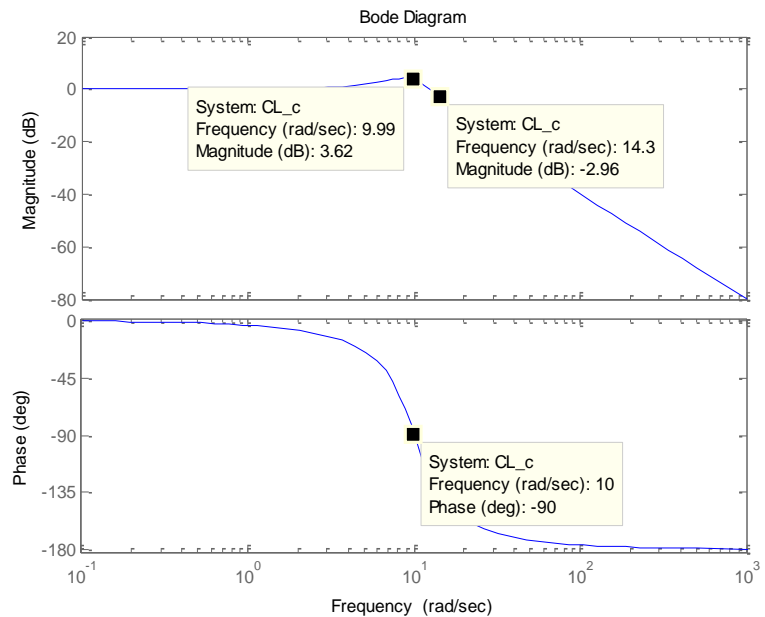


Bode diagram (b) – $\kappa=21.38$: data points from top to bottom indicate bandwidth BW, resonance peak M_r , and resonant frequency ω_r .

```
CL_c=G_c/(1+G_c)
BW = bandwidth(CL_c)
bode(CL_c)
```



Bode diagram (c) – $k=100$: data points from top to bottom indicate resonance peak M_r , bandwidth BW, and resonant frequency ω_r .



10-3) If $u(t) = U \sin(\omega t)$ is the input, then

$$G(j\omega) = \frac{j\omega + \frac{1}{A_1}}{j\omega + \frac{1}{A_2}} = \frac{A_2(1 + A_1j\omega)}{A_1(1 + A_2j\omega)}$$

where

$$|G(j\omega)| = \frac{A_2}{A_1} \sqrt{\frac{1 + A_1^2\omega^2}{1 + A_2^2\omega^2}}$$

and

$$\angle G(j\omega) = \tan^{-1} A_1\omega - \tan^{-1} A_2\omega = \Phi$$

Therefore:

$$y(t) = \frac{UA_2}{A_1} \sqrt{\frac{1 + A_1^2\omega^2}{1 + A_2^2\omega^2}} \sin(\omega t + \tan^{-1} A_1\omega - \tan^{-1} A_2\omega)$$

As a result:

$$\begin{cases} \text{if } A_1 > A_2 \rightarrow \Phi > 0 \rightarrow \text{the network is a lead network} \\ \text{if } A_1 < A_2 \rightarrow \Phi < 0 \rightarrow \text{the network is a lag network} \end{cases}$$

10-4 (a) $M_r = 2.944$ (9.38 dB) $\omega_r = 3$ rad/sec BW = 4.495 rad/sec

(b) $M_r = 15.34$ (23.71 dB) $\omega_r = 4$ rad/sec BW = 6.223 rad/sec

(c) $M_r = 4.17$ (12.4 dB) $\omega_r = 6.25$ rad/sec BW = 9.18 rad/sec

(d) $M_r = 1$ (0 dB) $\omega_r = 0$ rad/sec BW = 0.46 rad/sec

(e) $M_r = 1.57$ (3.918 dB) $\omega_r = 0.82$ rad/sec BW = 1.12 rad/sec

(f) $M_r = \infty$ (unstable) $\omega_r = 1.5$ rad/sec BW = 2.44 rad/sec

(g) $M_r = 3.09$ (9.8 dB) $\omega_r = 1.25$ rad/sec BW = 2.07 rad/sec

(h) $M_r = 4.12$ (12.3 dB) $\omega_r = 3.5$ rad/sec BW = 5.16 rad/sec

10-5)

Maximum overshoot = 0.1 Thus, $\zeta = 0.59$

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = 1.05 \quad t_r = \frac{1-0.416\zeta+2.917\zeta^2}{\omega_n} = 0.1 \text{ sec}$$

Thus, minimum $\omega_n = 17.7$ rad/sec Maximum $M_r = 1.05$

$$\text{Minimum BW} = \omega_n \left((1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2} \right)^{1/2} = 20.56 \text{ rad/sec}$$

10-6)

Maximum overshoot = 0.2 Thus, $0.2 = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$ $\zeta = 0.456$

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = 1.232 \quad t_r = \frac{1-0.416\zeta+2.917\zeta^2}{\omega_n} = 0.2 \quad \text{Thus, minimum } \omega_n = 14.168 \text{ rad/sec}$$

Maximum $M_r = 1.232$ Minimum BW = $\left((1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2} \right)^{1/2} = 18.7$ rad/sec

10-7) Maximum overshoot = 0.3 Thus, $0.3 = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$ $\zeta = 0.358$

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = 1.496 \quad t_r = \frac{1-0.416\zeta+2.917\zeta^2}{\omega_n} = 0.2 \quad \text{Thus, minimum } \omega_n = 6.1246 \text{ rad/sec}$$

$$\text{Maximum } M_r = 1.496 \quad \text{Minimum BW} = \left((1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2} \right)^{1/2} = 1.4106 \text{ rad/sec}$$

10-8) (a)

$$G(j\omega) = \frac{0.5K}{-0.375\omega^2 + j(\omega - 0.25\omega^3)}$$

At the gain crossover:

$$|G(j\omega)| = \frac{0.5K}{\sqrt{(0.375^2\omega^4) + (\omega^2 - 0.25\omega^3)^2}} = 1$$

Therefore:

$$(0.375)^2\omega^4 + (\omega - 0.25\omega^3)^2 - 0.25K^2 = 0$$

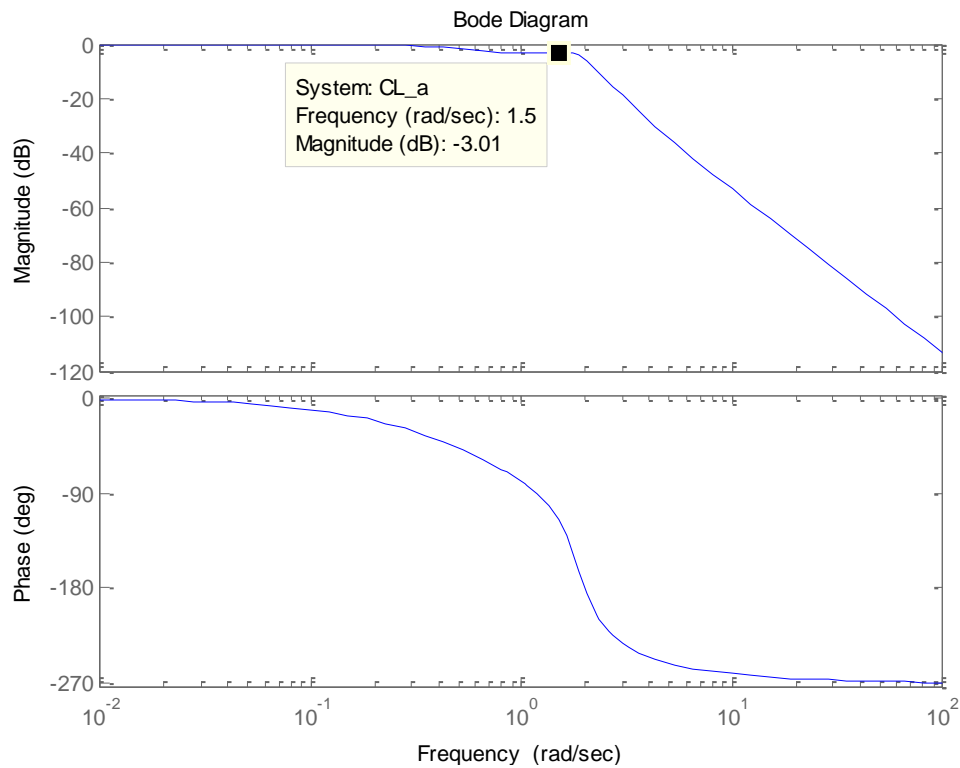
$$\text{at } \omega = 1.5 \Rightarrow K = 2.138$$

(b)

MATLAB code:

```
%solving for k:
syms kc
omega=1.5
sol=eval(solve('0.25*kc^2=0.7079^2*((-0.25*omega^3+omega)^2+(-0.375*omega^2+0.5*kc)^2)',kc))
%plotting bode with K=1.0370
s = tf('s')
K=1.0370;
num_G_a= 0.5*K;
den_G_a=s*(0.25*s^2+0.375*s+1);
G_a=num_G_a/den_G_a;
CL_a = G_a/(1+G_a)
BW = bandwidth(CL_a)
bode(CL_a);
```

Bode diagram: data point shows -3dB point at 1.5 rad/sec frequency which is the closed loop bandwidth



10-9) $\theta = \sin^{-1}\left(\frac{1}{Mp}\right) = \sin^{-1}\left(\frac{1}{2.2}\right) \approx 27^\circ$

$$\alpha = 90 - \theta = 63^\circ$$

$$OA = -\frac{M^2}{M^2 - 1} = -1.26$$

Therefore:

$$\begin{aligned} AB &= \frac{M}{M^2 - 1} \cos \alpha \\ &= \left(\frac{M}{M^2 - 1}\right) \cos(90 - \theta) \\ &= \frac{M}{M^2 - 1} \sin \alpha \\ &= \frac{M}{M^2 - 1} \left(\frac{1}{M}\right) = \frac{1}{M^2 - 1} \end{aligned}$$

As a result:

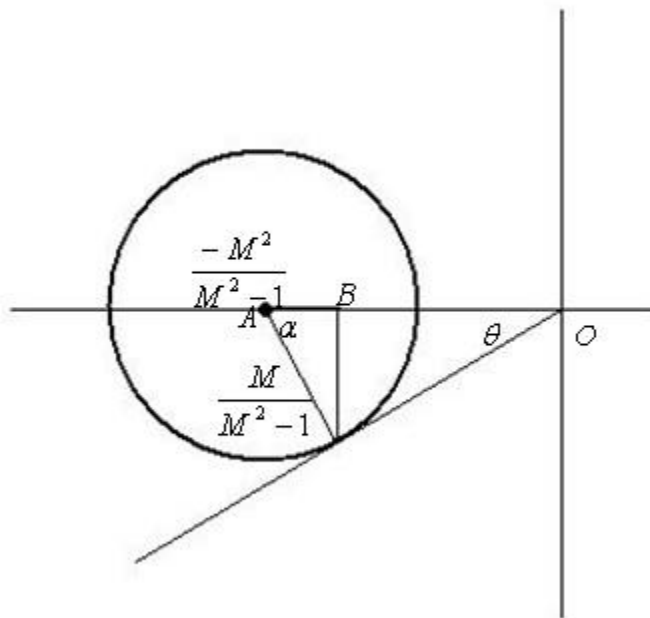
$$OB = -\frac{M^2}{M^2 - 1} - \frac{1}{M^2 - 1} = -1$$

Therefore:

$$|G(j\omega)|_{s=-1} = -0.54$$

To change the crossover frequency requires adding gain as:

$$K = -\frac{1}{-0.54} = 1.85$$

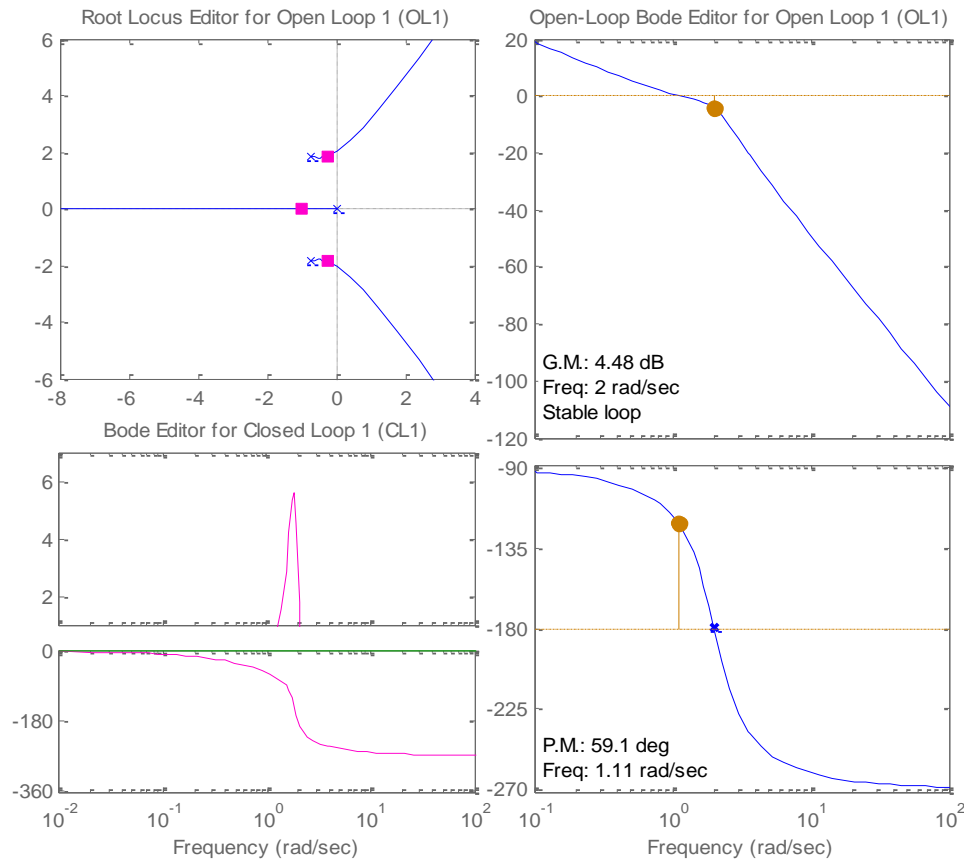


(b) MATLAB code:

```
s = tf('s')
% (b)
K = 0.95*2;
num_G_a = 0.5*K;
den_G_a = s*(0.25*s^2+0.375*s+1);
G_a = num_G_a/den_G_a;
CL_a = G_a/(1+G_a);
bode(CL_a);
figure(2);
sisotool
```

Peak mag = 2.22 can be converted to dB units by: $20 \cdot \log(2.22) = 6.9271$ dB

By using sisotool and importing the loop transfer function, the overall gain (0.5K) was changed until the magnitude of the resonance in Bode was about 6.9 dB. At $0.5K \approx 0.95$ or $K=1.9$, this resonance peak was achieved as can be seen in the BODE diagram of the following figure:



10-10)

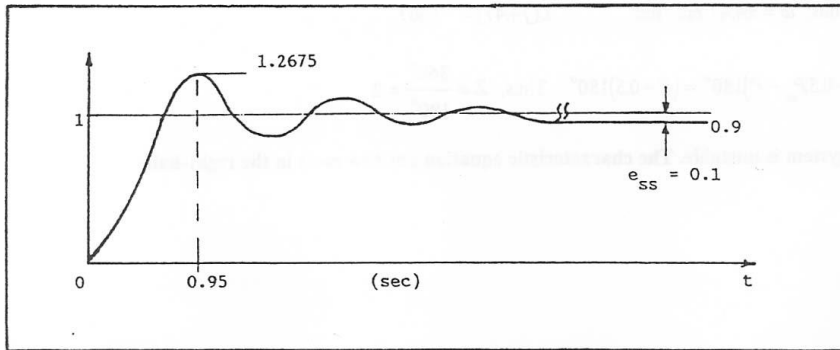
$$M_r = 1.4 = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \quad \text{Thus, } \zeta = 0.387 \quad \text{Maximum overshoot} = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} = 0.2675 \text{ (26.75\%)}$$

$$\omega_r = 3 \text{ rad/sec} = \omega_n \sqrt{1-2\zeta^2} = 0.8367\omega_n \text{ rad/sec} \quad \omega_n = \frac{3}{0.8367} = 3.586 \text{ rad/sec}$$

$$t_{\max} = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} = \frac{\pi}{3.586 \sqrt{1-(0.387)^2}} = 0.95 \text{ sec} \quad \text{At } \omega = 0, \quad |M| = 0.9.$$

This indicates that the steady-state value of the unit-step response is 0.9.

Unit-step Response:



10-11) a) The closed loop transfer function is:

$$\frac{Y(s)}{X(s)} = \frac{GH}{1 + GH} = \frac{K}{10s^2 + s + K} = \frac{K}{s^2 + 0.1s + 0.1K}$$

$$\text{as } \frac{1}{2\xi\sqrt{1-\xi^2}} = 1.4, \text{ which means } \xi = 0.387$$

According to the transfer function: $\xi \omega_n = 0.1 \Rightarrow \omega_n = 0.129 \text{ rad/s}$

As $\omega_n^2 = 0.1K$; then, $K = 10 \omega_n^2 = 0.1669$

$$\text{b) } \omega_R = \omega_0 \sqrt{1 - 2\xi^2} = 0.108 \frac{\text{rad}}{\text{s}}$$

$$M_p = \exp\left(-\frac{\pi\xi}{\sqrt{1-\xi^2}}\right) = 0.268$$

$$PM = \angle GH(j\omega_g) - 180^\circ$$

$$|GH(j\omega)|_{\omega=\omega_g} = 1 \Rightarrow \frac{K}{|(j\omega)(10j\omega+1)|} = \frac{K}{\sqrt{100\omega_g^4 + \omega_g^2}} = 1$$

As $K = 0.1664$, then $100\omega_g^4 + \omega_g^2 = 0.0277$

which means $\omega_g = 0.1104$

Accordingly $PM = 42^\circ$

As $M(\omega) = \left|\frac{Y(j\omega)}{X(j\omega)}\right| > \left(\frac{\sqrt{2}}{2}\right)K$, then $\omega_b = 0.179 \frac{\text{rad}}{\text{s}}$

10-12)

T	BW (rad/sec)	M_r
0	1.14	1.54
0.5	1.17	1.09
1.0	1.26	1.00
2.0	1.63	1.09
3.0	1.96	1.29
4.0	2.26	1.46
5.0	2.52	1.63

10-13)

T	BW (rad/sec)	M_r
0	1.14	1.54
0.5	1.00	2.32
1.0	0.90	2.65
2.0	0.74	2.91
3.0	0.63	3.18
4.0	0.55	3.37
5.0	0.50	3.62

10-14) The Routh array is:

S^3	0.25	1
S^2	0.375	0.5K
S^1	1-1/3	0
S^0	0.5K	

Therefore:

$$GH \approx \frac{0.5K}{0.375s^2 + \left(1 - \frac{K}{3}\right)s + 0.5K}$$

As $H(\omega) = |GH(j\omega_b)| > \frac{\sqrt{2}}{k}K$, if GH is rearranged as:

$$GH \approx \frac{1}{\frac{0.75}{k}s^2 + 2\left(\frac{1}{k} - \frac{1}{3}\right)s + 1}$$

then

$$\frac{\omega_b^2}{\omega_n^2} = (1 - \xi^2) + \sqrt{2 - 4\xi^2(1 - \xi^2)}$$

which gives

$$\omega_b^2 = \omega_n^2 \left[(1 - 2\xi^2) + \sqrt{2 - 4\xi^2(1 - \xi^2)} \right] = (1.5)^2$$

where $\omega_n^2 = \frac{K}{0.75}$ and $\xi^2 = \frac{1}{0.75k}$

therefore, $K = 2.146$, $\omega_n = 1.692$ and $\xi = 0.6213$

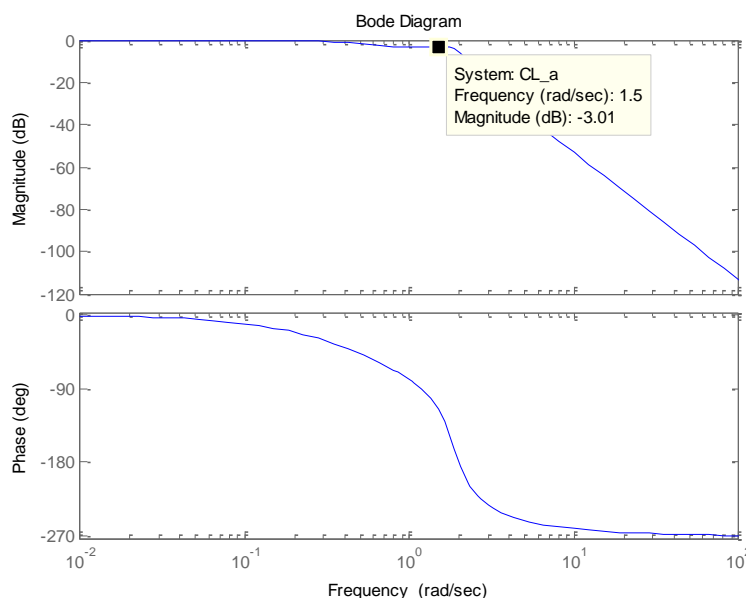
(c)

MATLAB code:

```
s = tf('s')
%c)
K = 1.03697;
num_G_a = 0.5*K;
den_G_a =
s*(0.25*s^2+0.375*s+1
);
%create closed-loop
system
G_a =
num_G_a/den_G_a;
CL_a = G_a/(1+G_a)
bode(CL_a);
```

Notes:

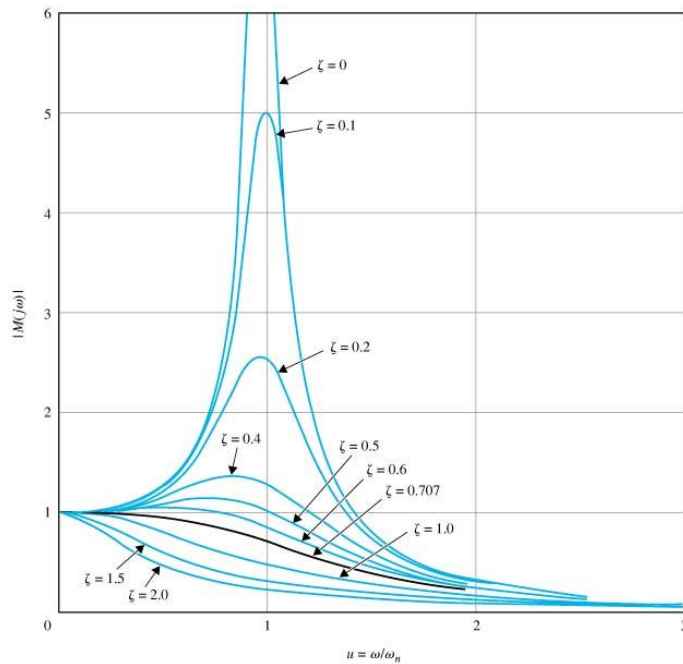
Bode diagram:



1 - BW is verified by finding -3dB point at Freq = 1.5 rad/sec in the Bode graph at calculated k.

2- By comparison to diagram of typical 2nd order poles with different damping ratios, damping ratio is approximated as:

$$\xi = \sim .707$$



10-15 (a)

$$L(s) = \frac{20}{s(1+0.1s)(1+0.5s)} \quad P_\omega = 1, \quad P = 0$$

$$\text{When } \omega = 0: \angle L(j\omega) = -90^\circ \quad |L(j\omega)| = \infty \quad \text{When } \omega = \infty: \angle L(j\omega) = -270^\circ \quad |L(j\omega)| = 0$$

$$L(j\omega) = \frac{20}{-0.6\omega^2 + j\omega(1-0.05\omega^2)} = \frac{20[-0.6\omega^2 - j\omega(1-0.05\omega^2)]}{0.36\omega^4 + \omega^2(1-0.05\omega^2)^2} \quad \text{Setting } \text{Im}[L(j\omega)] = 0$$

$$1 - 0.05\omega^2 = 0 \quad \text{Thus, } \omega = \pm 4.47 \text{ rad/sec} \quad L(j4.47) = -1.667$$

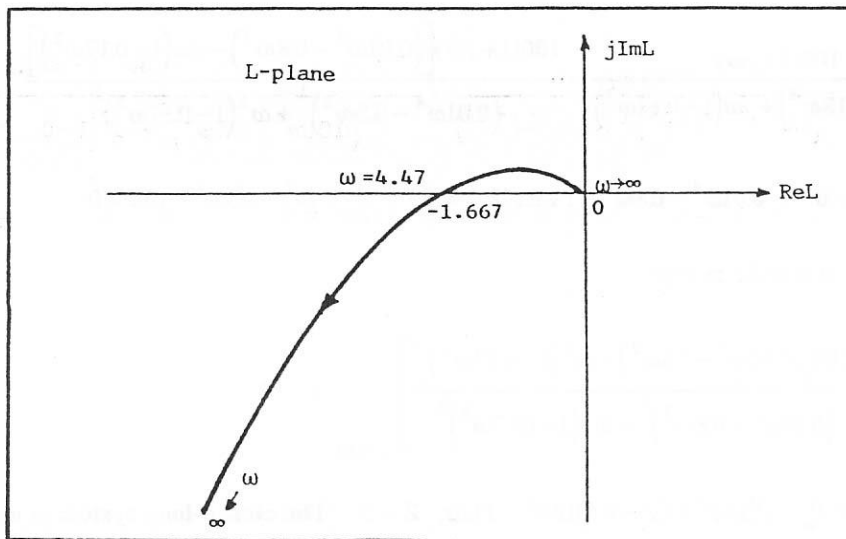
$$\Phi_{11} = 270^\circ = (Z - 0.5P_\omega - P)180^\circ = (Z - 0.5)180^\circ \quad \text{Thus, } Z = \frac{360^\circ}{180^\circ} = 2$$

The closed-loop system is unstable. The characteristic equation has two roots in the right-half s -plane.

MATLAB code:

```
s = tf('s')
%a)
figure(1);
num_G_a= 20;
den_G_a=s*(0.1*s+1)*(0.5*s+1);
G_a=num_G_a/den_G_a;
nyquist(G_a)
```

Nyquist Plot of $L(j\omega)$:



(b)

$$L(s) = \frac{10}{s(1+0.1s)(1+0.5s)}$$

Based on the analysis conducted in part (a), the intersect of the negative

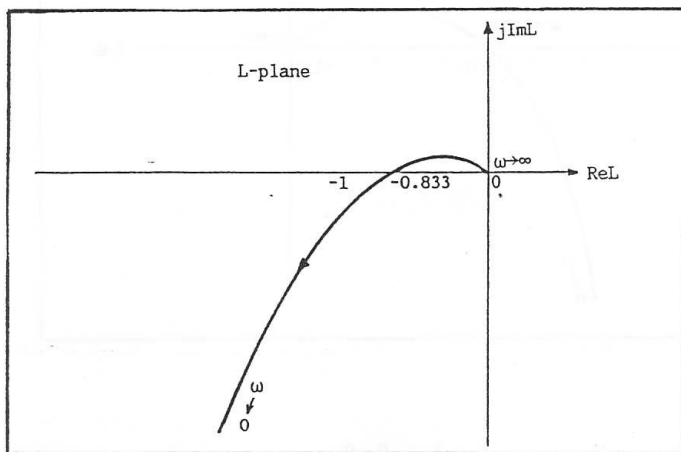
real axis by the $L(j\omega)$ plot is at -0.8333 , and the corresponding ω is 4.47 rad/sec.

$$\Phi_{11} = -90^\circ = \angle -0.5P_\omega - P \angle 180^\circ = 180Z - 90^\circ \quad \text{Thus, } Z = 0. \quad \text{The closed-loop system is stable.}$$

MATLAB code:

```
s = tf('s')
%b)
figure(1);
num_G_a= 10;
den_G_a=s*(0.1*s+1)*(0.5*s+1);
G_a=num_G_a/den_G_a;
nyquist(G_a)
```

Nyquist Plot of $L(j\omega)$:



(c)

$$L(s) = \frac{100(1+s)}{s(1+0.1s)(1+0.2s)(1+0.5s)} \quad P_\omega = 1, \quad P = 0.$$

$$\text{When } \omega = 0: \angle L(j0) = -90^\circ \quad |L(j0)| = \infty \quad \text{When } \omega = \infty: \angle L(j\infty) = -270^\circ \quad |L(j\infty)| = 0$$

$$\text{When } \omega = \infty: \angle L(j\omega) = -270^\circ \quad |L(j\omega)| = 0 \quad \text{When } \omega = \infty: \angle L(j\omega) = -270^\circ \quad |L(j\omega)| = 0$$

$$L(j\omega) = \frac{100(1+j\omega)}{(0.01\omega^4 - 0.8\omega^2) + j\omega(1-0.17\omega^2)} = \frac{100(1+j\omega)[(0.01\omega^4 - 0.8\omega^2) - j\omega(1-0.17\omega^2)]}{(0.01\omega^4 - 0.8\omega^2)^2 + \omega^2(1-0.17\omega^2)^2}$$

$$\text{Setting } \text{Im}[L(j\omega)] = 0 \quad 0.01\omega^4 - 0.8\omega^2 - 1 + 0.17\omega^2 = 0 \quad \omega^4 - 63\omega^2 - 100 = 0$$

$$\text{Thus, } \omega^2 = 64.55 \quad \omega = \pm 8.03 \text{ rad/sec}$$

$$L(j8.03) = \left(\frac{100[(0.01\omega^4 - 0.8\omega^2) + \omega^2(1-0.17\omega^2)]}{(0.01\omega^4 - 0.8\omega^2)^2 + \omega^2(1-0.17\omega^2)^2} \right) \bigg|_{\omega=8.03} = -10$$

$$\Phi_{11} = 270^\circ = (Z - 0.5P_\omega - P)180^\circ = (Z - 0.5)180^\circ \quad \text{Thus, } Z = 2 \quad \text{The closed-loop system is}$$

unstable.

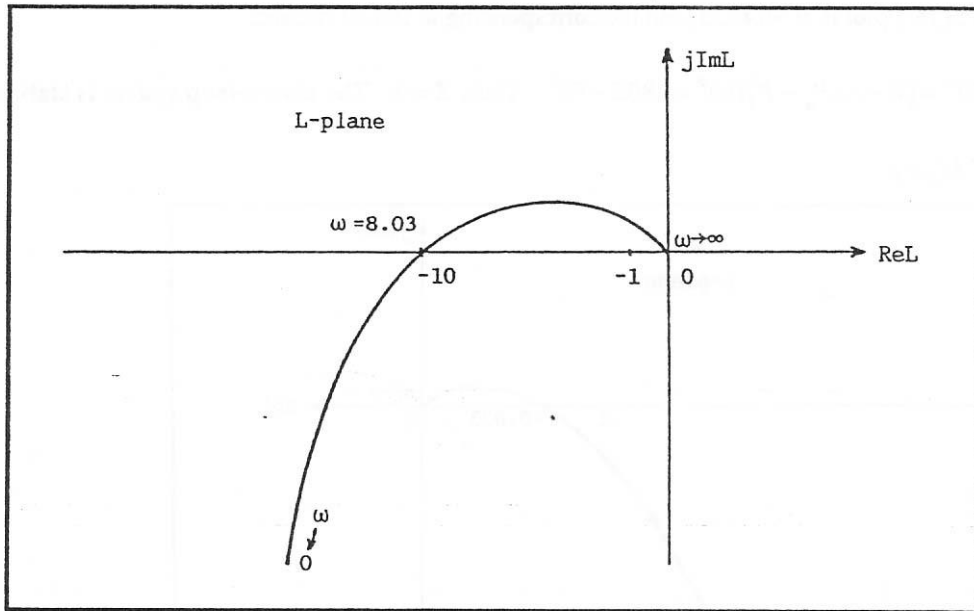
The characteristic equation has two roots in the right-half s-plane.

MATLAB code:

```
s = tf('s')
%c)
figure(1);
num_G_a = 100*(s+1);
```

```
den_G_a=s*(0.1*s+1)*(0.2*s+1)*(0.5*s+1);
G_a=num_G_a/den_G_a;
nyquist(G_a)
```

Nyquist Plot of $L(j\omega)$:



(d)

$$L(s) = \frac{10}{s^2(1+0.2s)(1+0.5s)} \quad P_\omega = 2 \quad P = 0$$

$$\text{When } \omega = 0: \angle L(j\omega) = -180^\circ \quad |L(j\omega)| = \infty \quad \text{When } \omega = \infty: \angle L(j\omega) = -360^\circ \quad |L(j\omega)| = 0$$

$$L(j\omega) = \frac{10}{(0.1\omega^4 - \omega^2) - j0.7\omega^3} = \frac{10(0.1\omega^4 - \omega^2 + j0.7\omega^3)}{(0.1\omega^4 - \omega^2)^2 + 0.49\omega^6}$$

Setting $\text{Im}[L(j\omega)] = 0$, $\omega = \infty$. The Nyquist plot of $L(j\omega)$ does not intersect the real axis except at the

origin where $\omega = \infty$.

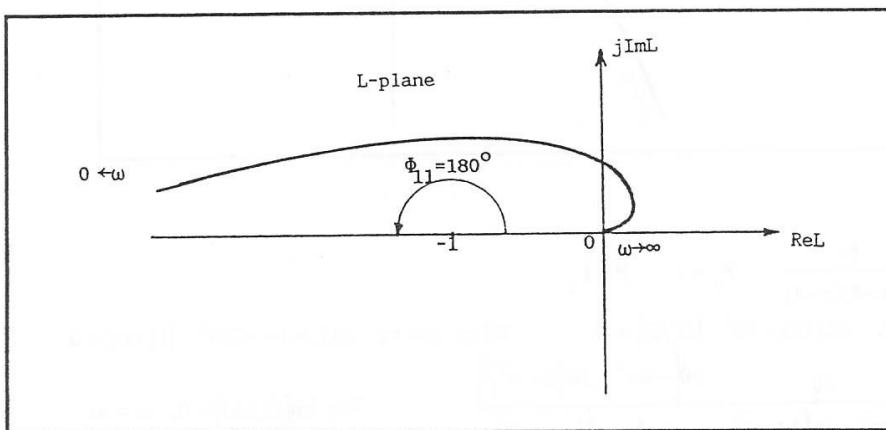
$$\Phi_{11} = (Z - 0.5P - P)180^\circ = (Z - 1)180^\circ \quad \text{Thus, } Z = 2.$$

The closed-loop system is unstable. The characteristic equation has two roots in the right-half s-plane.

MATLAB code:

```
s = tf('s')
% d)
figure(1);
num_G_a = 10;
den_G_a = s^2 * (0.2*s+1) * (0.5*s+1);
G_a = num_G_a/den_G_a;
nyquist(G_a)
```

Nyquist Plot of $L(j\omega)$:



10-15 (e)

$$L(s) = \frac{3(s+2)}{s(s^3+3s+1)} \quad P_\omega = 1 \quad P = 2$$

$$\text{When } \omega = 0: \angle L(j0) = -90^\circ \quad |L(j0)| = \infty \quad \text{When } \omega = \infty: \angle L(j\infty) = -270^\circ \quad |L(j\infty)| = 0$$

$$L(j\omega) = \frac{3(j\omega+2)}{(\omega^4-3\omega^2)+j\omega} = \frac{3(j\omega+2)[(\omega^4-3\omega^2)-j\omega]}{(4^4-3\omega^2)^2+\omega^2} \quad \text{Setting } \text{Im}[L(j\omega)] = 0,$$

$$\omega^4 - 3\omega^2 - 2 = 0 \quad \text{or} \quad \omega^2 = 3.56 \quad \omega = \pm 1.89 \text{ rad/sec.} \quad L(j1.89) = 3$$

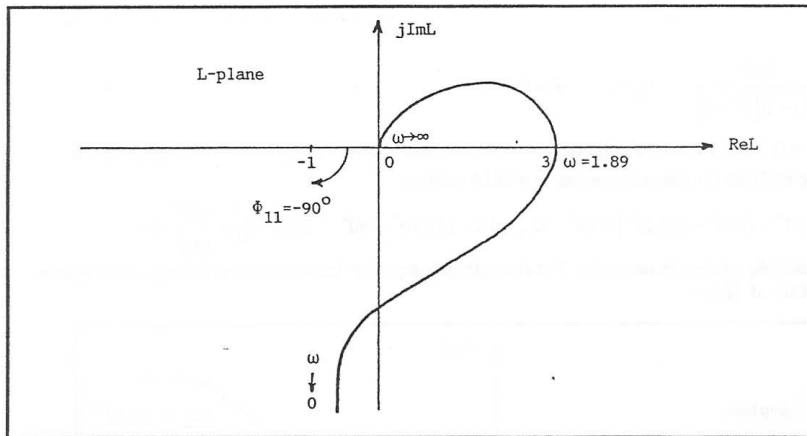
$$\Phi_{11} = (Z - 0.5P_\omega - P)180^\circ = (Z - 2.5)180^\circ = -90^\circ \quad \text{Thus, } Z = 2$$

The closed-loop system is unstable. The characteristic equation has two roots in the right-half s-plane.

MATLAB code:

```
s = tf('s')
%e)
figure(1);
num_G_a = 3*(s+2);
den_G_a = s*(s^3+3*s+1);
G_a = num_G_a/den_G_a;
nyquist(G_a)
```

Nyquist Plot of $L(j\omega)$:

**10-15 (f)**

$$L(s) = \frac{0.1}{s(s+1)(s^2+s+1)} \quad P_\omega = 1 \quad P = 0$$

$$\text{When } \omega = 0: \angle L(j0) = -90^\circ \quad |L(j0)| = \infty \quad \text{When } \omega = \infty: \angle L(j\infty) = -360^\circ \quad |L(j\infty)| = 0$$

$$L(j\omega) = \frac{0.1}{(\omega^4 - 2\omega^2) + j\omega(1 - 2\omega^2)} = \frac{0.1[(\omega^4 - 2\omega^2) - j\omega(1 - 2\omega^2)]}{(\omega^4 - 2\omega^2)^2 + \omega^2(1 - 2\omega^2)^2} \quad \text{Setting } \text{Im}[L(j\omega)] = 0$$

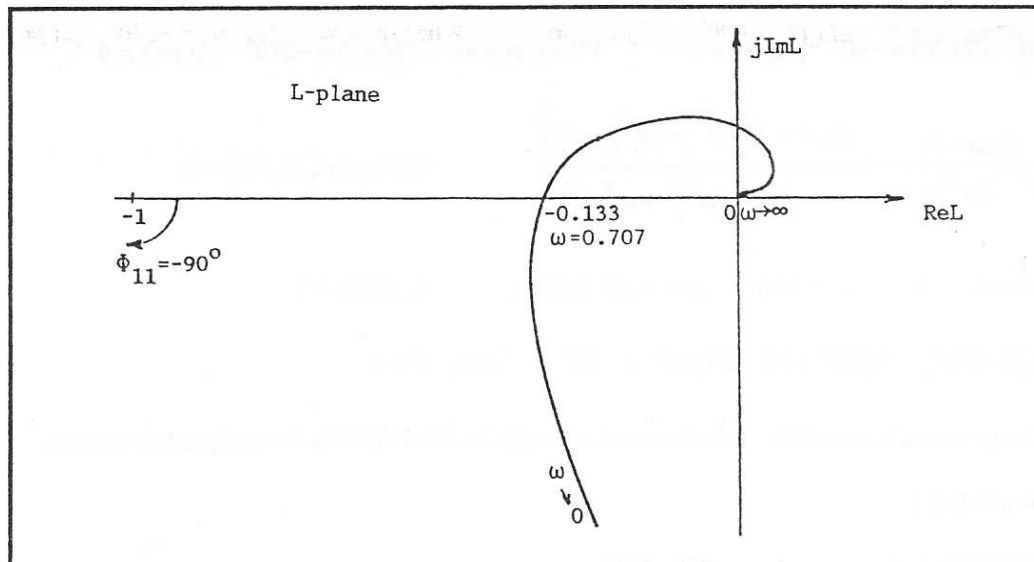
$$\omega = \infty \quad \text{or} \quad \omega^2 = 0.5 \quad \omega = \pm 0.707 \text{ rad/sec} \quad L(j0.707) = -0.1333$$

$$\Phi_{11} = (Z - 0.5P_\omega - P)180^\circ = (Z - 0.5)180^\circ = -90^\circ \quad \text{Thus, } Z = 0 \quad \text{The closed-loop system is stable.}$$

MATLAB code:

```
s = tf('s')
%f)
figure(1);
num_G_a = 0.1;
den_G_a = s*(s+1)*(s^2+s+1);
G_a = num_G_a/den_G_a;
nyquist(G_a)
```

Nyquist Plot of $L(j\omega)$:

**10-15 (g)**

$$L(s) = \frac{100}{s(s+1)(s^2+2)} \quad P_o = 3 \quad P = 0$$

$$\text{When } \omega = 0: \angle L(j0) = -90^\circ \quad |L(j0)| = \infty \quad \text{When } \omega = \infty: \angle L(j\infty) = -360^\circ \quad |L(j\infty)| = 0$$

The phase of $L(j\omega)$ is discontinuous at $\omega = 1.414$ rad/sec.

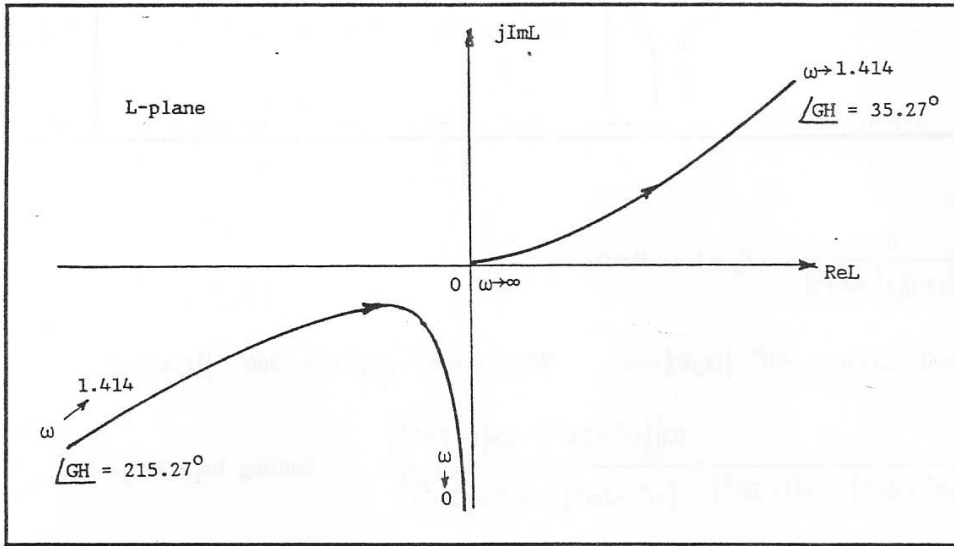
$$\Phi_{11} = 35.27^\circ + (270^\circ - 215.27^\circ) = 90^\circ \quad \Phi_{11} = (Z - 1.5)180^\circ = 90^\circ \quad \text{Thus, } P_{11} = \frac{360^\circ}{180^\circ} = 2$$

The closed-loop system is unstable. The characteristic equation has two roots in the right-half s -plane.

MATLAB code:

```
s = tf('s')
%g)
figure(1);
num_G_a= 100;
den_G_a=s*(s+1)*(s^2+2);
G_a=num_G_a/den_G_a;
nyquist(G_a)
```

Nyquist Plot of $L(j\omega)$:

**10-15 (h)**

$$L(s) = \frac{10(s+10)}{s(s+1)(s+100)} \quad P_\omega = 1 \quad P = 0$$

$$\text{When } \omega = 0: \angle L(j0) = -90^\circ \quad |L(j0)| = \infty \quad \text{When } \omega = \infty: \angle L(j\infty) = -180^\circ \quad |L(j\infty)| = 0$$

$$L(j\omega) = \frac{10(j\omega + 10)}{-101\omega^2 + j\omega(100 - \omega^2)} = \frac{10(j\omega + 10)[-101\omega^2 - j\omega(100 - \omega^2)]}{10201\omega^4 + \omega^2(100 - \omega^2)^2}$$

Setting $\text{Im}[L(j\omega)] = 0$, $\omega = 0$ is the only solution. Thus, the Nyquist plot of $L(j\omega)$ does not intersect the real axis, except at the origin.

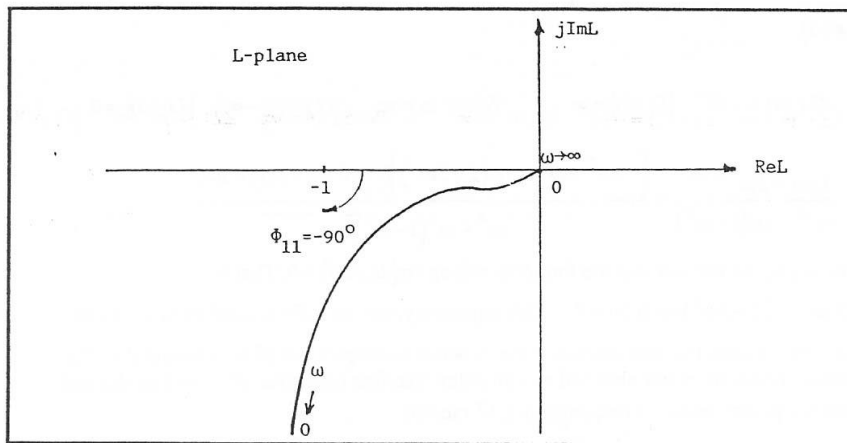
$$\Phi_{11} = (Z - 0.5P_\omega - P)180^\circ = (Z - 0.5)180^\circ = -90^\circ \quad \text{Thus, } Z = 0.$$

The closed-loop system is stable.

MATLAB code:

```
s = tf('s')
%h)
figure(1);
num_G_a= 10*(s+10);
den_G_a=s*(s+1)*(s+100);
G_a=num_G_a/den_G_a;
nyquist(G_a)
```

Nyquist Plot of $L(j\omega)$:



10-16

MATLAB code:

```
s = tf('s')
%a)
figure(1);
num_G_a= 1;
den_G_a=s*(s+2)*(s+10);
G_a=num_G_a/den_G_a;
nyquist(G_a)

%b)
figure(2);
num_G_b= 1*(s+1);
den_G_b=s*(s+2)*(s+5)*(s+15);
G_b=num_G_b/den_G_b;
nyquist(G_b)

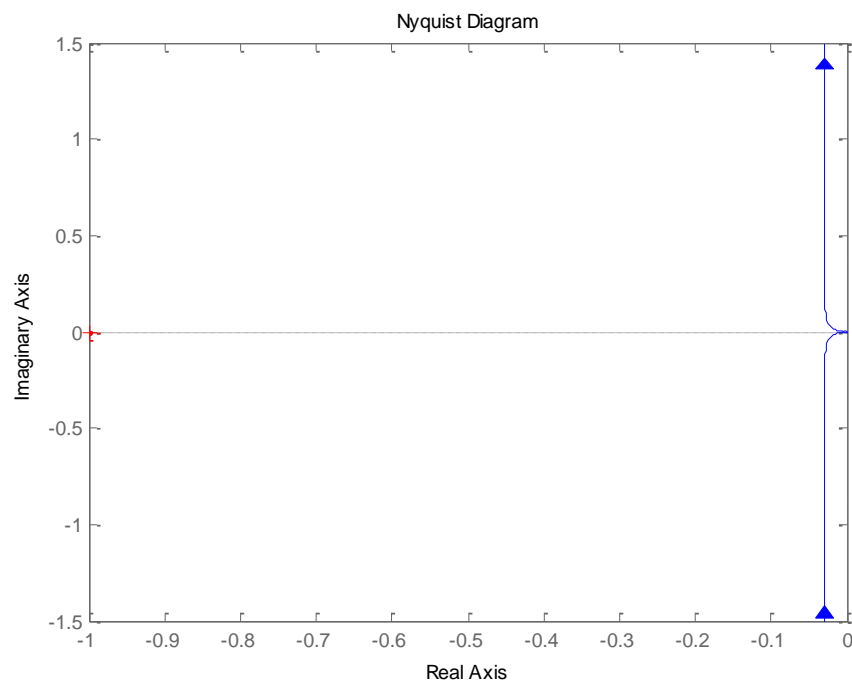
%c)
figure(3);
num_G_c= 1;
den_G_c=s^2*(s+2)*(s+10);
G_c=num_G_c/den_G_c;
nyquist(G_c)

%d)
figure(4);
num_G_d= 1;
den_G_d=(s+2)^2*(s+5);
G_d=num_G_d/den_G_d;
```

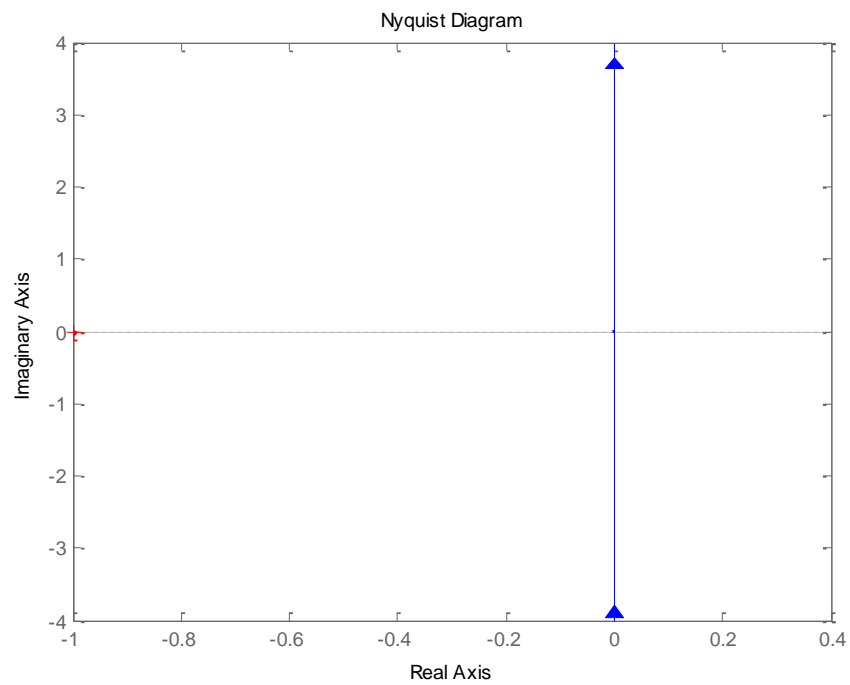
```
nyquist(G_d)

%e)
figure(5);
num_G_e= 1*(s+5)*(s+1);
den_G_e=(s+50)*(s+2)^3;
G_e=num_G_e/den_G_e;
nyquist(G_e)
```

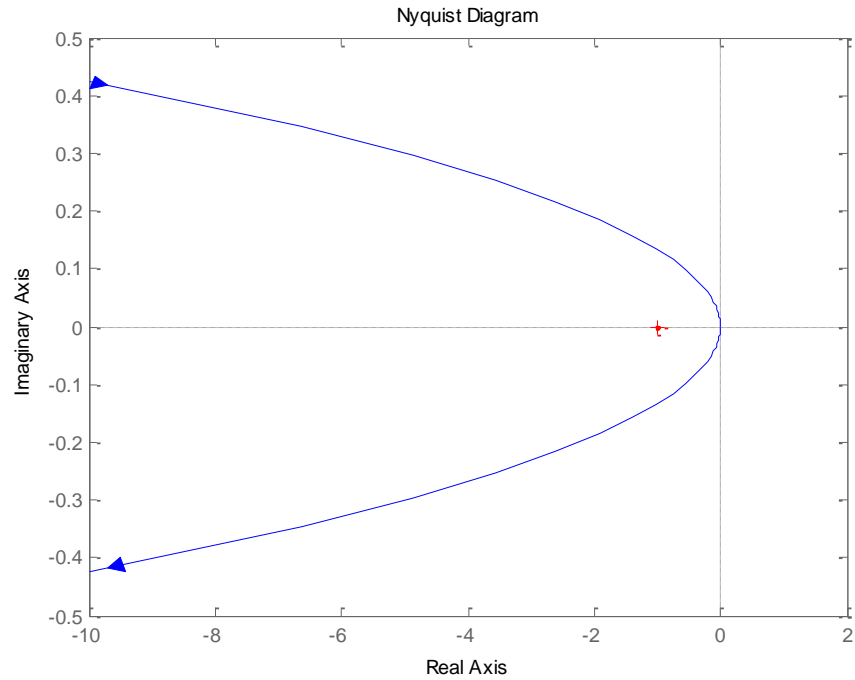
Nyquist graph, part(a):



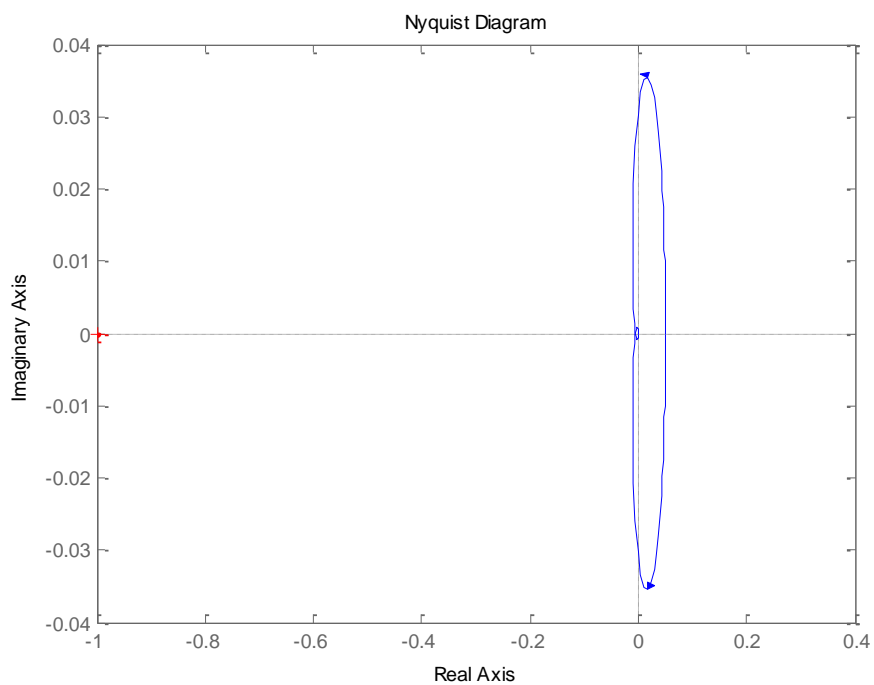
Nyquist graph, part(b):



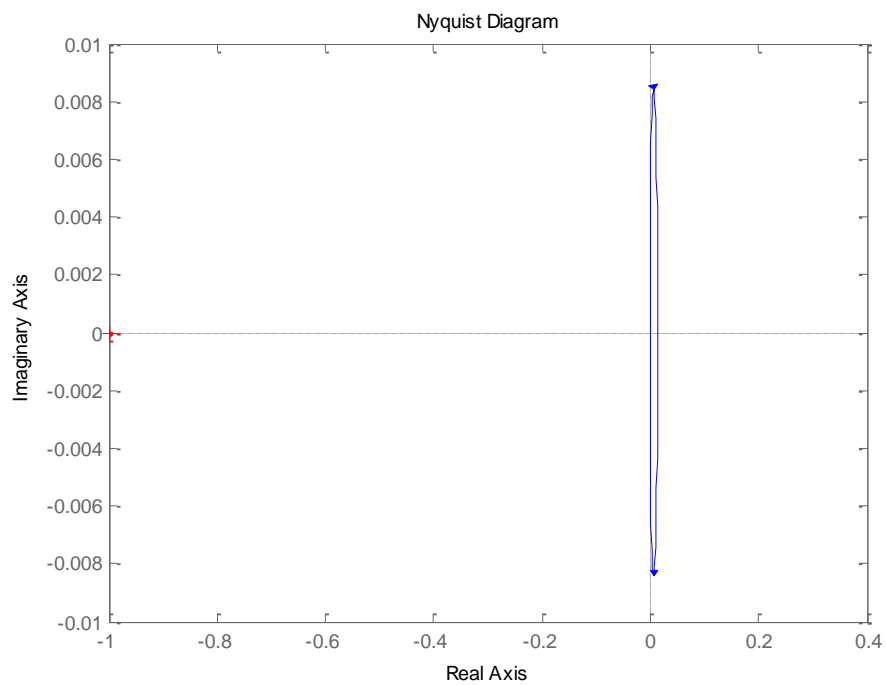
Nyquist graph, part(c):



Nyquist graph, part(d):



Nyquist graph, part (e):



10-17 (a)

$$G(s) = \frac{K}{(s+5)^2} \quad P_\omega = 0 \quad P = 0$$

$$\angle G(j0) = 0^\circ \quad (K > 0) \quad \angle G(0) = 180^\circ \quad (K < 0) \quad |G(j0)| = \frac{K}{25}$$

$$G(j\infty) = -180^\circ \quad (K > 0) \quad \angle G(j\infty) = 0^\circ \quad (K < 0) \quad |G(j\infty)| = 0$$

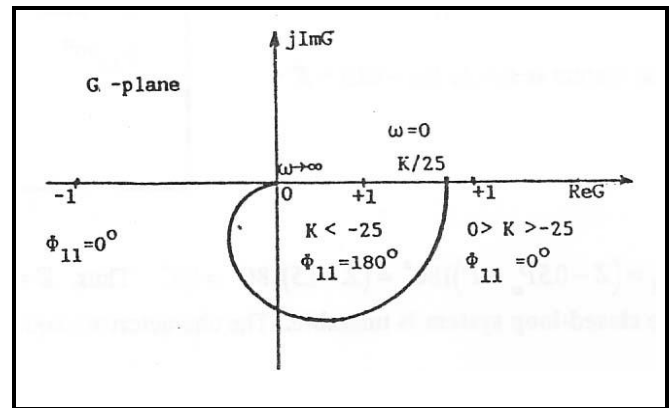
For stability, $Z = 0$.

$$\Phi_{11} = -(0.5P_\omega + P)180^\circ = 0^\circ$$

$$0 < K < \infty \quad \Phi_{11} = 0^\circ \quad \text{Stable}$$

$$K < -25 \quad \Phi_{11} = 180^\circ \quad \text{Unstable}$$

$$-25 < K < 0 \quad \Phi_{11} = 0^\circ \quad \text{Stable}$$



The system is stable for $-25 < K < \infty$.

10-17 (b)

$$G(s) = \frac{K}{(s+5)^3} \quad P_\omega = 0 \quad P = 0$$

$$\angle G(j0) = 0^\circ \quad (K > 0) \quad \angle G(0) = 180^\circ \quad (K < 0) \quad |G(j0)| = \frac{K}{125}$$

$$G(j\infty) = -270^\circ \quad (K > 0) \quad \angle G(j\infty) = 270^\circ \quad (K < 0) \quad |G(j\infty)| = 0$$

For stability, $Z = 0$.

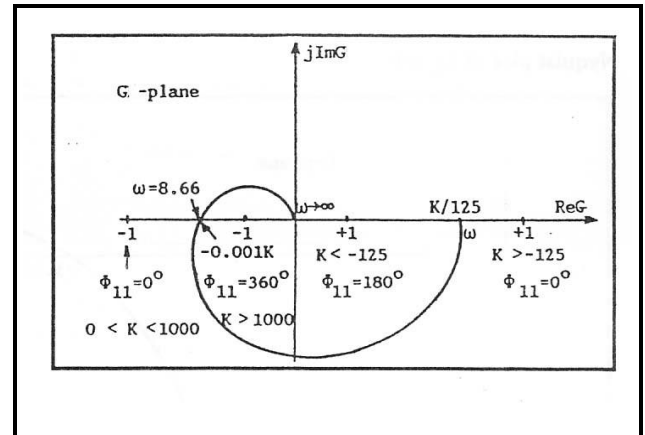
$$\Phi_{11} = -(0.5P_\omega + P)180^\circ = 0^\circ$$

$$0 < K < 1000 \quad \Phi_{11} = 0^\circ \quad \text{Stable}$$

$$K > 1000 \quad \Phi_{11} = 360^\circ \quad \text{Unstable}$$

$$K < -125 \quad \Phi_{11} = 180^\circ \quad \text{Unstable}$$

$$-125 < K < 0 \quad \Phi_{11} = 0^\circ \quad \text{Stable}$$



The system is stable for $-125 < K < 0$.

10-17 (c)

$$G(s) = \frac{K}{(s+5)^4} \quad P_\omega = P = 0$$

$$\angle G(j0) = 0^\circ \quad (K > 0) \quad \angle G(0) = 180^\circ \quad (K < 0) \quad |G(j0)| = \frac{K}{625}$$

$$G(j\infty) = 0^\circ \quad (K > 0) \quad \angle G(j\infty) = 180^\circ \quad (K < 0) \quad |G(j\infty)| = 0$$

For stability, $Z = 0$.

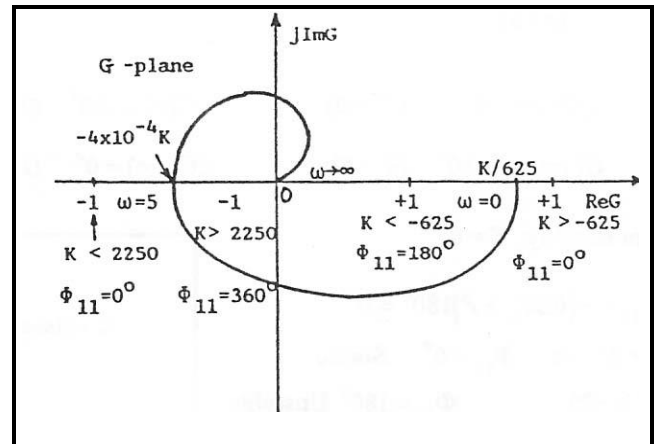
$$\Phi_{11} = -(0.5P_\omega + P)180^\circ = 0^\circ$$

$$0 < K < 2500 \quad \Phi_{11} = 0^\circ \quad \text{Stable}$$

$$K > 2500 \quad \Phi_{11} = 360^\circ \quad \text{Unstable}$$

$$K < -625 \quad \Phi_{11} = 180^\circ \quad \text{Unstable}$$

$$-625 < K < 0 \quad \Phi_{11} = 0^\circ \quad \text{Stable}$$



The system is stable for $-625 < K < 2500$.

10-18) The characteristic equation:

$$1 + \frac{K}{(s+1)(s^2+2s+2)} = 0$$

or

$$s^3 + 3s^2 + 4s + K + 2 = 0$$

if $K > 0$, the cross real axis at $s = 0.1$. \Rightarrow For stability $-10 < K < 10$

if $K < 0$, the Nyquist cross the real axis at $s = 0.5$. So, for stability, $-2 < K < 2$

therefore, the range of stability for the system is $-2 < K < 10$

MATLAB code:

```
s = tf('s')
```

```
K=1
```

```
G= K/(s^2+2*s+2);
```

```
H=1/(s+1);
```

```
GH=G*H;
```

```
sisotool
```

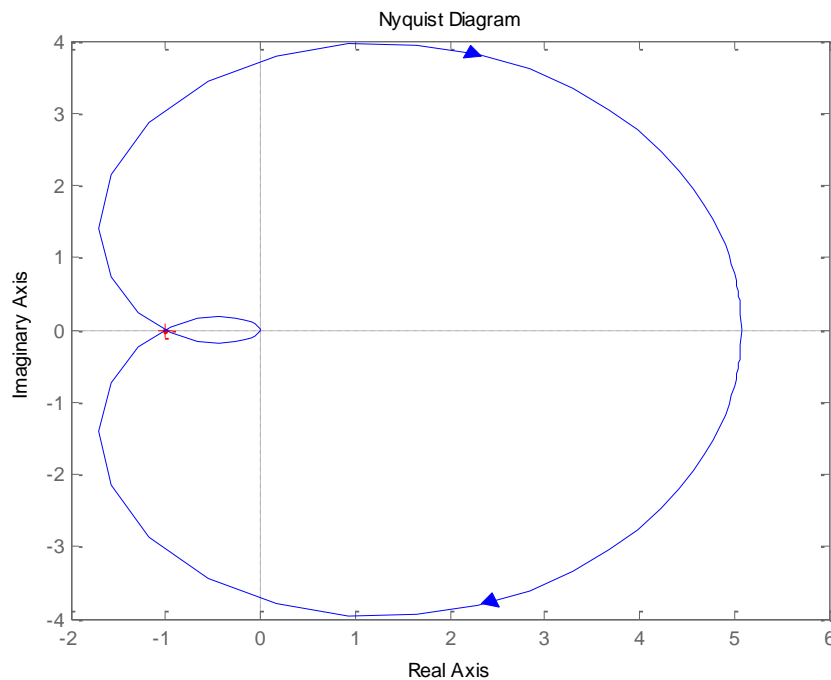
```

K=10.15
G2= K/(s^2+2*s+2);
H2=1/(s+1);
GH2=G2*H2;
nyquist(GH2)
xlim([-1.5,.5])
ylim([-1,1])

```

After generating the feed-forward (G) and feedback (H) transfer functions in the MATLAB code, these transfer functions are imported to sisotool. Nyquist diagram is added to the results of sisotool. The overall gain of the transfer function is changed until Nyquist diagram passes through $-1+0j$ point. Higher values of K resulted in unstable Nyquist diagram. Therefore $K < 10.15$ determines the range of stability for the closed loop system.

Nyquist at margin of stability:



10-19)

$$s(s^3 + 2s^2 + s + 1) + K(s^2 + s + 1) = 0$$

$$L_{eq}(s) = \frac{K(s^2 + s + 1)}{s(s^3 + 2s^2 + s + 1)} \quad P_\omega = 1 \quad P = 0 \quad L_{eq}(j0) = \infty \angle -90^\circ \quad L_{eq}(j\infty) = 0 \angle 180^\circ$$

$$L_{eq}(j\omega) = \frac{K[(1 - \omega^2) + j\omega]}{(\omega^4 - \omega^2) + j\omega(1 - 2\omega^2)} = \frac{K[-(\omega^6 + \omega^4) - j\omega(\omega^4 - 2\omega^2 + 1)]}{(\omega^4 - \omega^2)^2 + \omega^2(1 - 2\omega^2)^2}$$

$$\text{Setting } \text{Im}[L_{eq}(j\omega)] = 0$$

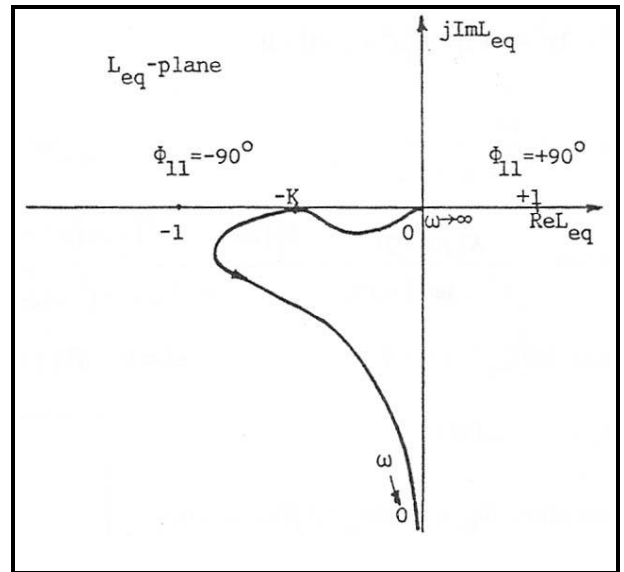
$$\omega^4 - 2\omega^2 + 1 = 0$$

Thus, $\omega = \pm 1$ rad/sec are the real solutions.

$$L_{eq}(j1) = -K$$

For stability,

$$\Phi_{11} = -(0.5P_\omega + P)180^\circ = -90^\circ$$



When $K = 1$ the system is marginally stable.

$$K > 0 \quad \Phi_{11} = -90^\circ \quad \text{Stable}$$

$$K < 0 \quad \Phi_{11} = +90^\circ \quad \text{Unstable}$$

Routh Tabulation

s^4	1	$K+1$	K
s^3	2	$K+1$	
s^2	$\frac{K+1}{2}$	K	$K > -1$
s^1	$\frac{K^2 - 2K + 1}{K+1} = \frac{(K-1)^2}{K+1}$		
s^0	K		$K > 0$

When $K = 1$ the coefficients of the s^1 row are all zero. The auxiliary equation is $s^2 + 1 = 0$. The solutions are $\omega = \pm 1$ rad/sec. Thus the Nyquist plot of $L_{eq}(j\omega)$ intersects the -1 point when $K = 1$, when $\omega = \pm 1$ rad/sec. **The system is stable for $0 < K < \infty$, except at $K = 1$.**

10-20) Solution is similar to the previous problem. Let's use Matlab as an alternative approach

MATLAB code:

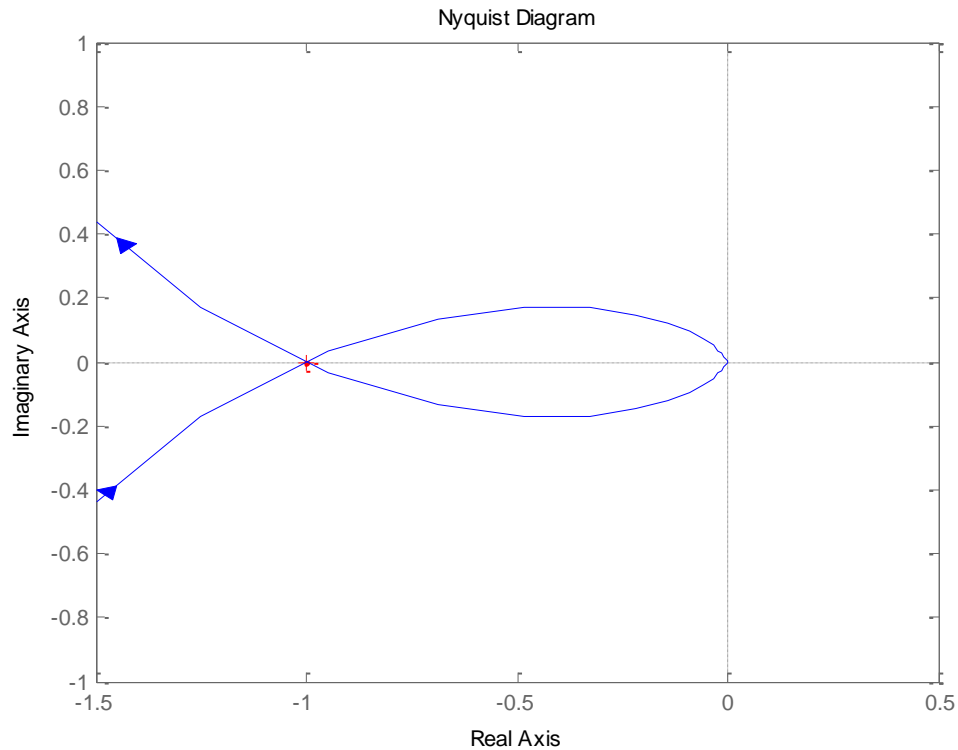
```
s = tf('s')

figure(1);
K=8.09
num_GH= K;
den_GH=(s^3+3*s^2+3*s+1);
GH=num_GH/den_GH;
nyquist(GH)
xlim([-1.5, .5])
ylim([-1, 1])

sisotool;
```

After generating the loop transfer function and analyzing Nyquist in MATLAB sisotool, it was found that for values of K higher than ~ 8.09 , the closed loop system is unstable. Following is the Nyquist diagram at margin of stability.

Part(a), Nyquist at margin of stability:



Part(b), Verification by Routh-Hurwitz criterion:

Using Routh criterion, the coefficient table is as follows:

S^3	1	3
S^2	3	$K+1$
S^1	$(8-K)/3$	0
S^0	$K+1$	0

The system is stable if the content of the 1st column is positive:

$$(8-K)/3 > 0 \rightarrow K < 8$$

$$K+1>0 \rightarrow K>-1$$

which is consistent with the results of the Nyquist diagrams.

10-21)

Parabolic error constant $K_a = \lim_{s \rightarrow 0} s^2 G(s) = \lim_{s \rightarrow 0} 10(K_p + K_D s) = 10K_p = 100$ Thus $K_p = 10$

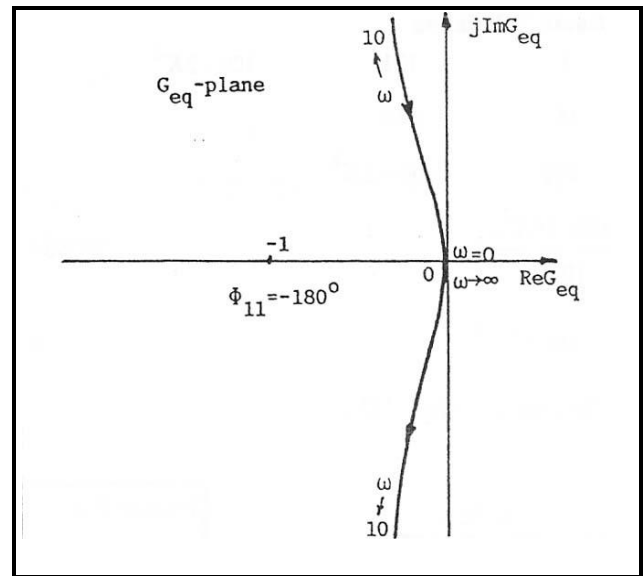
Characteristic Equation: $s^2 + 10K_D s + 100 = 0$

$$G_{eq}(s) = \frac{10K_D s}{s^2 + 100} \quad P_\omega = 2 \quad P = 0$$

For stability,

$$\Phi_{11} = -(0.5P_\omega + P)180^\circ = -180^\circ$$

The system is stable for $0 < K_D < \infty$.



10-22 (a) The characteristic equation is $1 + G(s) - G(s) - 2[G(s)]^2 = 1 - 2[G(s)]^2 = 0$

$$G_{eq}(s) = -2[G(s)]^2 = \frac{-2K^2}{(s+4)^2(s+5)^2} \quad P_\omega = 0 \quad P = 0$$

$$G_{eq}(j\omega) = \frac{-2K^2}{(400 - 120\omega^2 + \omega^4) + j\omega(360 - 18\omega^2)} = \frac{-2K^2[(400 - 120\omega^2 + \omega^2) - j\omega(360 - 18\omega^2)]}{(400 - 120\omega^2 + \omega^2) + \omega^2(360 - 18\omega^2)^2}$$

$$G_{eq}(j0) = \frac{K^2}{200} \angle 180^\circ \quad G_{eq}(j\infty) = 0 \angle 180^\circ \quad \text{Setting } \text{Im}[G_{eq}(j\omega)] = 0$$

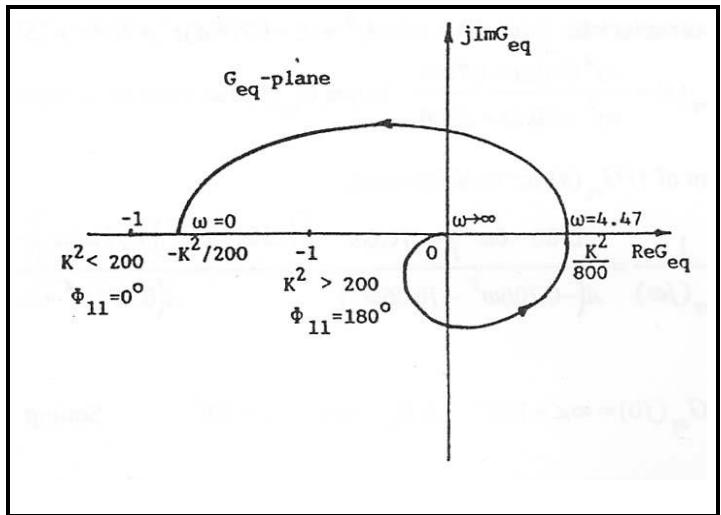
$$\omega = 0 \quad \text{and} \quad \omega = \pm 4.47 \text{ rad/sec} \quad G_{eq}(j4.47) = \frac{K^2}{800}$$

For stability,

$$\Phi_{11} = -(0.5P_\omega + P)180^\circ = 0^\circ$$

The system is stable for $K^2 < 200$

$$\text{or } |K| < \sqrt{200}$$



10-22 (b)

Characteristic Equation: $s^4 + 18s^3 + 121s^2 + 360s + 400 - 2K^2 = 0$

Routh Tabulation

s^4	1	121	$400 - 2K^2$	
s^3	18	360		
s^2	101	$400 - 2K^2$		
s^1	$\frac{29160 - 36K^2}{101}$			$29160 + 36K^2 > 0$
s^0	$400 - 2K^2$			$K^2 < 200$

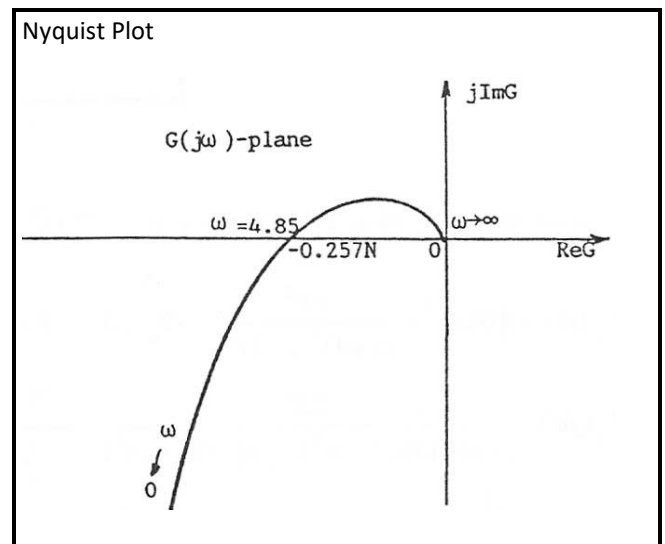
Thus for stability, $|K| < \sqrt{200}$

10-23 (a)

$$G(s) = \frac{83.33N}{s(s+2)(s+11.767)}$$

For stability, $N < 3.89$

Thus $N < 3$ since N must be an integer.



(b)

$$G(s) = \frac{2500}{s(0.06s + 0.706)(As + 100)}$$

Characteristic Equation: $0.06As^3 + (6 + 0.706A)s^2 + 70.6s + 2500 = 0$

$$G_{eq}(s) = \frac{As^2(0.06s + 0.706)}{6s^2 + 70.6s + 2500} \quad \text{Since } G_{eq}(s) \text{ has more zeros than poles, we should sketch the Nyquist}$$

plot of $1/G_{eq}(s)$ for stability study.

$$\frac{1}{G_{eq}(j\omega)} = \frac{(2500 - 6\omega^2) + j70.6\omega}{A(-0.706\omega^2 - j0.06\omega^3)} = \frac{[(2500 - 6\omega^2) + j70.6\omega](-0.706\omega^2 + j0.06\omega^3)}{A(0.498\omega^4 + 0.0036\omega^6)}$$

$$1/G_{eq}(j0) = \infty \angle -180^\circ \quad 1/G_{eq}(j\infty) = 0 \angle -90^\circ \quad \text{Setting } \text{Im} \left[\frac{1}{G_{eq}(j\omega)} \right] = 0$$

$$100.156 - 0.36\omega^2 = 0 \quad \omega = \pm 16.68 \text{ rad/sec}$$

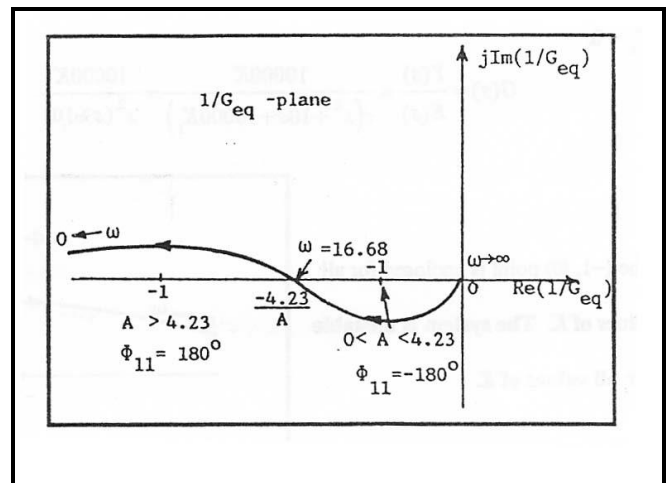
$$\frac{1}{G_{eq}(j16.68)} = \frac{-4.23}{A}$$

For stability,

$$\Phi_{11} = -(0.5P_\omega + P)180^\circ = -180^\circ$$

For $A > 4.23$ $\Phi_{11} = 180^\circ$ **Unstable**

For $0 < A < 4.23$ $\Phi_{11} = -180^\circ$ **Stable**



The system is stable for $0 < A < 4.23$.

(c)

$$G(s) = \frac{2500}{s(0.06s + 0.706)(50s + K_o)}$$

Characteristic Equation: $s(0.06s + 0.706)(50s + K_o) + 2500 = 0$

$$G_{eq}(s) = \frac{K_o s(0.06s + 0.706)}{3s^3 + 35.3s^2 + 2500} \quad P_\omega = 0 \quad P = 0 \quad G_{eq}(j0) = 0 \angle 90^\circ \quad G_{eq}(j\infty) = 0 \angle -90^\circ$$

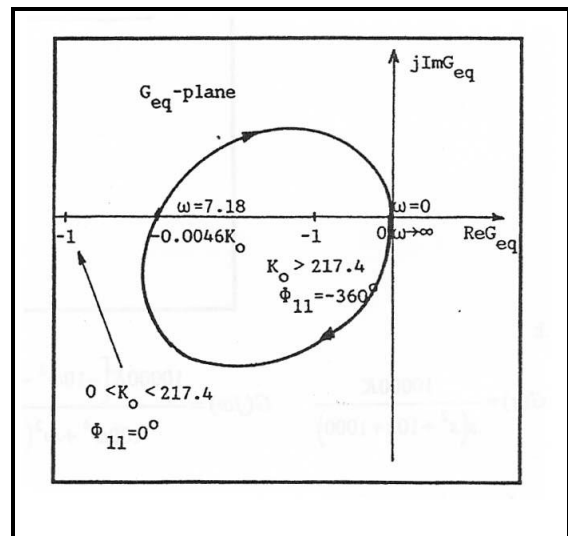
$$G_{eq}(j\omega) = \frac{K_o(-0.06\omega^3 + 0.706j\omega)}{(2500 - 35.3\omega^2) - j3\omega^3} = \frac{K_o(-0.06\omega^2 + 0.706j\omega)[(2500 - 35.3\omega^2) + j3\omega^3]}{(2500 - 35.3\omega^2)^2 + 9\omega^6}$$

Setting $\text{Im}[G_{eq}(j\omega)] = 0 \quad \omega^4 + 138.45\omega^2 - 9805.55 = 0 \quad \omega^2 = 51.6 \quad \omega = \pm 7.18 \text{ rad/sec}$

$$G_{eq}(j7.18) = -0.004K_o$$

For stability, $\Phi_{11} = -(0.5P_\omega + P)180^\circ = 0^\circ$

For stability, $0 < K_o < 217.4$



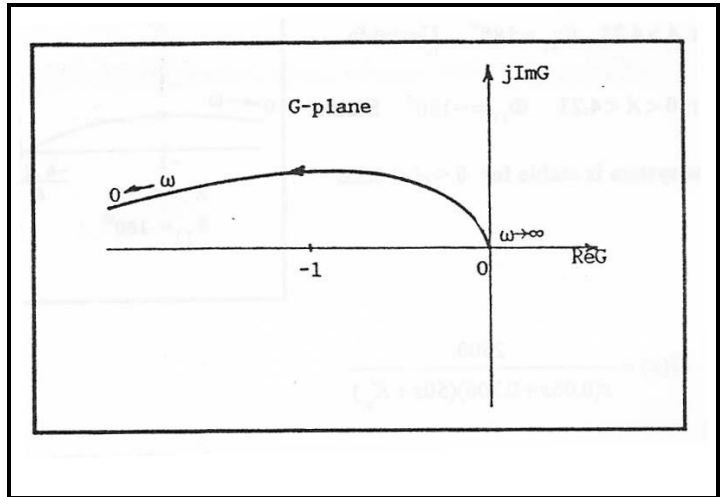
10-24 (a) $K_t = 0$:

$$G(s) = \frac{Y(s)}{E(s)} = \frac{10000K}{s(s^2 + 10s + 10000K_t)} = \frac{10000K}{s^2(s + 10)}$$

The $(-1, j0)$ point is enclosed for all

values of K . **The system is unstable**

for all values of K .



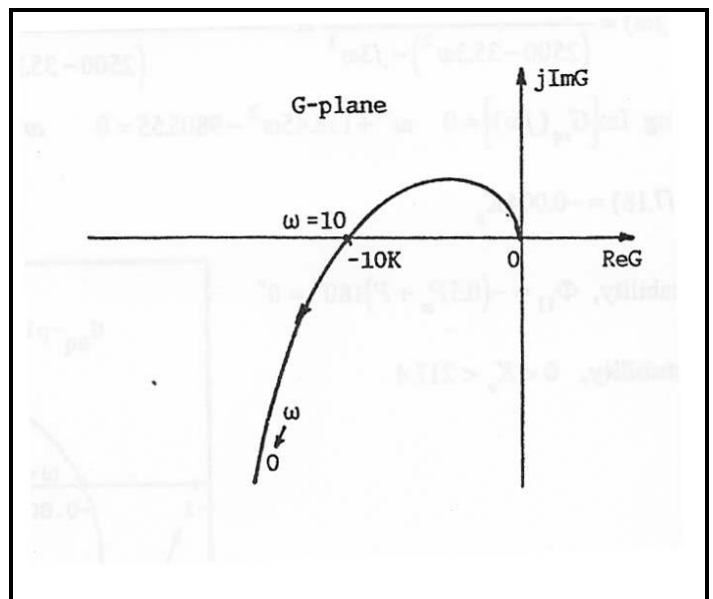
(b) $K_t = 0.01$:

$$G(s) = \frac{10000K}{s(s^2 + 10s + 100)} \quad G(j\omega) = \frac{10000K[-10\omega^2 - j\omega(100 - \omega^2)]}{100\omega^4 + \omega^2(100 - \omega^2)^2}$$

Setting $\text{Im}[G(j\omega)] = 0 \quad \omega^2 = 100$

$\omega = \pm 10 \text{ rad/sec} \quad G(j10) = -10K$

The system is stable for $0 < K < 0.1$

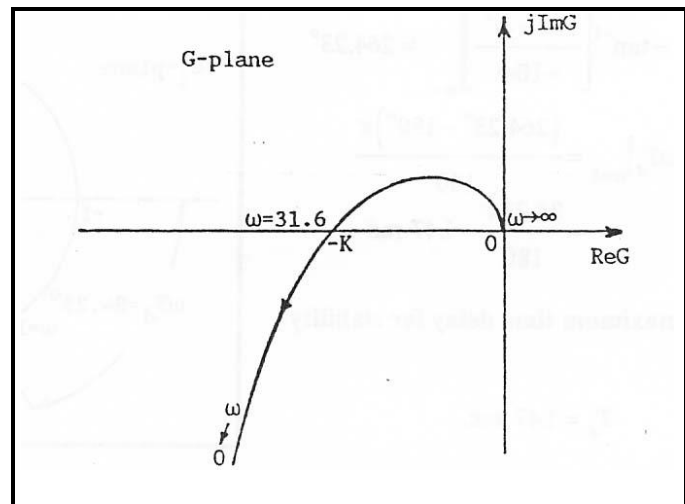


(c) $K_t = 0.1$:

$$G(s) = \frac{10000K}{s(s^2 + 10s + 1000)} \quad G(j\omega) = \frac{10000K[-10\omega^2 - j\omega(1000 - \omega^2)]}{100\omega^4 + \omega^2(1000 - \omega^2)^2}$$

Setting $\text{Im}[G(j\omega)] = 0 \quad \omega^2 = 100 \quad \omega = \pm 31.6 \text{ rad/sec} \quad G(j31.6) = -K$

For stability, $0 < K < 1$



10-25) The characteristic equation for $K = 10$ is:

$$s^3 + 10s^2 + 10,000K_t s + 100,000 = 0$$

$$G_{eq}(s) = \frac{10,000K_t s}{s^3 + 10s^2 + 100,000} \quad P_\omega = 0 \quad P = 2$$

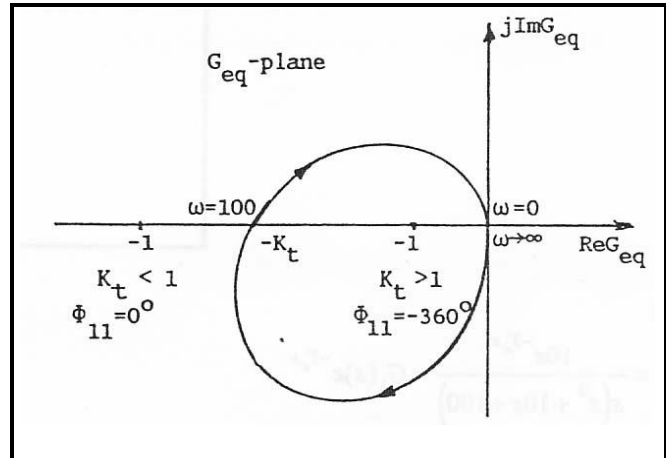
$$G_{eq}(j\omega) = \frac{10,000K_t j\omega}{100,000 - 10\omega^2 - j\omega^3} = \frac{10,000K_t[-\omega^4 + j\omega(10,000 - 10\omega^2)]}{(10,000 - 10\omega^2)^2 + \omega^6} \quad \text{Setting } \text{Im}[G_{eq}(j\omega)] = 0$$

$$\omega = 0, \quad \omega^2 = 10,000$$

$$\omega = \pm 100 \text{ rad/sec} \quad G_{eq}(j100) = -K_t$$

For stability,

$$\Phi_{11} = -(0.5P_\omega + P)180^\circ = -360^\circ$$



The system is stable for $K_t > 0$.

10-26)

$$\frac{Y(s)}{X(s)} = \frac{KK_f}{Js^2 + (a + KK_f)s + KK_f}$$

$$\text{a) } K_f = 0 \Rightarrow \frac{Y(s)}{X(s)} = \frac{K}{Js^2 + as + K} = \frac{K}{s^2 + s + K}$$

$$\text{b) } K_f = 0.1 \Rightarrow \frac{Y(s)}{X(s)} = \frac{0.1K}{Js^2 + (a + 0.1K)s + 0.1K} = \frac{0.1K}{s^2 + (1 + 0.1K)s + 0.1K}$$

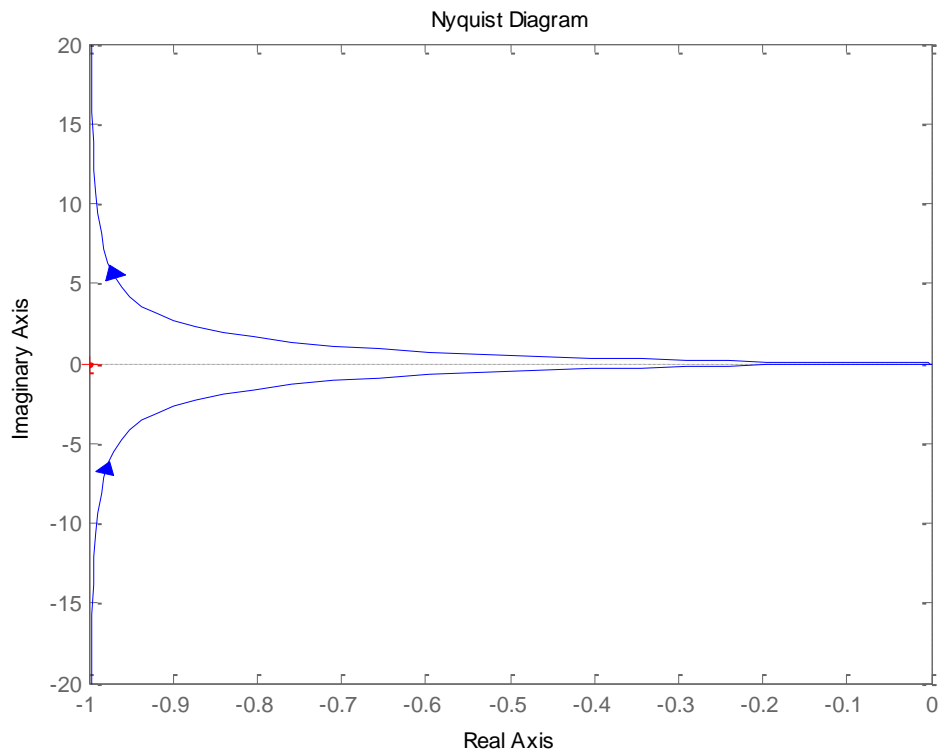
$$\text{c) } K_f = 0.2 \Rightarrow \frac{Y(s)}{X(s)} = \frac{0.2K}{s^2 + (-1 + 0.2K)s + 0.2K}$$

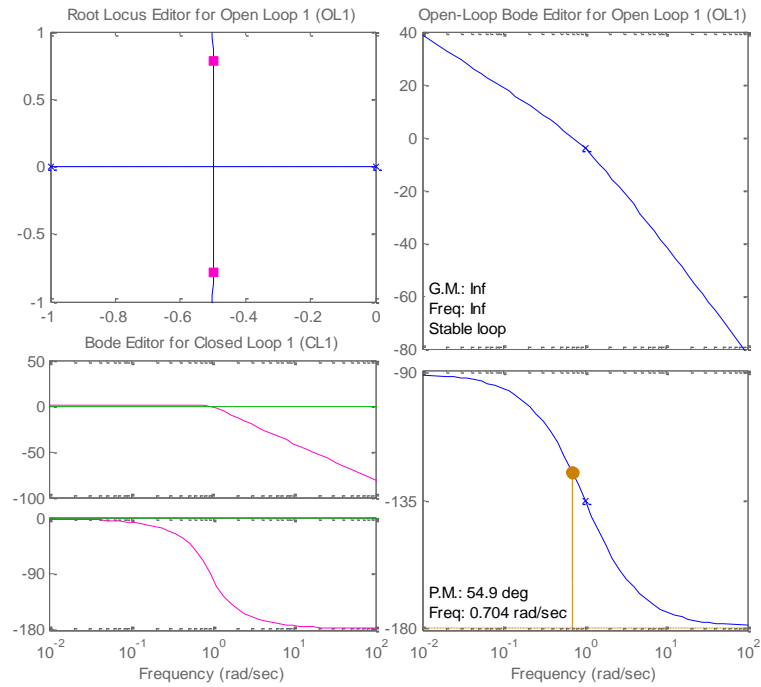
MATLAB code:

```
s = tf('s')
figure(1);
J=1;
B=1;
K=1;
Kf=0
G1= K/(J*s+B);
CL1=G1/(1+G1*Kf);
H2 = 1;
G1G2 = CL1/s;
L_TF=G1G2*H2;
nyquist(L_TF)
```

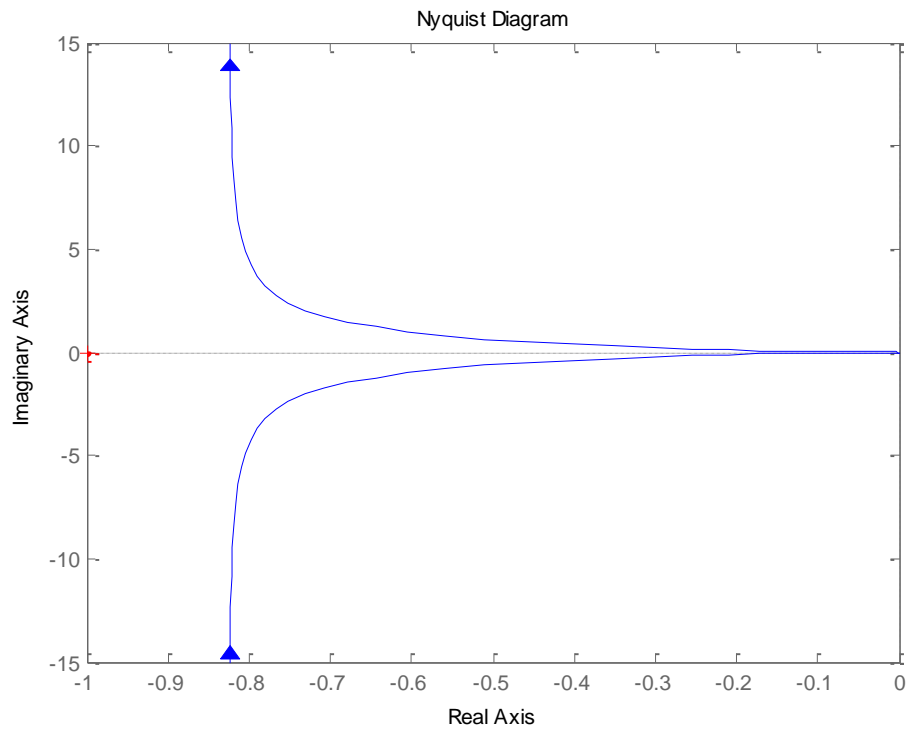
```
sisotool
```

Part (a), $K_f=0$: by plotting the Nyquist diagram in sisotool and varying the gain, it was observed that all values of gain (K) will result in a stable system. Location of poles in root locus diagram of the second figure will also verify that.

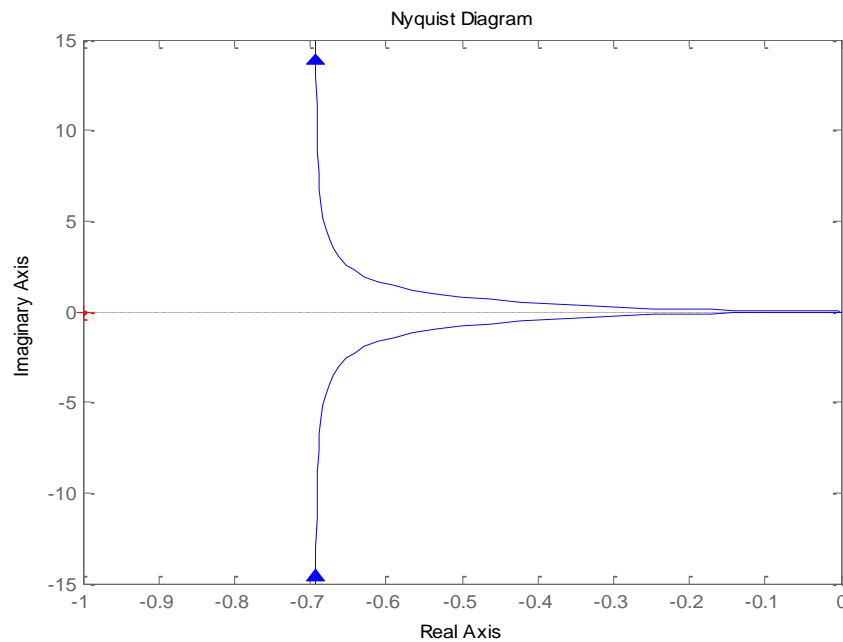




Part (b), $K_f = 0.1$: The result and approach is similar to part (a), a sample of Nyquist diagram is presented for his case as follows:



Part (c), $K_f=0.2$: The result and approach is similar to part (a), a sample of Nyquist diagram is presented for this case as follows:



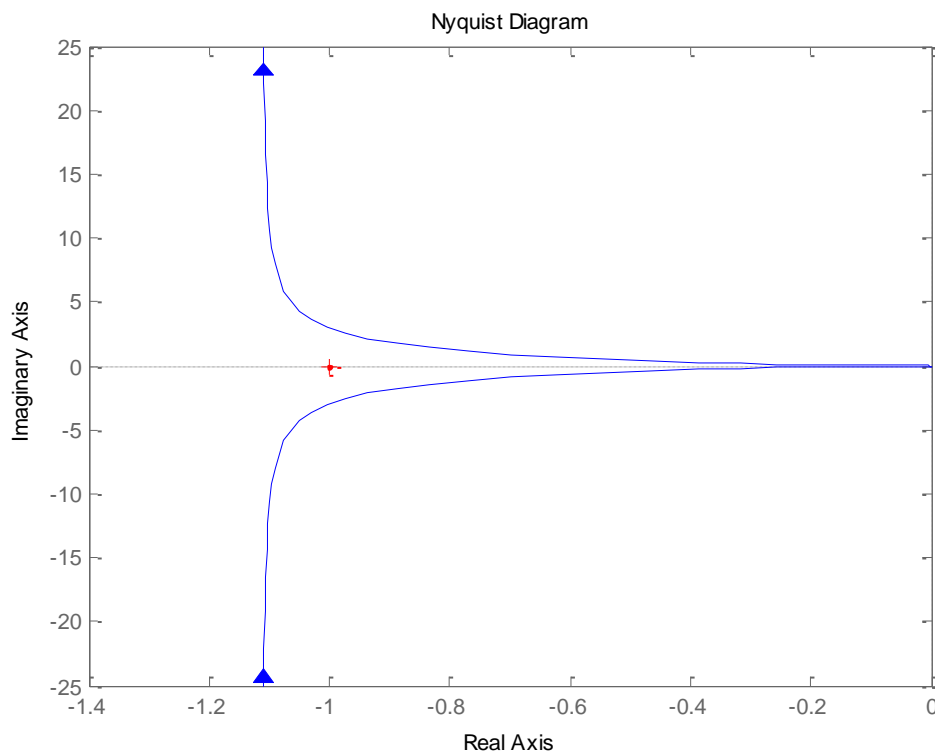
10-27)

$$\frac{Y(s)}{X(s)} = \frac{10K_f}{s^2 + (1 + 10K_f)s + 10K_f}$$

MATLAB code:

```
s = tf('s')
figure(1);
J=1;
B=1;
K=10;
Kf=0.2
G1= K/(J*s+B);
CL1=G1/(1+G1*Kf);
H2 = 1;
G1G2 = CL1/s;
L_TF=G1G2*H2;
nyquist(L_TF)
```

After assigning $K=10$, different values of K_f has been used in the range of $0.01 < K < 10^4$. The Nyquist diagrams shows the stability of the closed loop system for all $0 < K < \infty$. A sample of Nyquist diagram is plotted as follows:



10-28) a) $K > 2 \Rightarrow$ system is stable

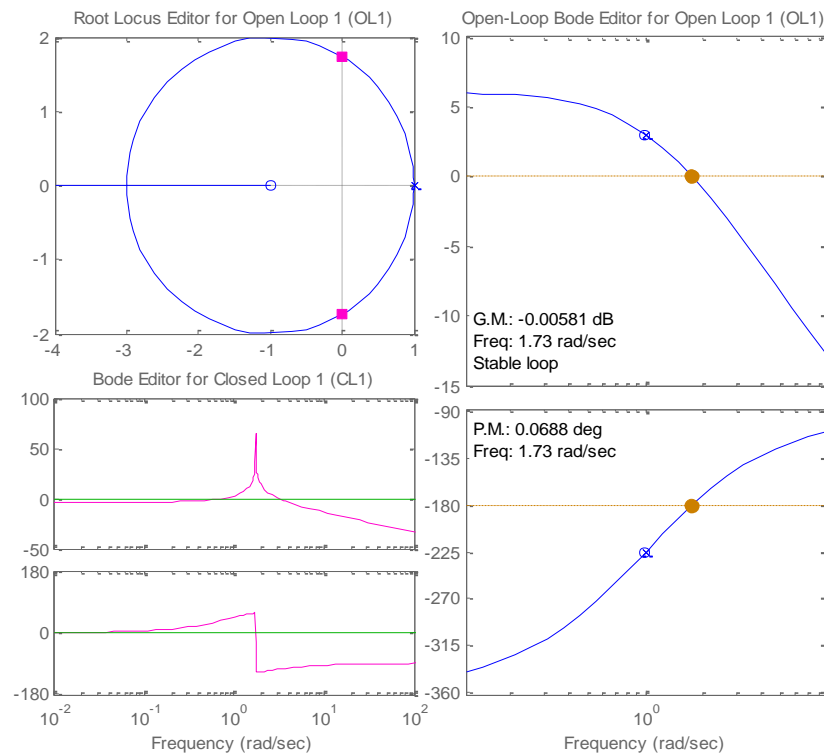
b) $0 < K < 1$ and $-2 < K < 0 \Rightarrow -2 < K < 1 \Rightarrow$ system is stable

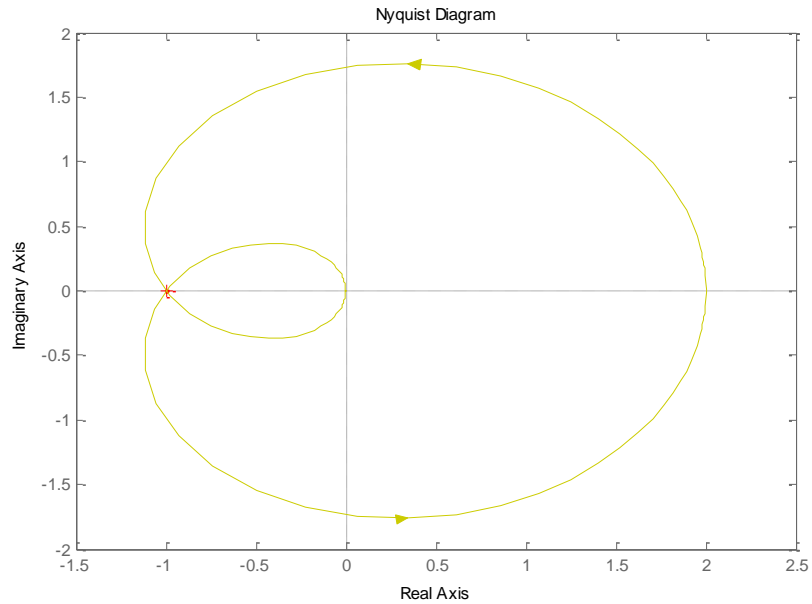
MATLAB code:

```
s = tf('s')
%a)
figure(1);
K=1
num_GH_a= K*(s+1);
den_GH_a=(s-1)^2;
GH_a=num_GH_a/den_GH_a;
nyquist(GH_a)
%b)
figure(2);
K=1
num_GH_b= K*(s-1);
den_GH_b=(s+1)^2;
GH_b=num_GH_b/den_GH_b;
nyquist(GH_b)
```

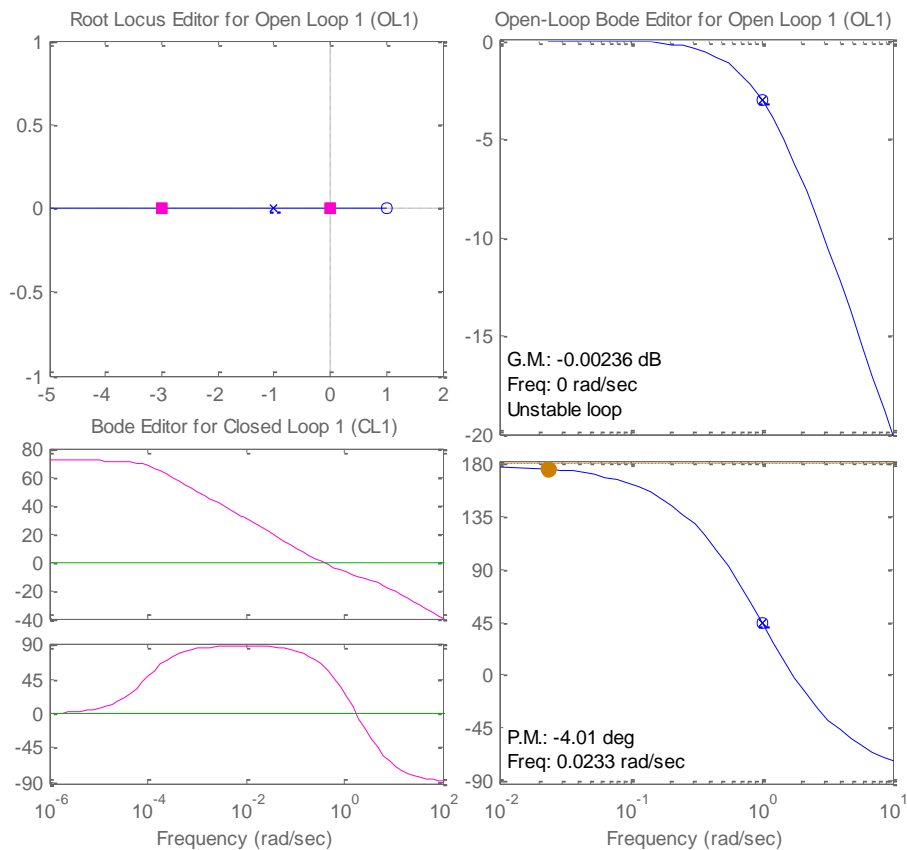
sisotool

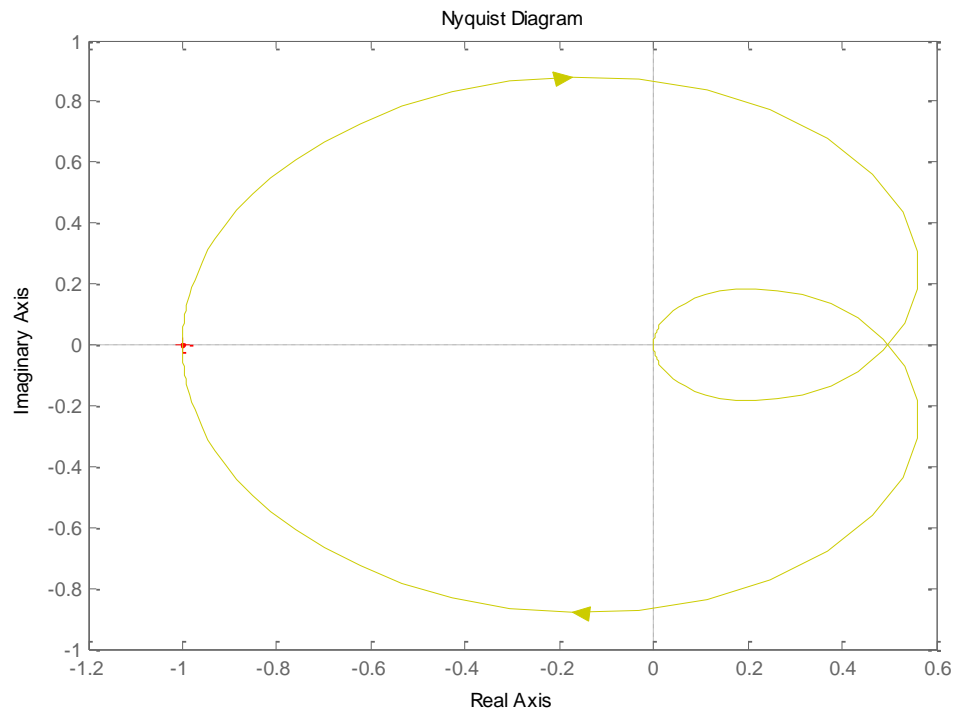
Part(a): Using MATLAB sisotool, the transfer function gain can be iteratively changed in order to obtain different phase margins. By changing the gain so that $PM=0$ (margin of stability), $K > 2$ resulted in stable Nyquist diagram for part(a). Following two figures illustrate the sisotool and Nyquist results at margin of stability for part (a).





Part(b): Similar methodology applied as in part (a). $K < 1$ results in closed loop stability. Following are sisotool and Nyquist results at margin of stability ($K=1$):



**10-29) (a)**

Let $G(s) = G_1(s)e^{-T_d s}$ Then $G_1(s) = \frac{100}{s(s^2 + 10s + 100)}$

Let $\left| \frac{100}{-10\omega^2 + j\omega(100 - \omega^2)} \right| = 1$ or $\frac{100}{\left[100\omega^4 + \omega^2(100 - \omega^2)^2 \right]^{1/2}} = 1$

Thus $100\omega^4 + \omega^2(100 - \omega^2)^2 = 10,000$ $\omega^6 - 100\omega^4 + 10,000\omega^2 - 10,000 = 0$

The real solution for ω are $\omega = \pm 1$ rad/sec.

$$\angle G_1(j\omega) = -\tan^{-1} \left[\frac{100 - \omega^2}{-10\omega} \right] \bigg|_{\omega=1} = 264.23^\circ$$

$$\begin{aligned} \text{Equating } \omega T_d \big|_{\omega=1} &= \frac{(264.23^\circ - 180^\circ)\pi}{180} \\ &= \frac{84.23\pi}{180} = 1.47 \text{ rad} \end{aligned}$$

Thus the maximum time delay for stability
is

$$T_d = 1.47 \text{ sec.}$$

(b) $T_d = 1 \text{ sec.}$

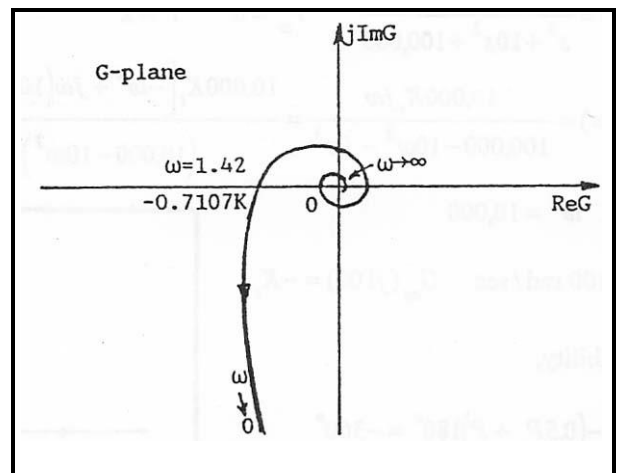
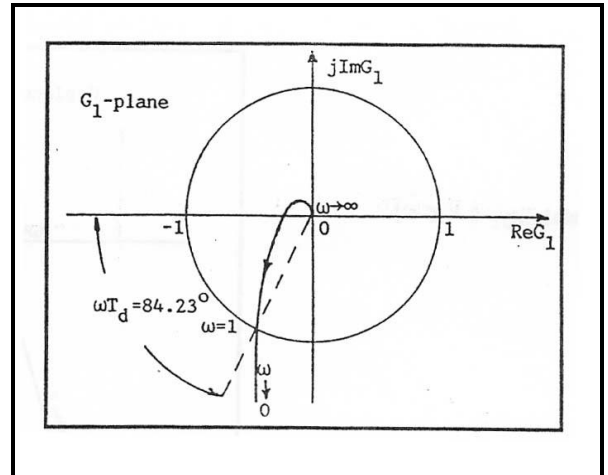
$$G(s) = \frac{100Ke^{-s}}{s(s^2 + 10s + 100)} \quad G(j\omega) = \frac{100Ke^{-j\omega}}{-10\omega^2 + j\omega(100 - \omega^2)}$$

At the intersect on the negative real axis, $\omega = 1.42 \text{ rad/sec.}$

$$G(j1.42) = -0.7107K.$$

The system is stable for

$$0 < K < 1.407$$



10-30 (a) $\kappa = 0.1$

$$G(s) = \frac{10e^{-T_d s}}{s(s^2 + 10s + 100)} = G_1(s)e^{-T_d s}$$

$$\text{Let } \left| \frac{10}{-10\omega^2 + j\omega(100 - \omega^2)} \right| = 1 \quad \text{or} \quad \frac{10}{\left[100\omega^4 + \omega^2(100 - \omega^2)^2 \right]^{1/2}} = 1$$

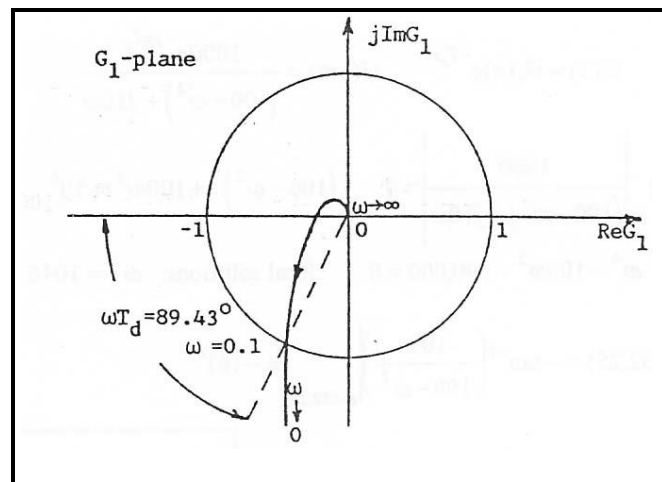
Thus $\omega^6 - 100\omega^4 + 10,000\omega^2 - 100 = 0$ The real solutions for ω is $\omega = \pm 0.1$ rad/sec.

$$\angle G_1(j0.1) = -\tan^{-1} \left[\frac{100 - \omega^2}{-10\omega} \right] \bigg|_{\omega=0.1} = 269.43^\circ$$

$$\text{Equate } \omega T_d \big|_{\omega=0.1} = \frac{(269.43^\circ - 180^\circ)\pi}{180^\circ} = 1.56 \text{ rad} \quad \text{We have } T_d = 15.6 \text{ sec.}$$

We have the maximum time delay

for stability is 15.6 sec.

**10-30 (b) $T_d = 0.1$ sec.**

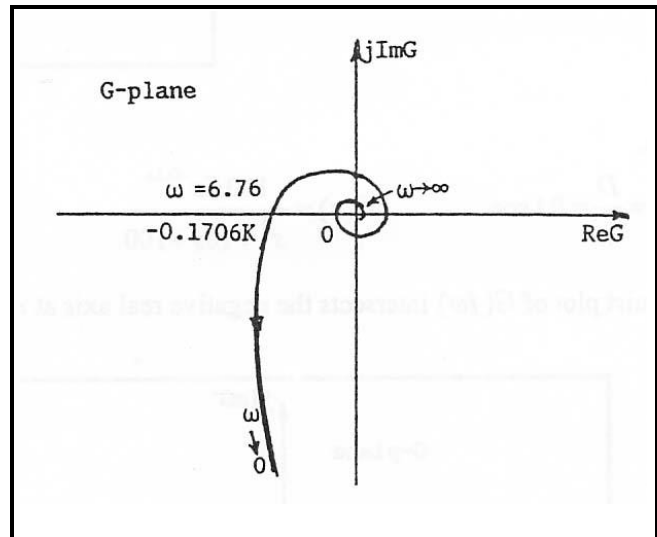
$$G(s) = \frac{100Ke^{-0.1s}}{s(s^2 + 10s + 100)} \quad G(j\omega) = \frac{100Ke^{-0.1j\omega}}{-10\omega^2 + j\omega(100 - \omega^2)}$$

At the intersect on the negative real axis,

$$\omega = 6.76 \text{ rad/sec. } G(j6.76) = -0.1706K$$

The system is stable for

$$0 < K < 5.86$$



10-31)

MATLAB code:

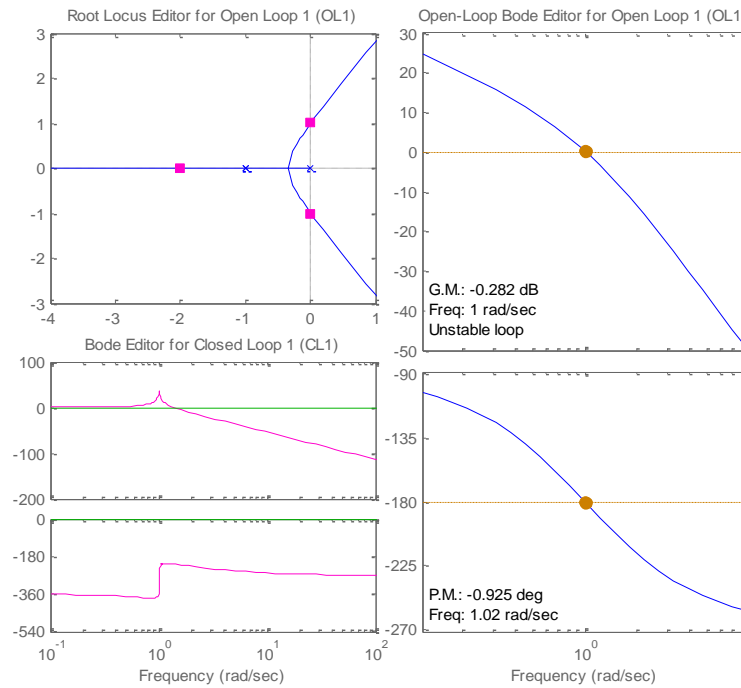
```
s = tf('s')
%a)

figure(1);
K=1
num_GH_a= K;
den_GH_a=s*(s+1)*(s+1);
GH_a=num_GH_a/den_GH_a;
nyquist(GH_a)

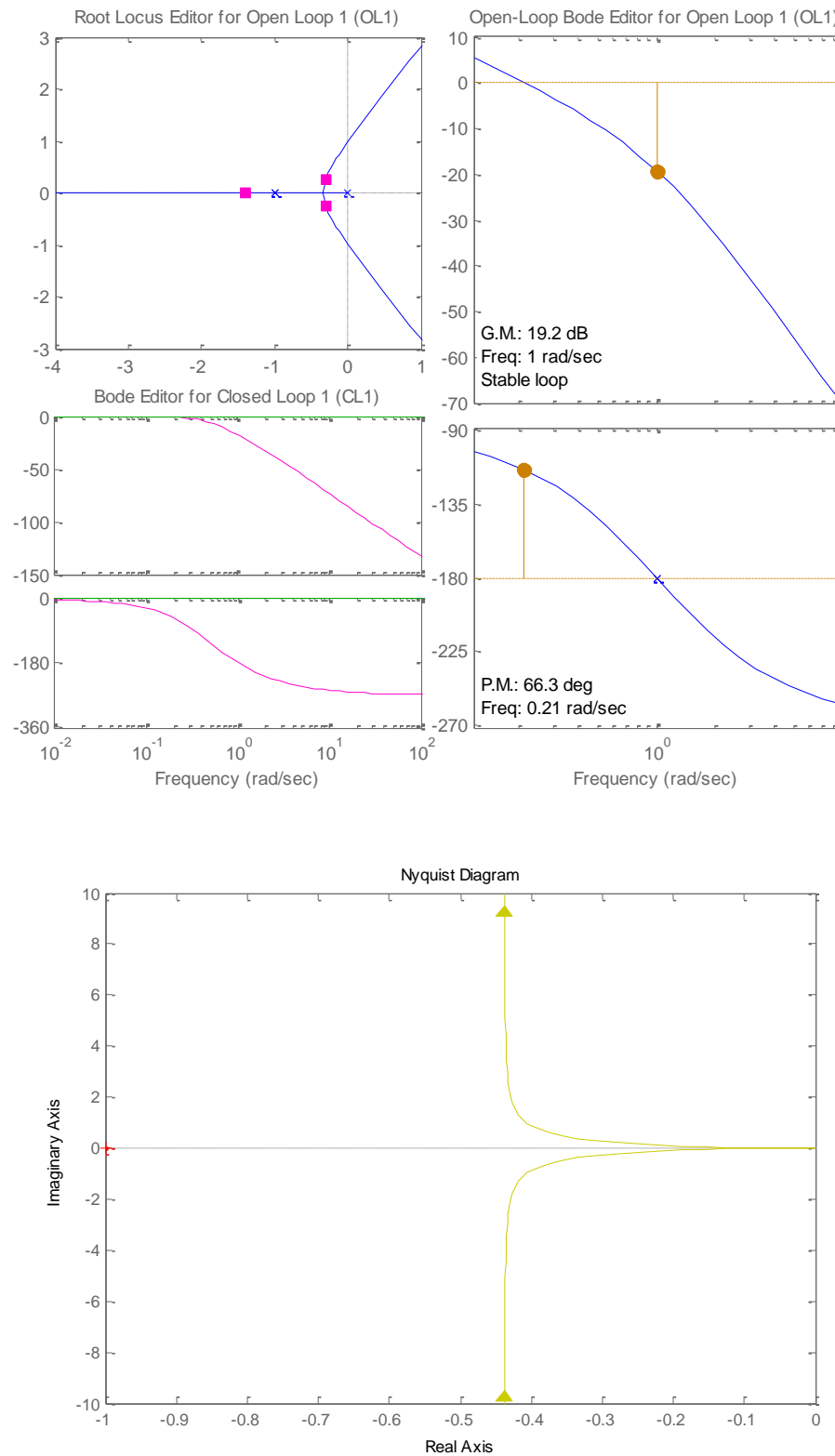
%b)
figure(2);
K=20
num_GH_b= K;
den_GH_b=s*(s+1)*(s+1);
GH_b=num_GH_b/den_GH_b;
nyquist(GH_b)

sisotool;
```

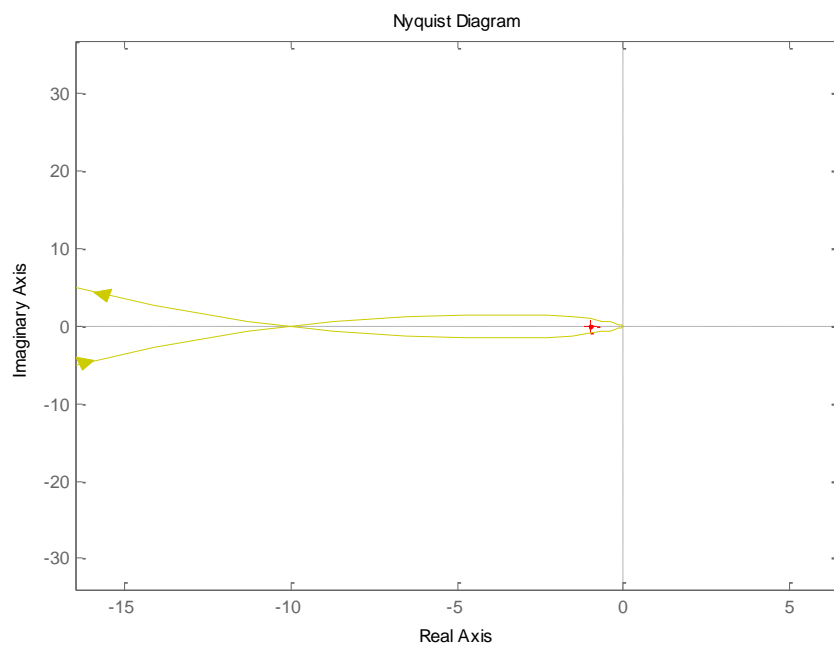
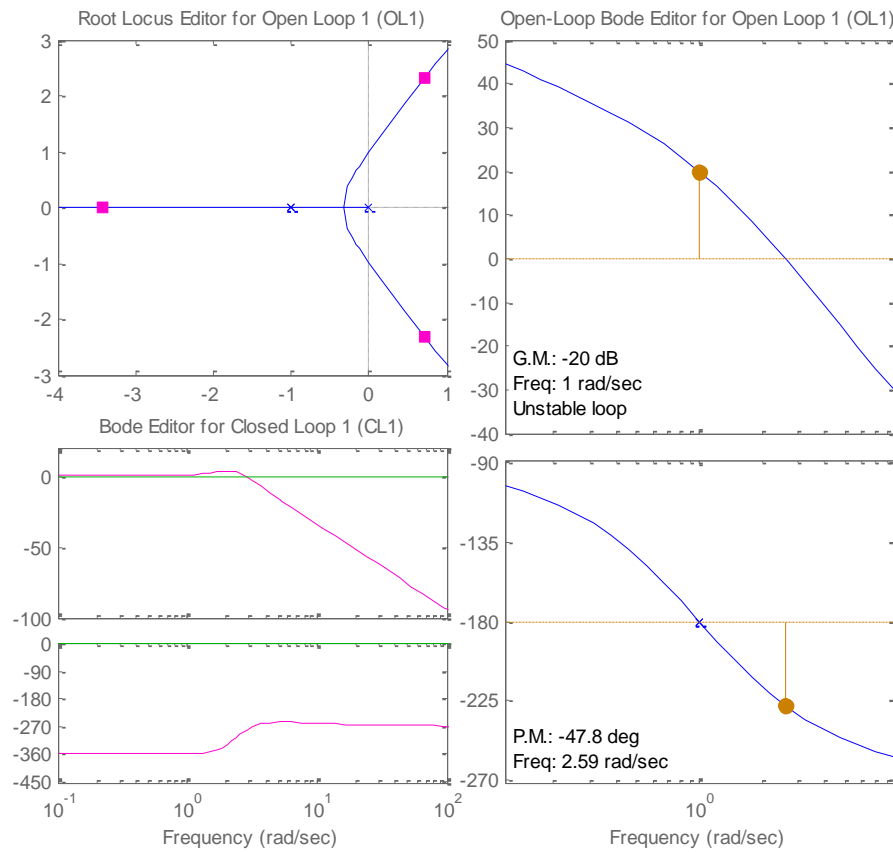
By using sisotool and importing the loop transfer function, different values of K has been tested which resulted in a stable system when $K < 2$, and unstable system for $K > 2$. Following diagrams correspond to margin of stability:



Part(a): small K resulted in stable system as shown below for $K=0.219$:



Part(b): Large K resulted in unstable system as shown below for $K=20$:



The system is stable for small value of K , since there is no encirclement of the $s = -1$

The system is unstable for large value of K , since the locus encirclement the $s = -1$ twice in CCW; which means two poles are in the right half s -plane.

10-32) (a) The transfer function (gain) for the sensor-amplifier combination is $10 \text{ V}/0.1 \text{ in} = 100 \text{ V/in}$. The velocity of flow of the solution is

$$v = \frac{10 \text{ in}^3 / \text{sec}}{0.1 \text{ in}} = 100 \text{ in/sec}$$

The time delay between the valve and the sensor is $T_d = D / v$ sec. The loop transfer function is

$$G(s) = \frac{100K e^{-T_d s}}{s^2 + 10s + 100}$$

(b) $K = 10$:

$$G(s) = G_1(s)e^{-T_d s} \quad G(j\omega) = \frac{1000e^{-j\omega T_d}}{(100 - \omega^2) + j10\omega}$$

$$\text{Setting } \left| \frac{1000}{(100 - \omega^2) + j10\omega} \right| = 1 \quad (100 - \omega^2)^2 + 100\omega^2 = 10^6$$

$$\text{Thus, } \omega^4 - 100\omega^2 - 990,000 = 0 \quad \text{Real solutions: } \omega^2 = 1046.2 \quad \omega = 32.35 \text{ rad/sec}$$

$$\angle G_1(j32.25) = -\tan^{-1}\left(\frac{10\omega}{100 - \omega^2}\right)_{\omega=32.25} = -161^\circ$$

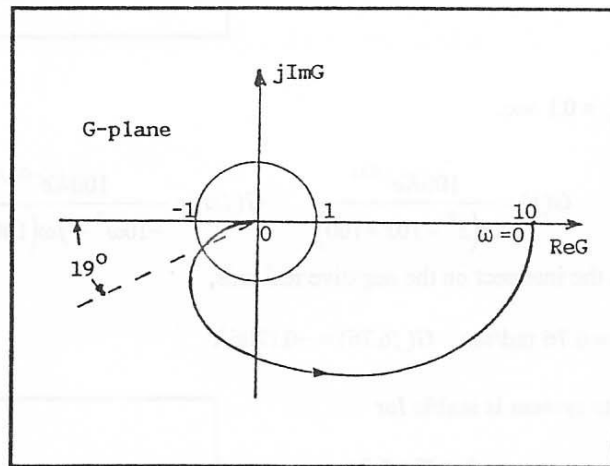
Thus,

$$32.35T_d = \frac{19^\circ \pi}{180^\circ} = 0.33 \text{ rad}$$

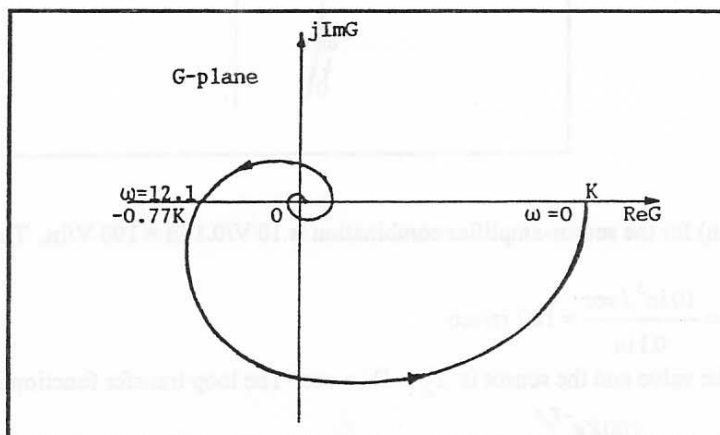
Thus,

$$T_d = 0.0103 \text{ sec}$$

$$\text{Maximum } D = vT_d = 100 \times 0.0103 = 103 \text{ in}$$

(c) $D = 10 \text{ in.}$

$$T_d = \frac{D}{v} = 0.1 \text{ sec} \quad G(s) = \frac{100Ke^{-0.1s}}{s^2 + 10s + 100}$$

The Nyquist plot of $G(j\omega)$ intersects the negative real axis at $\omega = 12.1 \text{ rad/sec.}$ 

10-33)

- (a)** The transfer function (gain) for the sensor-amplifier combination is $1 \text{ V}/0.1 \text{ in} = 10 \text{ V/in}$. The velocity of flow of the solutions is

$$v = \frac{10 \text{ in}^3 / \text{sec}}{0.1 \text{ in}} = 100 \text{ in} / \text{sec}$$

The time delay between the valve and sensor is $T_d = D / v$ sec. The loop transfer function is

$$G(s) = \frac{10Ke^{-T_d s}}{s^2 + 10s + 100}$$

(b) $K = 10$:

$$G(s) = G_1(s)e^{-T_d s} \quad G(j\omega) = \frac{100e^{-j\omega T_d}}{(100 - \omega^2) + j10\omega}$$

$$\text{Setting } \left| \frac{100}{(100 - \omega^2) + j10\omega} \right| = 1 \quad (100 - \omega^2)^2 + 100\omega^2 = 10,000$$

Thus, $\omega^4 - 100\omega^2 = 0$ Real solutions: $\omega = 0, \omega = \pm 10 \text{ rad/sec}$

$$\angle G_1(j10) = -\tan^{-1} \left(\frac{10\omega}{100 - \omega^2} \right) \bigg|_{\omega=10} = -90^\circ$$

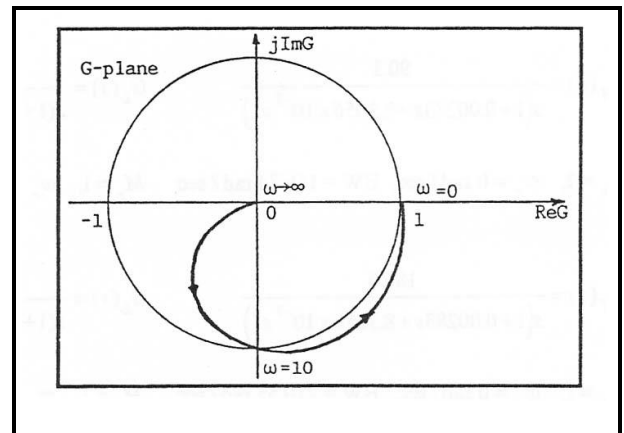
Thus,

$$10T_d = \frac{90^\circ \pi}{180^\circ} = \frac{\pi}{2} \text{ rad}$$

Thus,

$$T_d = \frac{\pi}{20} = 0.157 \text{ sec}$$

Maximum $D = vT_d = 100 \times 0.157 = 15.7 \text{ in}$

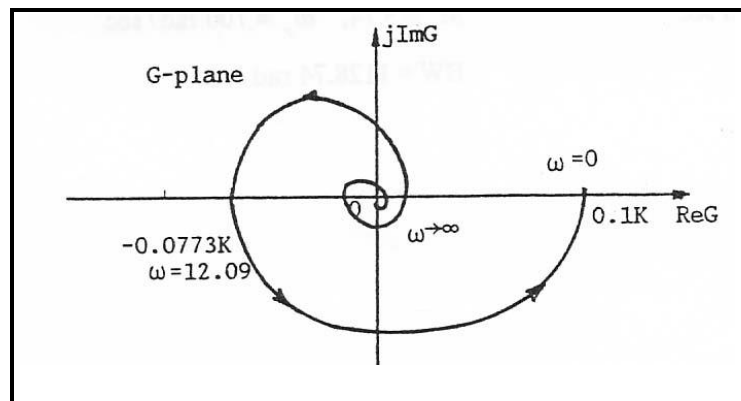


(c) $D = 10$ in.

$$T_d = \frac{D}{v} = \frac{10}{100} = 0.1 \text{ sec} \quad G(s) = \frac{10Ke^{-0.1s}}{s^2 + 10s + 100}$$

The Nyquist plot of $G(j\omega)$ intersects the negative real axis at $\omega = 12.09$ rad/sec. $G(j) = -0.0773K$

For stability, the maximum value of K is 12.94 .



10-34)

The system (GH) has zero poles in the right of s plane: $P=0$.

According to Nyquist criteria ($Z=N+P$), to ensure the stability which means the number of right poles of $1+GH=0$ should be zero ($Z=0$), we need N clockwise encirclements of Nyquist diagram about $-1+0j$ point. That is $N=-P$ or in other words, we need P counter-clockwise encirclement about $-1+0j$. In this case, we need 0 CCW encirclements.

10-34(a) According to Nyquist diagrams, this happens when $K < -1$. The three Nyquist diagrams are plotted with $K=-10$, $K=-1$, $K=10$ as examples:

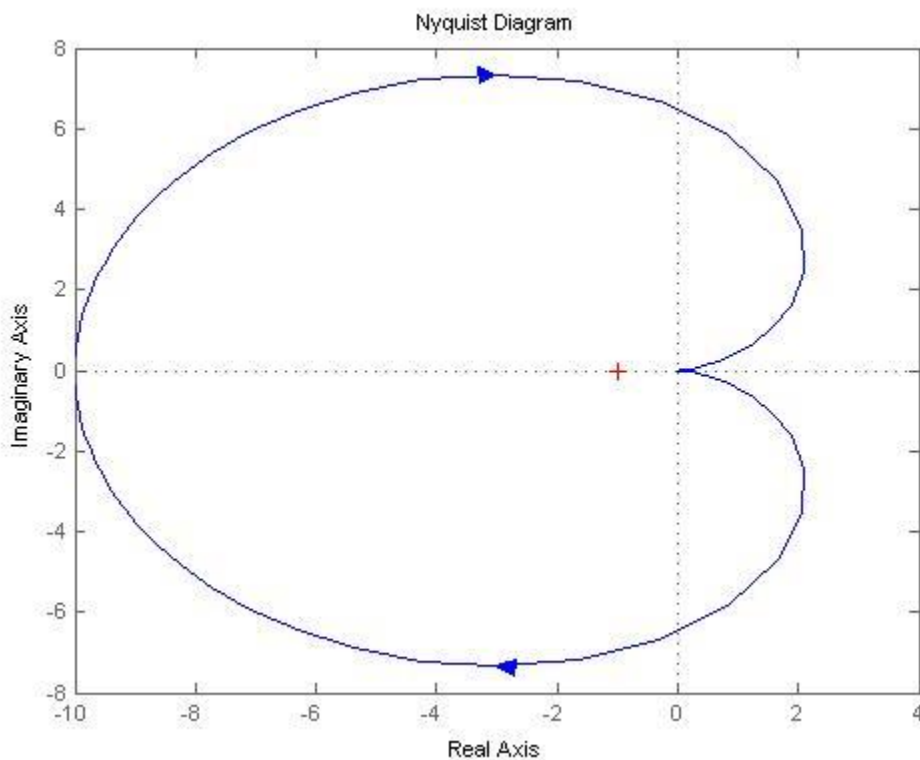
MATLAB code:

```
s = tf('s')
%a)
figure(1);
K=-10
num_G_a = K ;
```

```

den_G_a = (s+1);
num_H_a = (s+2);
den_H_a = (s^2+2*s+2);
G_a=num_G_a/den_G_a;
H_a = num_H_a/den_H_a;
OL_a = G_a*H_a
nyquist(OL_a)

```

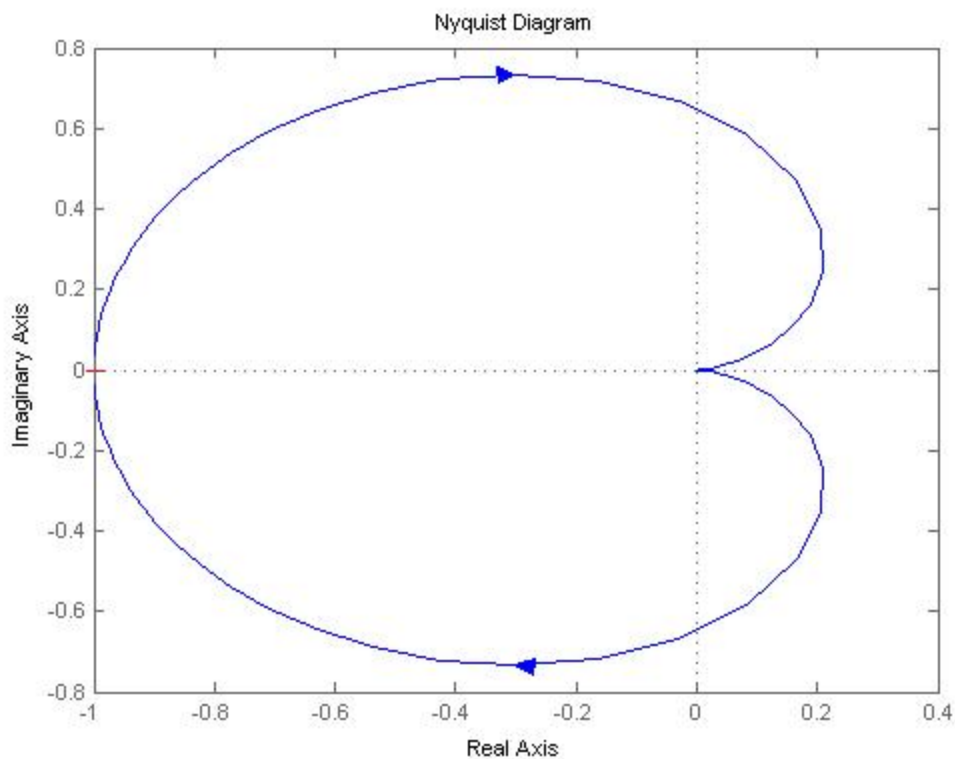


Case 1) Nyquist graph, $K=-10$: margin of stability $K < -1$ unstable

```

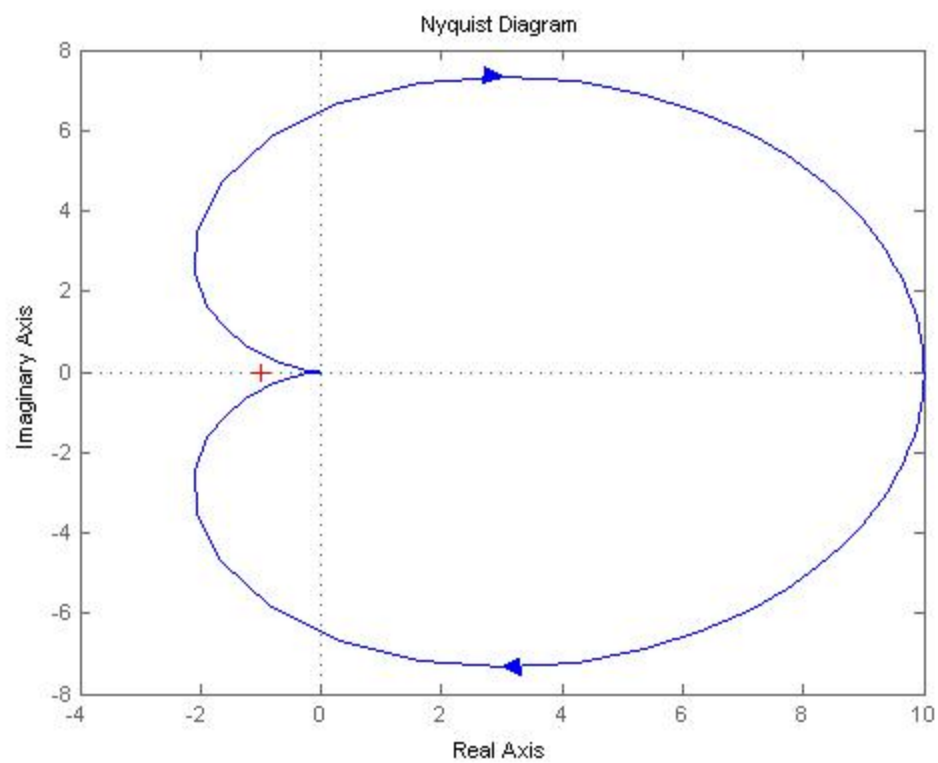
figure(2);
K=-1
num_G_a = K ;
den_G_a = (s+1);
num_H_a = (s+2);
den_H_a = (s^2+2*s+2);
G_a=num_G_a/den_G_a;
H_a = num_H_a/den_H_a;
%CL_a = G_a/(1 + G_a*H_a);
OL_a = G_a*H_a
nyquist(OL_a)

```



Case 2) Nyquist graph, $K=-1$: marginally unstable

```
figure(3);
K=10
num_G_a = K ;
den_G_a = (s+1);
num_H_a = (s+2);
den_H_a = (s^2+2*s+2);
G_a=num_G_a/den_G_a;
H_a = num_H_a/den_H_a;
%CL_a = G_a/(1 + G_a*H_a);
OL_a = G_a*H_a
nyquist(OL_a)
```



Case 3) Nyquist graph, $K=10$: stable case, $-1 < K$ no CCW encirclement about $-1+0j$ point

10-34 (b)

For $K < -1$ (unstable), there will be 1 real pole in the right hand side of s-plane for the closed loop system, by running the following code.

```
K=-10
num_G_a = K ;
den_G_a = (s+1) ;
num_H_a = (s+2) ;
den_H_a = (s^2+2*s+2) ;
G_a=num_G_a/den_G_a;
H_a = num_H_a/den_H_a;
OL_a = G_a*H_a
nyquist(OL_a)
CL=1/(1+OL_a)
pole(CL)
```

K =

-10

Transfer function:

$$\frac{-10 s - 20}{s^3 + 3 s^2 + 4 s + 2}$$

Transfer function:

$$\frac{s^3 + 3 s^2 + 4 s + 2}{s^3 + 3 s^2 - 6 s - 18}$$

ans =

2.4495
-3.0000
-2.4495

For $K = -1$ (marginally unstable), there will be 2 negative complex conjugate poles and a pole at zero for the closed loop system, by running the following code.

```
K=-1
num_G_a = K ;
den_G_a = (s+1) ;
num_H_a = (s+2) ;
den_H_a = (s^2+2*s+2) ;
```

```
G_a=num_G_a/den_G_a;
H_a = num_H_a/den_H_a;
OL_a = G_a*H_a
nyquist(OL_a)
CL=1/(1+OL_a)
pole(CL)
```

K =

-1

Transfer function:

$-s - 2$

 $s^3 + 3 s^2 + 4 s + 2$

Transfer function:

$s^3 + 3 s^2 + 4 s + 2$

 $s^3 + 3 s^2 + 3 s$

ans =

0

-1.5000 + 0.8660i

-1.5000 - 0.8660i

Note: you may also wish to use MATLAB `sisotool`.
See alternative solution to 10-38.

10-34(c) The Characteristic Equation is: $s^3 + 3s^2 + (4+K)s + 2+2K$

Using Routh criterions, the coefficient table is as follows:

s^3	1	$4+K$
s^2	3	$2K+2$
s^1	$K+10$	0
s^0	$2K+2$	0

The system is stable if the content of the 1st column is positive:

$$K+10 > 0 \rightarrow K > -10$$

$$2K+2 > 0 \rightarrow K > -1$$

which is consistent with the results of the Nyquist diagrams. For $K > -1$ system is **stable**.

10-35) (a) $M_r = 2.06$, $\omega_r = 9.33$ rad/sec, BW = 15.2 rad/sec**(b)**

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = 2.06 \quad \zeta^4 - \zeta^2 + 0.0589 = 0 \quad \text{The solution for } \zeta < 0.707 \text{ is } \zeta = 0.25.$$

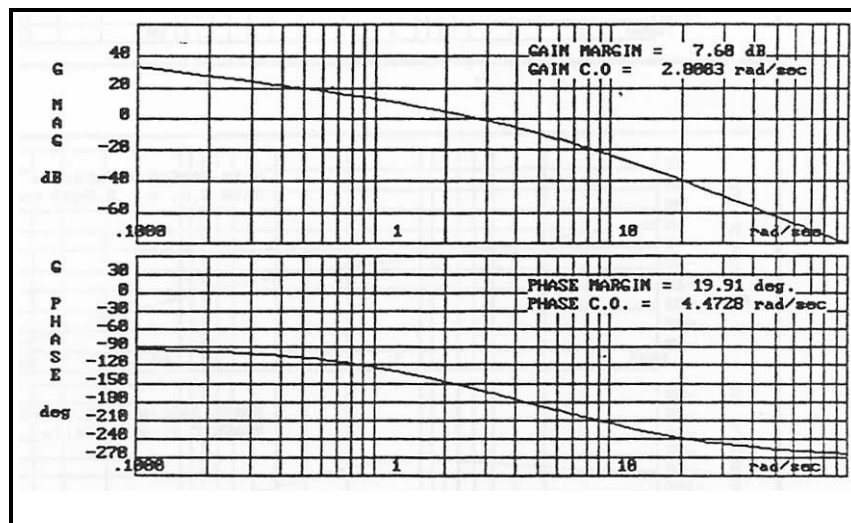
$$\omega_r \sqrt{1-2\zeta^2} = 9.33 \text{ rad/sec} \quad \text{Thus } \omega_n = \frac{9.33}{0.9354} = 9.974 \text{ rad/sec}$$

$$G_L(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)} = \frac{99.48}{s(s + 4.987)} = \frac{19.94}{s(1 + 0.2005s)} \quad \text{BW} = 15.21 \text{ rad/sec}$$

10-36) Assuming a unity feedback loop ($H=1$), $G(s)H(s)=G(s)$

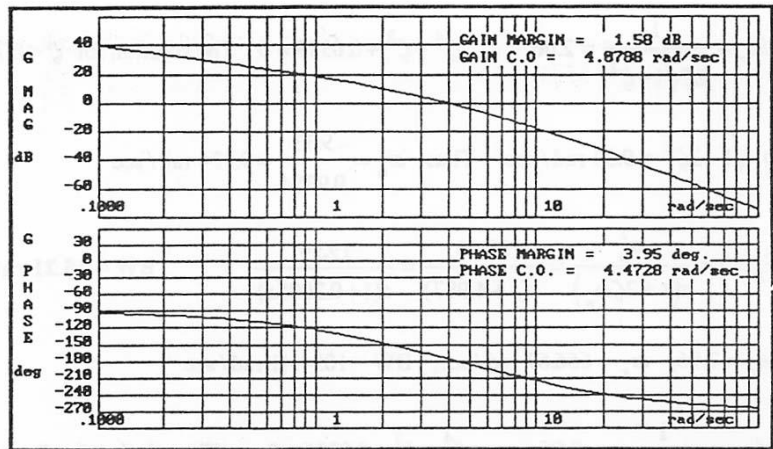
(a)

$$G(s) = \frac{5}{s(1+0.5s)(1+0.1s)}$$



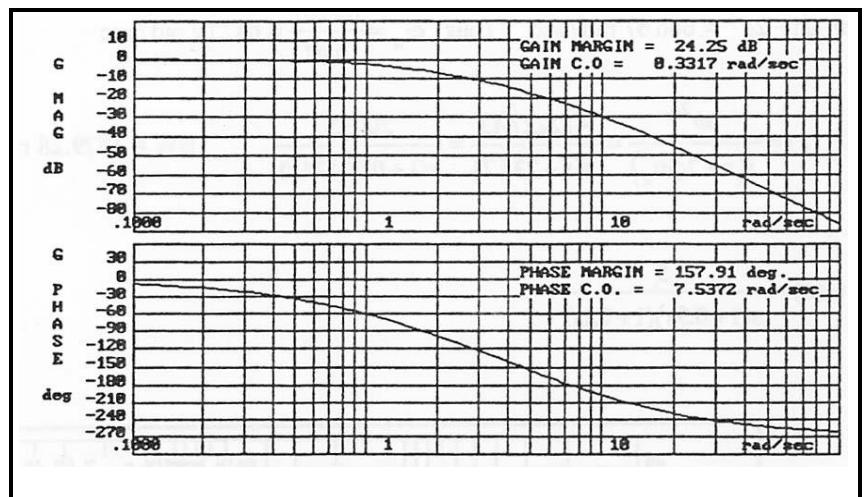
10-36 (b)

$$G(s) = \frac{10}{s(1+0.5s)(1+0.1s)}$$



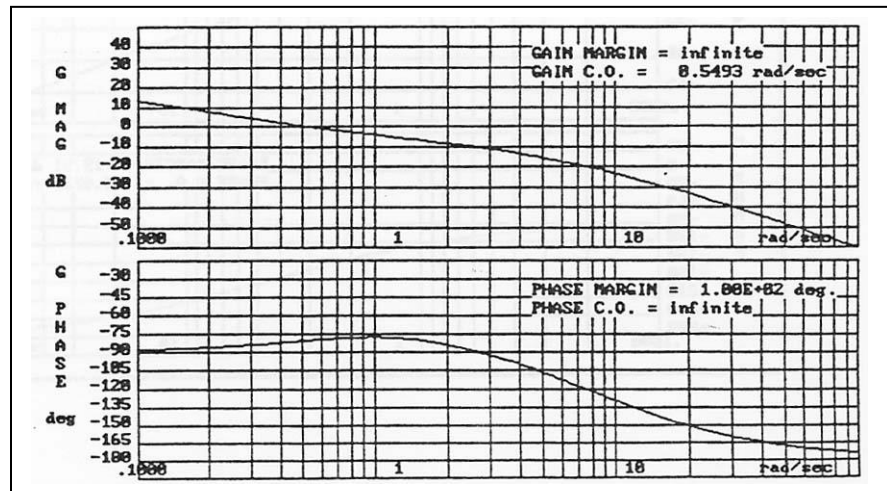
(c)

$$G(s) = \frac{500}{(s+1.2)(s+4)(s+10)}$$



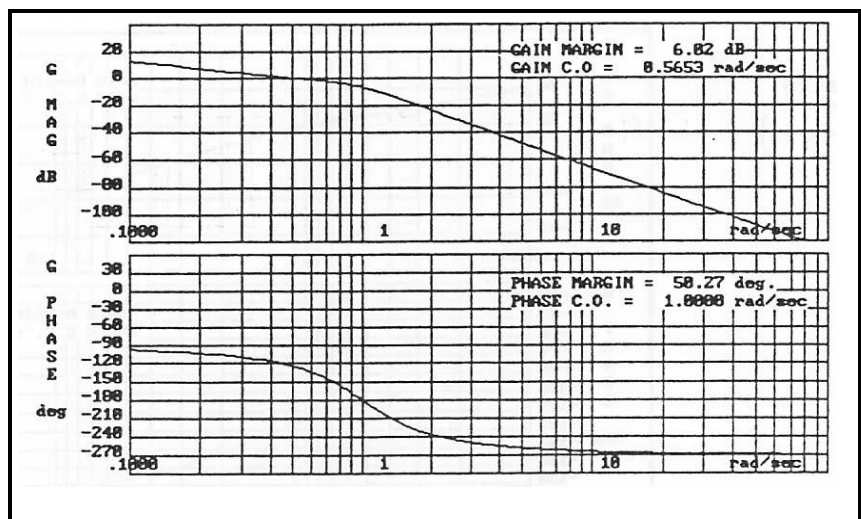
(d)

$$G(s) = \frac{10(s+1)}{s(s+2)(s+10)}$$



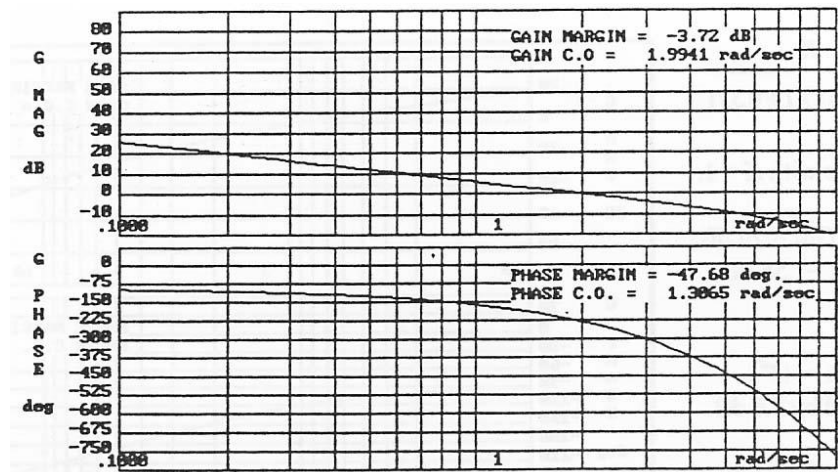
10-36 (e)

$$G(s) = \frac{0.5}{s(s^2 + s + 1)}$$



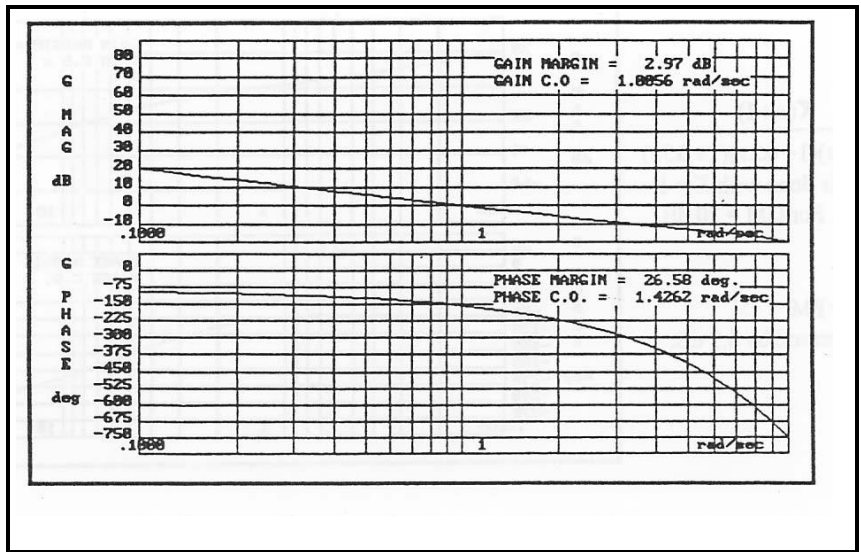
(f)

$$G(s) = \frac{100e^{-s}}{s(s^2 + 10s + 50)}$$



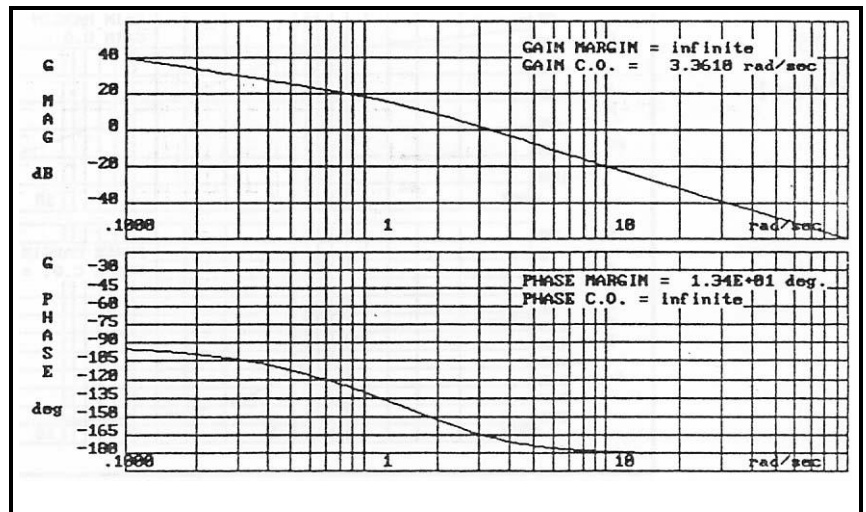
(g)

$$G(s) = \frac{100e^{-s}}{s(s^2 + 10s + 100)}$$



10-36 (h)

$$G(s) = \frac{10(s+5)}{s(s^2 + 5s + 5)}$$



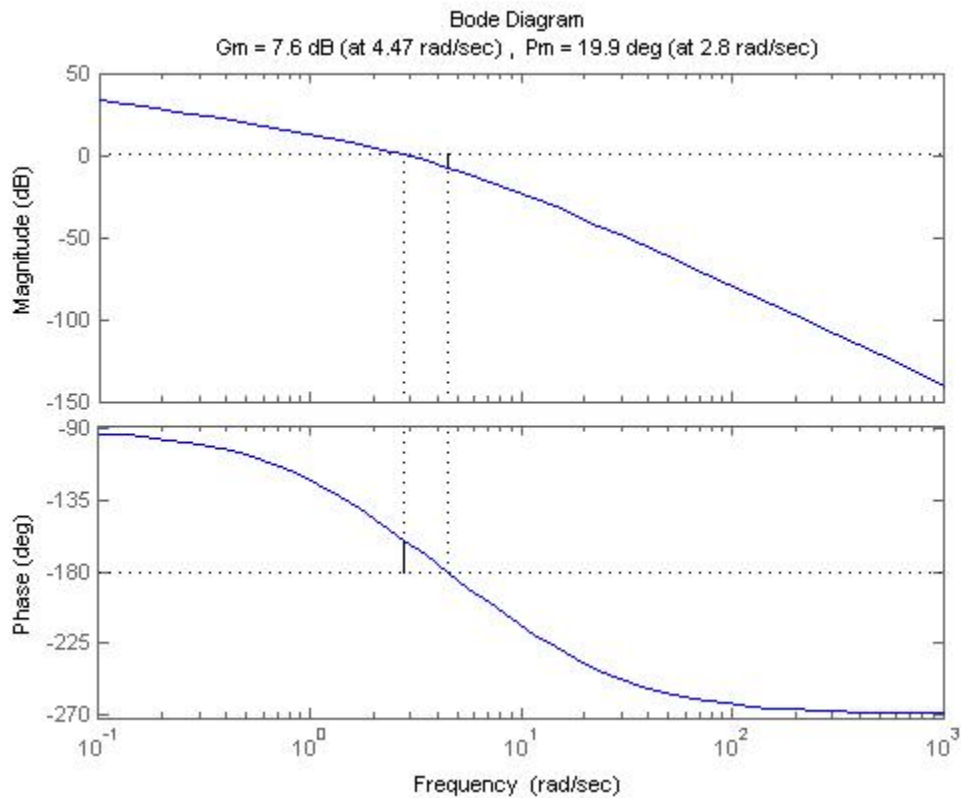
(a)

MATLAB code:

```

s = tf('s')
num_G_a= 5;
den_G_a=s*(0.5*s+1)*(0.1*s+1);
G_a=num_G_a/den_G_a
margin(G_a)

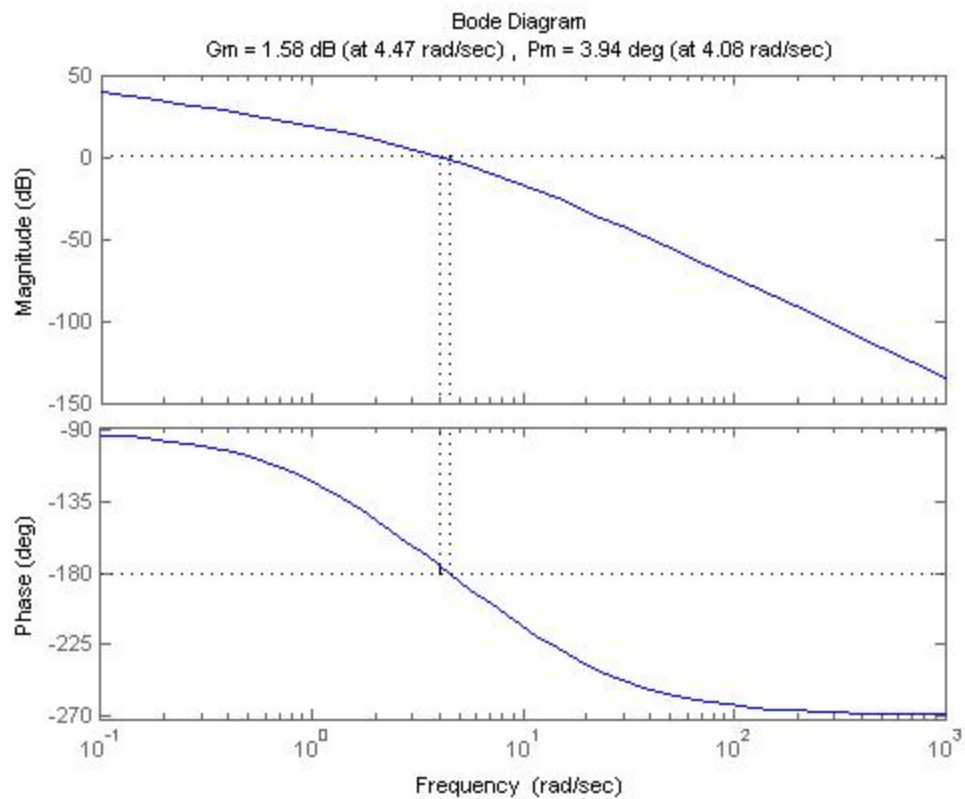
```

Bode diagram:**(b)****MATLAB code:**

```

s = tf('s')
num_G_a= 10;
den_G_a=s*(1+0.5*s)*(1+0.1*s);
G_a=num_G_a/den_G_a
margin(G_a)

```



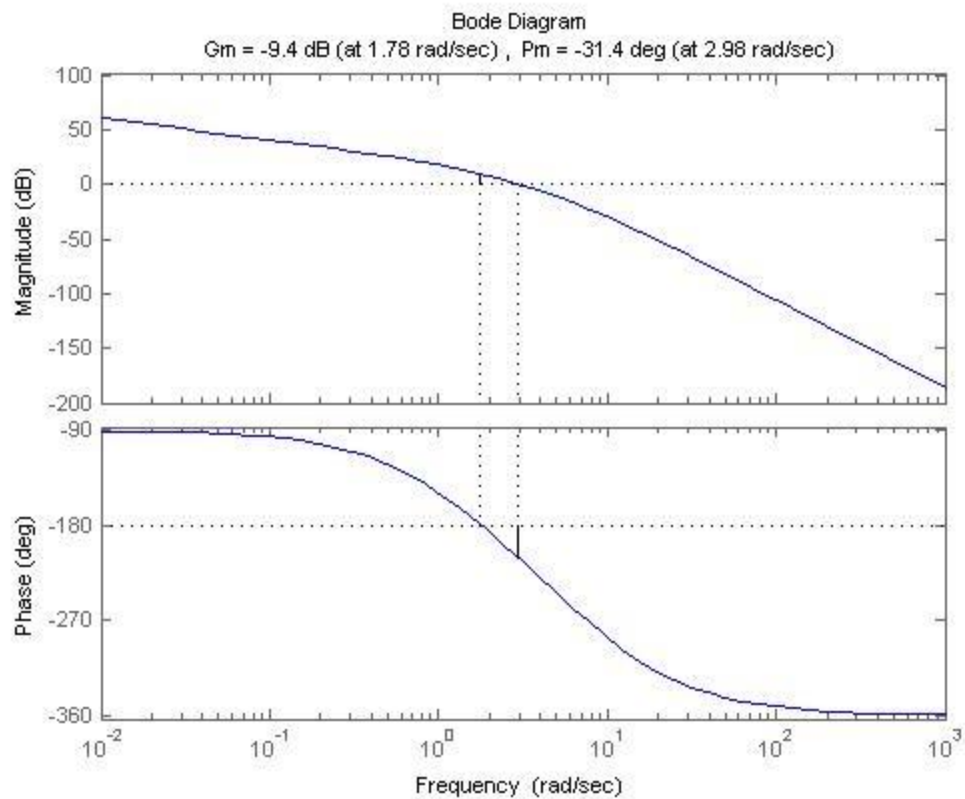
(c)

MATLAB code:

```

s = tf('s')
num_G_a= 500;
den_G_a=s*(s+1.2)*(s+4)*(s+10);
G_a=num_G_a/den_G_a
margin(G_a)

```



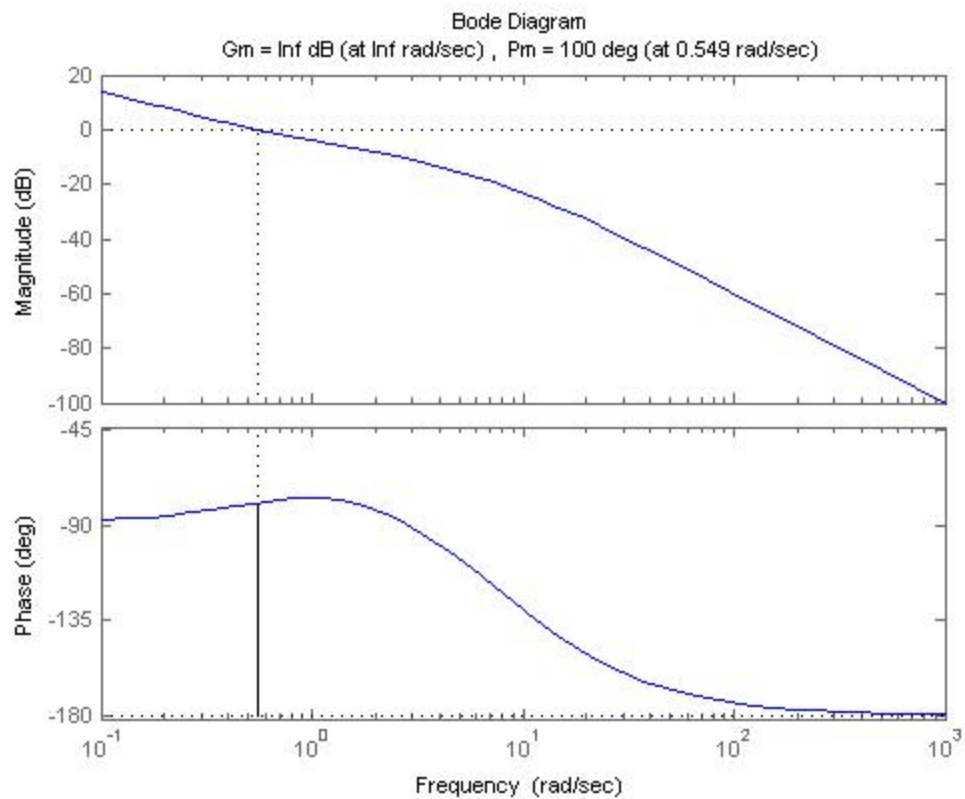
(d)

MATLAB code:

```

s = tf('s')
num_G_a= 10*(s+1);
den_G_a=s*(s+2)*(s+10);
G_a=num_G_a/den_G_a
margin(G_a)

```



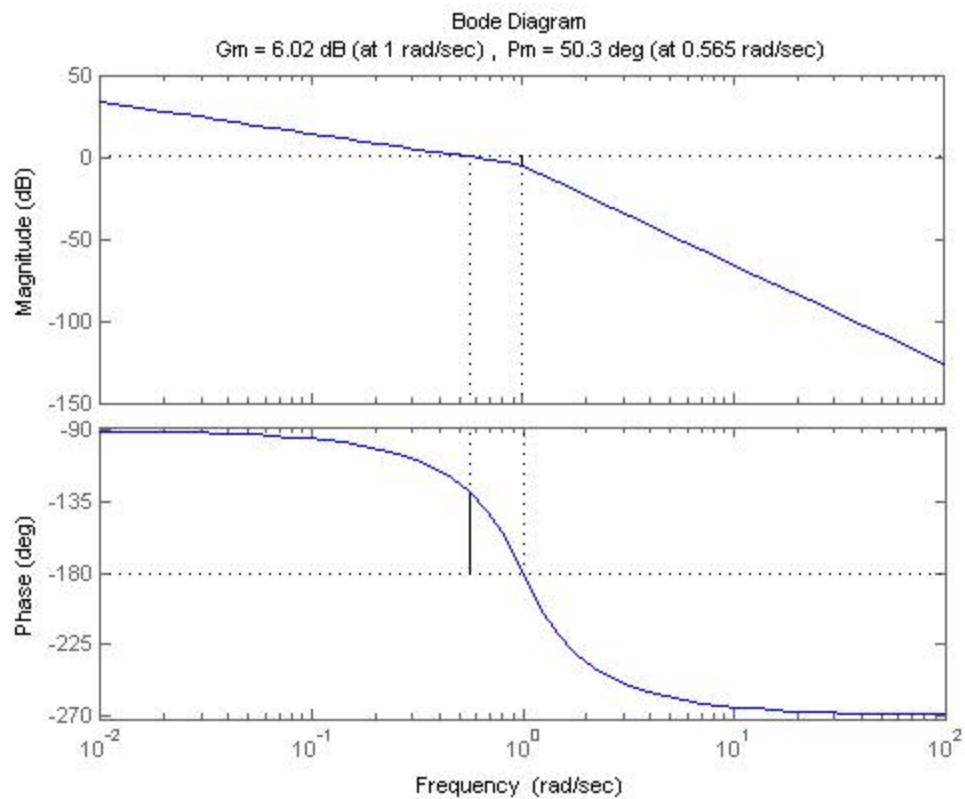
(e)

MATLAB code:

```

s = tf('s')
num_G_a= 0.5;
den_G_a=s*(s^2+s+1);
G_a=num_G_a/den_G_a
margin(G_a)

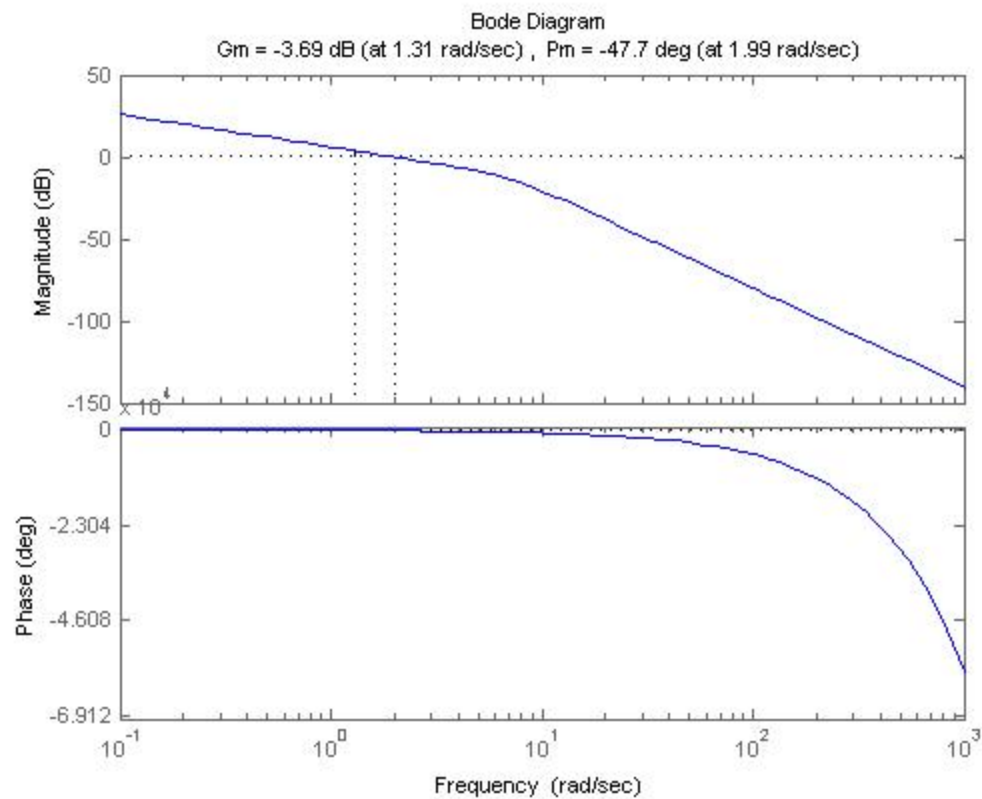
```



(f)

MATLAB code:

```
s = tf('s')
num_G_a= 100*exp(-s);
den_G_a=s*(s^2+10*s+50);
G_a=num_G_a/den_G_a
margin(G_a)
```



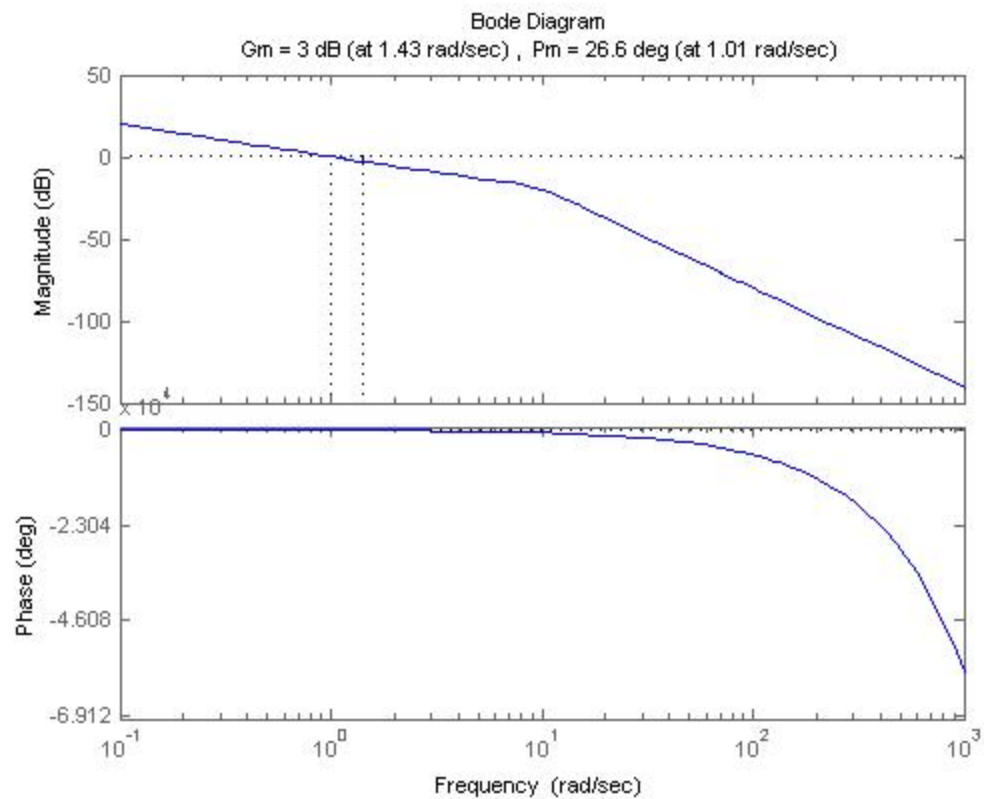
(g)

MATLAB code:

```

s = tf('s')
num_G_a= 100*exp(-s);
den_G_a=s*(s^2+10*s+100);
G_a=num_G_a/den_G_a
margin(G_a)

```



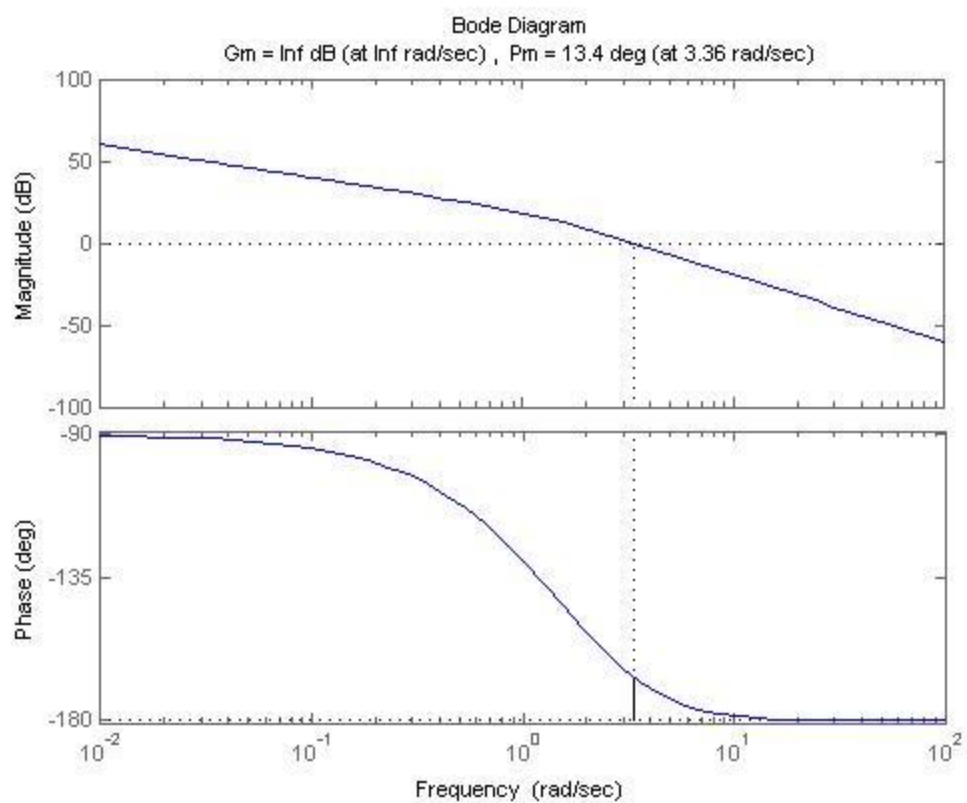
(h)

MATLAB code:

```

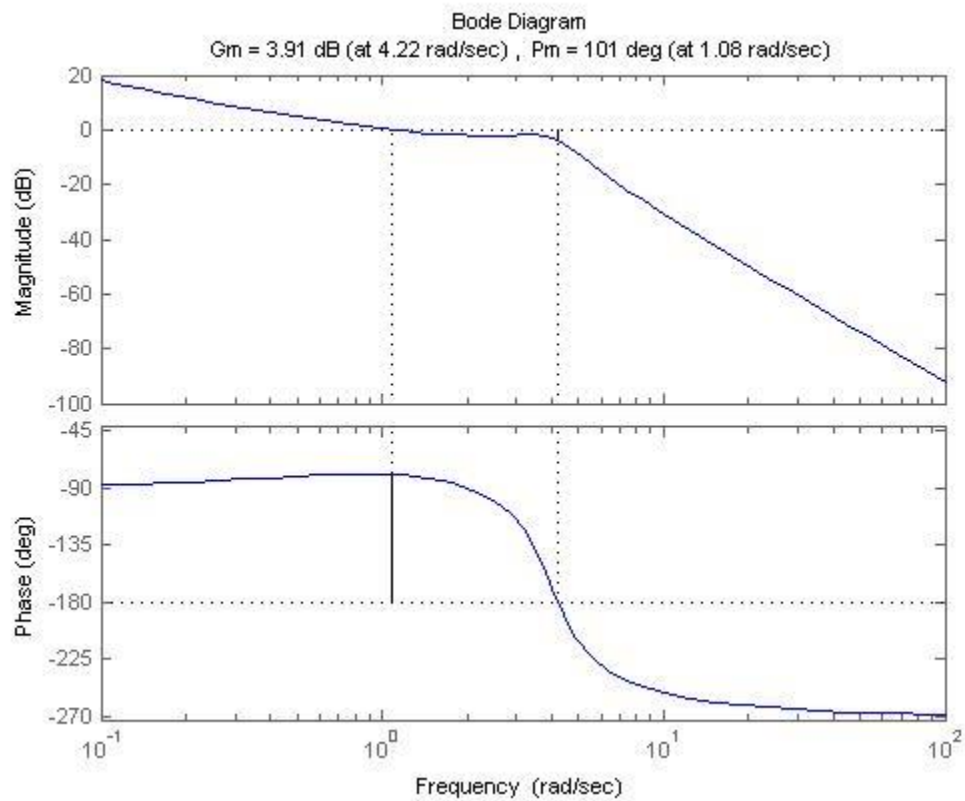
s = tf('s')
num_G_a= 10*(s+5);
den_G_a=s*(s^2+5*s+5);
G_a=num_G_a/den_G_a
margin(G_a)

```



10-37)**MATLAB code:**

```
s = tf('s')
num_GH_a= 25*(s+1);
den_GH_a=s*(s+2)*(s^2+2*s+16);
GH_a=num_GH_a/den_GH_a
margin(GH_a)
```



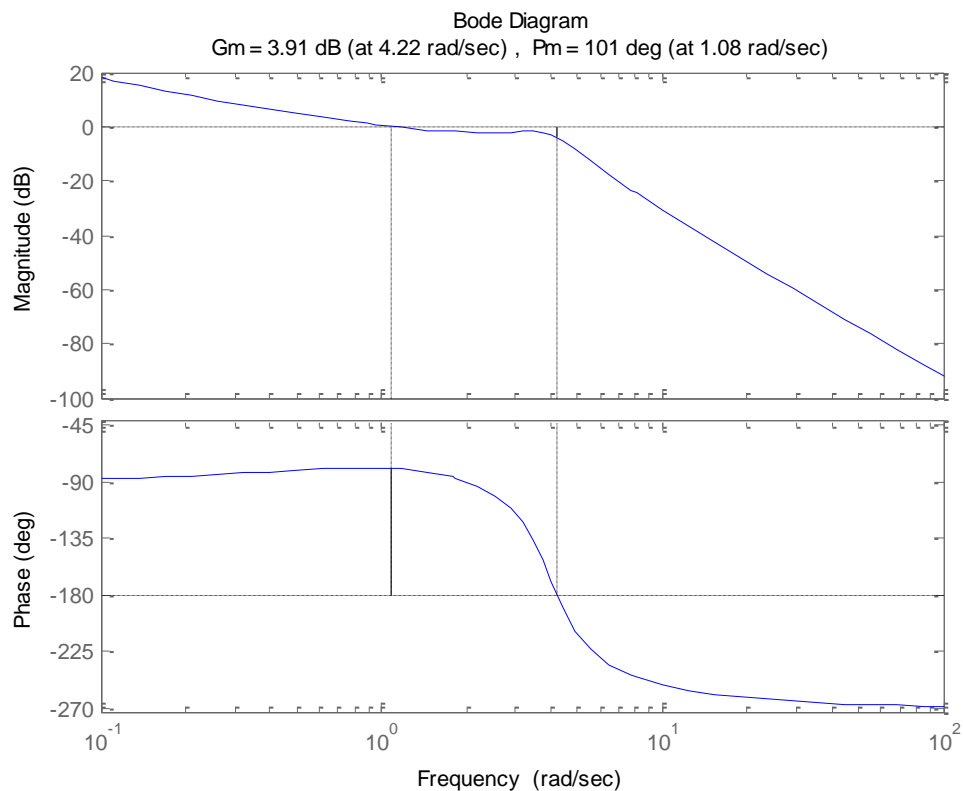
10-38) MATLAB code:

```

s = tf('s')
num_G_a= 25*(s+1);
den_G_a=s*(s+2)*(s^2+2*s+16);
G_a=num_G_a/den_G_a
margin(G_a)

```

Bode diagram: PM=101 deg, GM=3.91 dB @ 4.22 rad/sec

**10-38 Alternative solution**

MATLAB code:

```

s = tf('s')

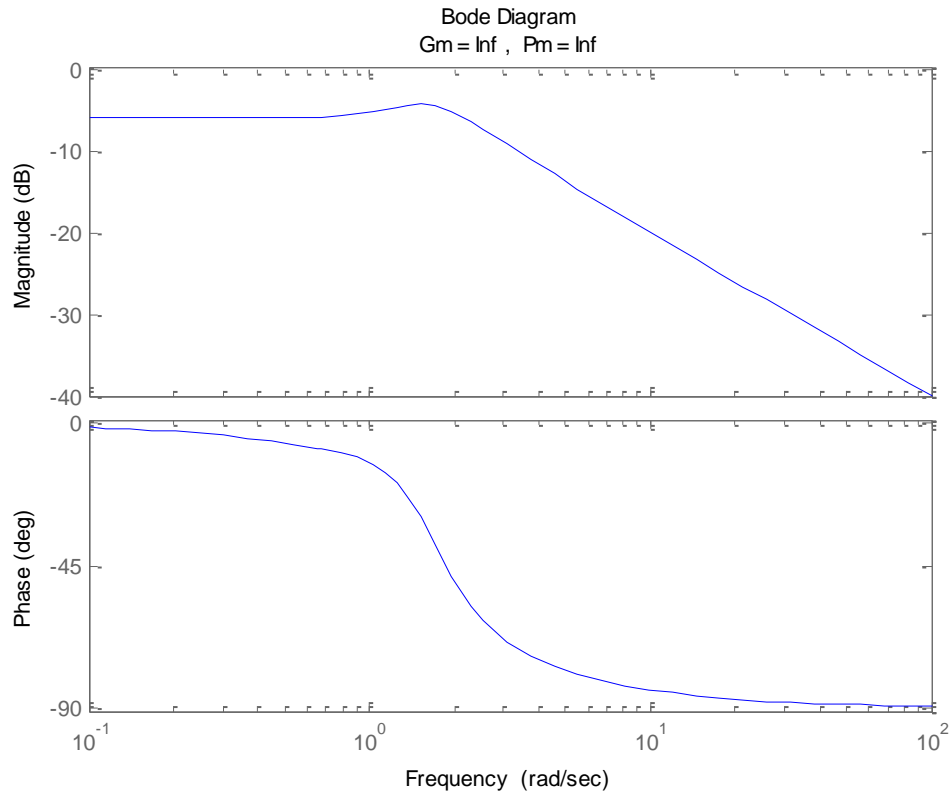
%a)
figure(1);
num_G_a = 1 ;
den_G_a = (s+1);
num_H_a = (s+2);
den_H_a = (s^2+2*s+2);

```

```
G_a=num_G_a/den_G_a;
H_a = num_H_a/den_H_a;
CL_a = G_a/(1 + G_a*H_a);
margin(CL_a)
```

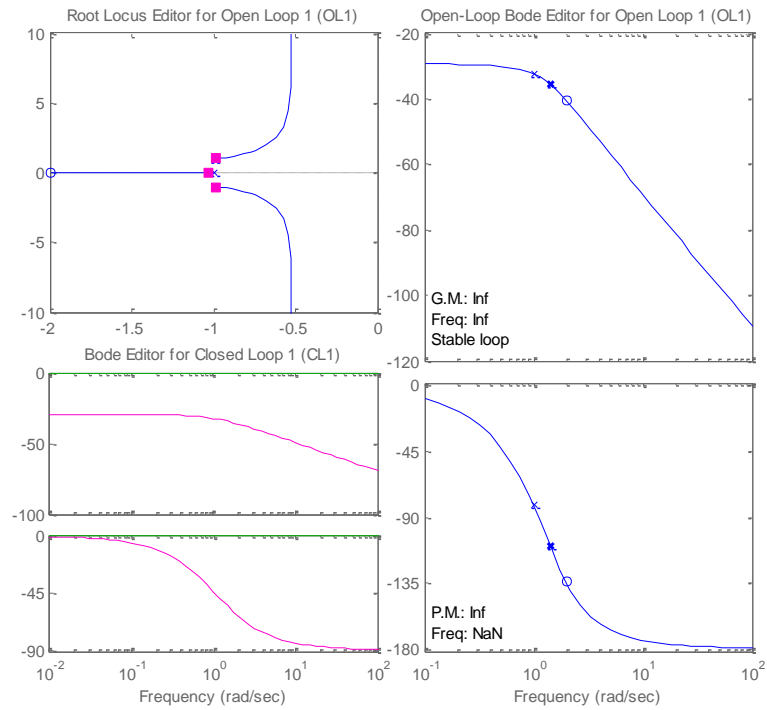
```
sisotool
```

Bode diagram: for k=1

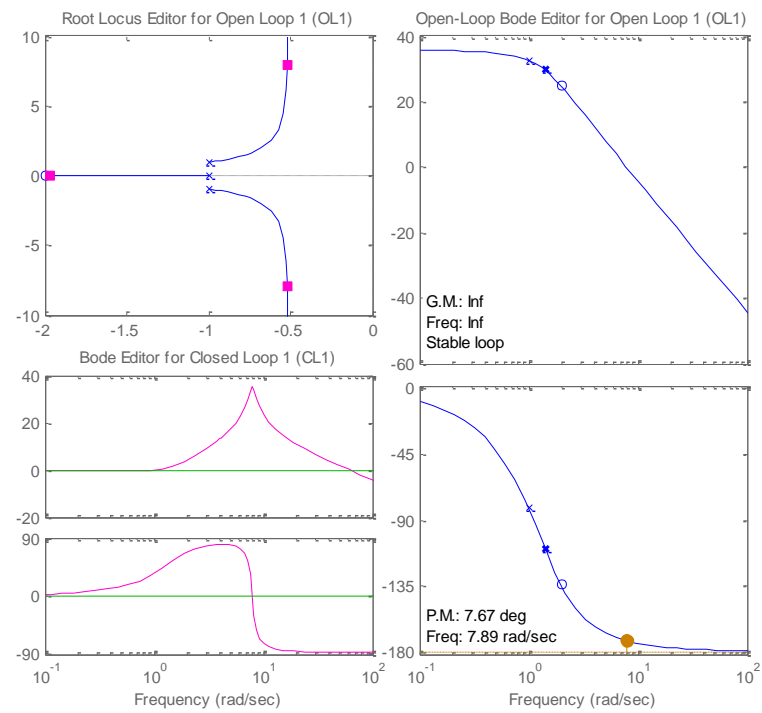


Using MATLAB sisotool, the transfer function gain can be iteratively changed in order to obtain different phase margins. By changing the gain K between very small and very big numbers, it was found that the closed loop system are stable (positive PM) **for every positive K in this system.**

K=0.034



K=59.9



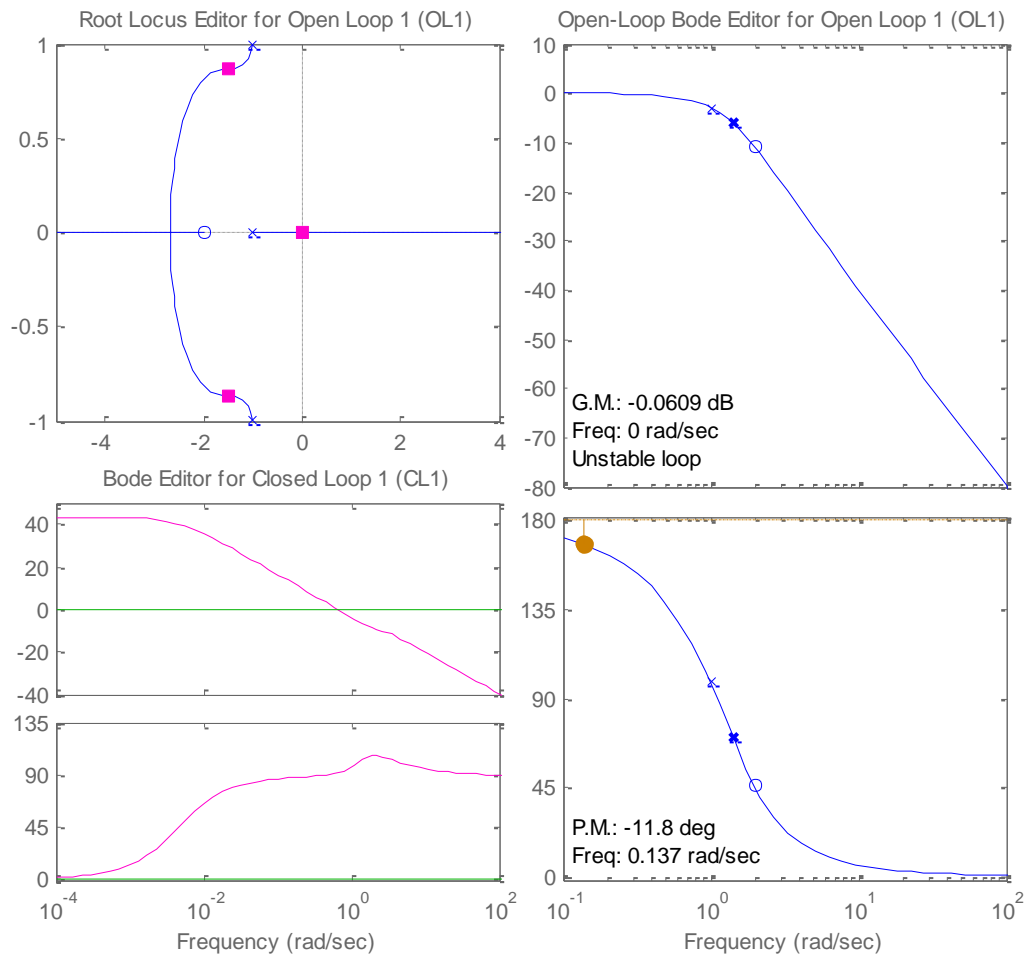
In order to test the negative range of K , -1 was multiplied to the loop transfer function through the following code, and sisotool was used again.

```
figure(1);
num_G_a = -1 ;
den_G_a = (s+1);
num_H_a = (s+2);
den_H_a = (s^2+2*s+2);
G_a=num_G_a/den_G_a;
H_a = num_H_a/den_H_a;
CL_a = G_a/(1 + G_a*H_a);
margin(CL_a)
```

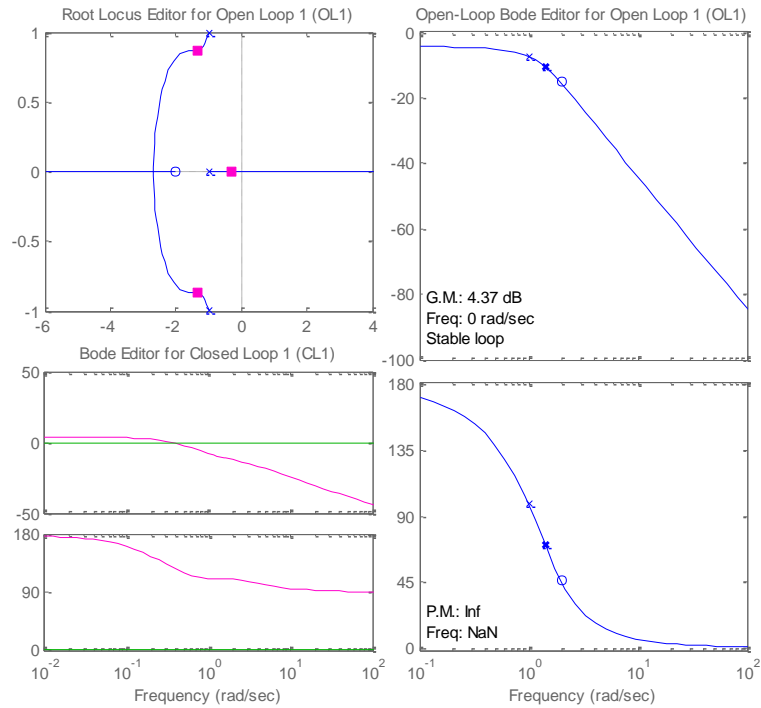
```
sisotool
```

at $K=-1$, margin of stability is observed as $PM \approx 0$:

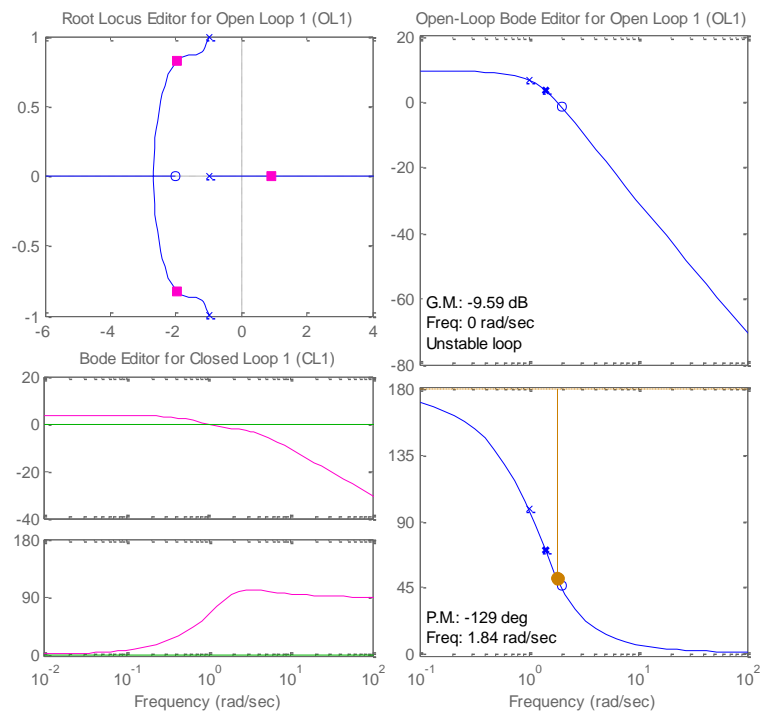
K= -1



The system is stable for $K > -1$ as follows: **K= -0.6**



And the system is unstable for $K < -1$: $K = -3$



*Combining the individual ranges for K , the system will be stable in the range of $K > -1$

10-39 See sample MATLAB code in Part e. The MATLAB codes are identical to problem 10-36.

(a)

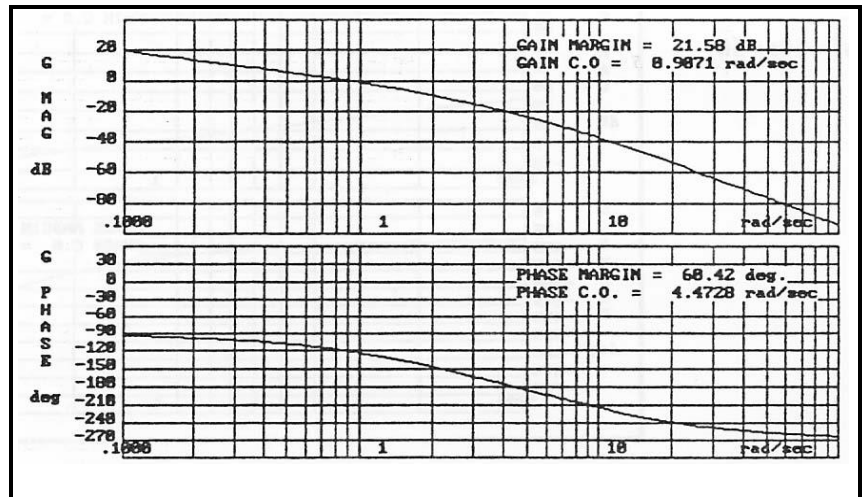
$$G(s) = \frac{K}{s(1+0.1s)(1+0.5s)}$$

The Bode plot is done with $K = 1$.

GM = 21.58 dB For GM = 20 dB,

K must be reduced by -1.58 dB.

Thus $K = 0.8337$



PM = 60.42°. For PM = 45°

K should be increased by 5.6 dB.

Or, $K = 1.91$

(b)

$$G(s) = \frac{K(s+1)}{s(1+0.1s)(1+0.2s)(1+0.5s)}$$

The

Bode plot is done with $K = 1$.

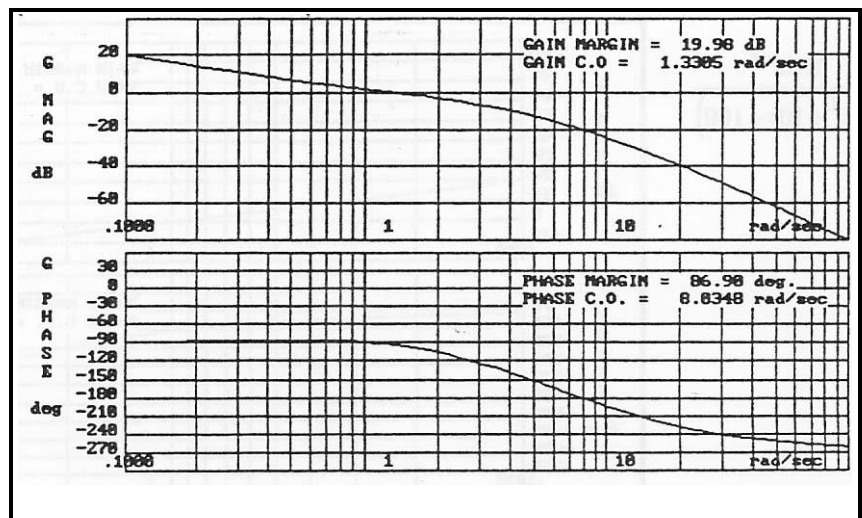
GM = 19.98 dB. For GM = 20 dB,

$K \cong 1$.

PM = 86.9°. For PM = 45°

K should be increased by 8.9 dB.

Or, $K = 2.79$.



8-39 (c) See the top plot

$$G(s) = \frac{K}{(s+3)^3}$$

The Bode plot is done with $K = 1$.

GM = 46.69 dB

PM = infinity.

For GM = 20 dB K can be

increased by 26.69 dB or $K = 21.6$.

For PM = 45 deg. K can be

increased by 28.71 dB, or

$K = 27.26$.

(d) See the middle plot

$$G(s) = \frac{K}{(s+3)^4}$$

The Bode plot is done with $K = 1$.

GM = 50.21 dB

PM = infinity.

For GM = 20 dB K can be

increased by 30.21 dB or $K = 32.4$

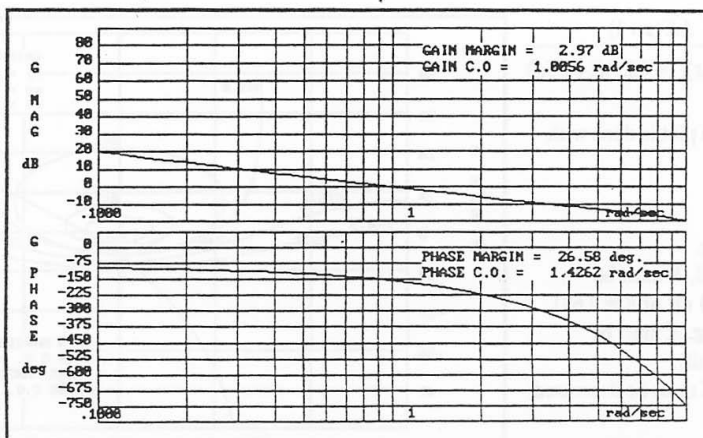
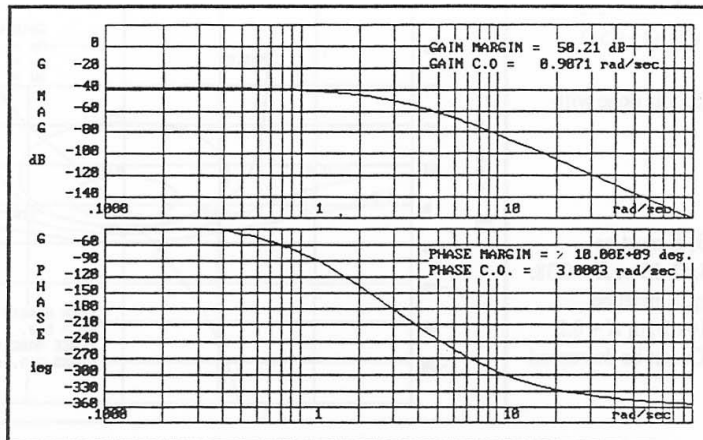
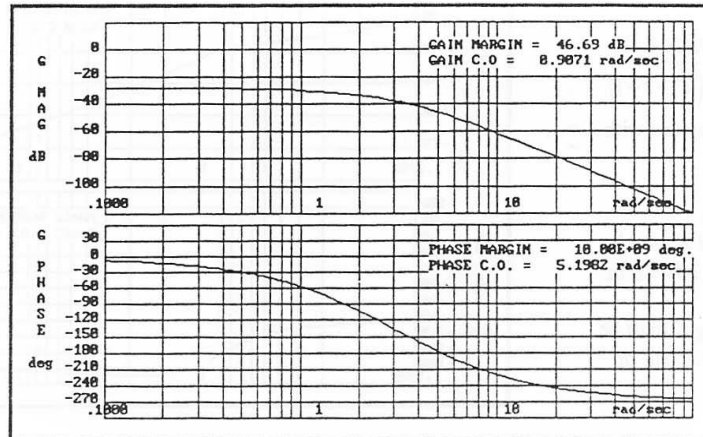
For PM = 45 deg. K can be

increased by 38.24 dB, or

$K = 81.66$

(e) See the bottom plot

The Bode plot is done with $K = 1$.



$$G(s) = \frac{Ke^{-s}}{s(1+0.1s+0.01s^2)}$$

GM=2.97 dB; PM = 26.58 deg

For GM = 20 dB K must be

decreased by -17.03 dB or

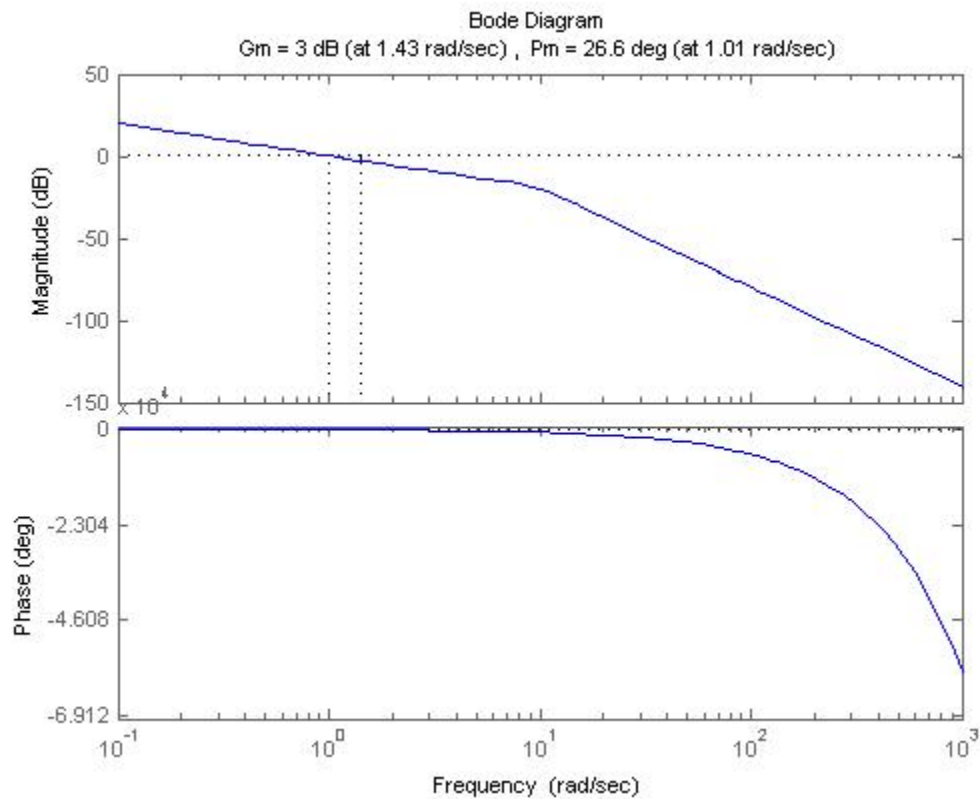
$K = 0.141$.

For PM = 45 deg. K must be

decreased by -2.92 dB or $K = 0.71$.

MATLAB code:

```
s = tf('s')
num_G_a= exp(-s);
den_G_a=s*(0.01*s^2+0.1*s+1);
G_a=num_G_a/den_G_a
margin(G_a)
```



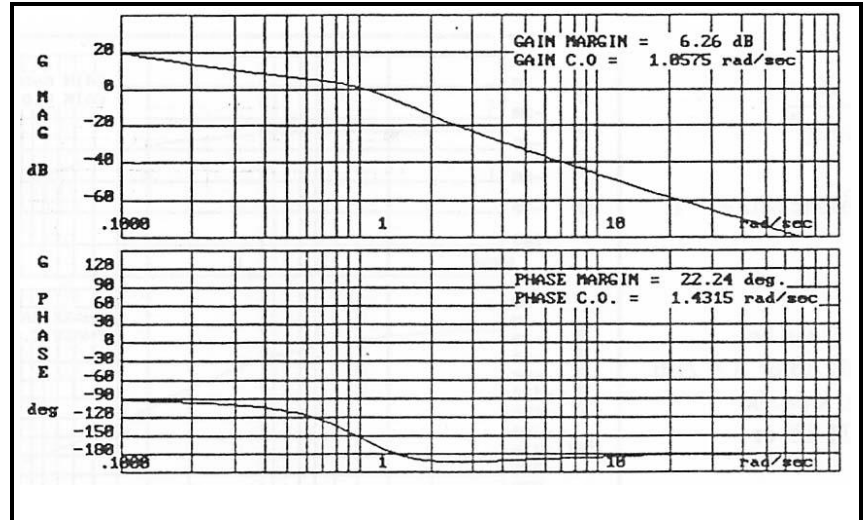
10-39 (f)

$$G(s) = \frac{K(1+0.5s)}{s(s^2+s+1)}$$

The Bode plot is done with $K = 1$.

GM = 6.26 dB

PM = 22.24 deg



For GM = 20 dB K must be decreased by -13.74 dB or

$K = 0.2055$.

For PM = 45 deg K must be decreased by -3.55 dB or

$K = 0.665$.

10-40 (a)

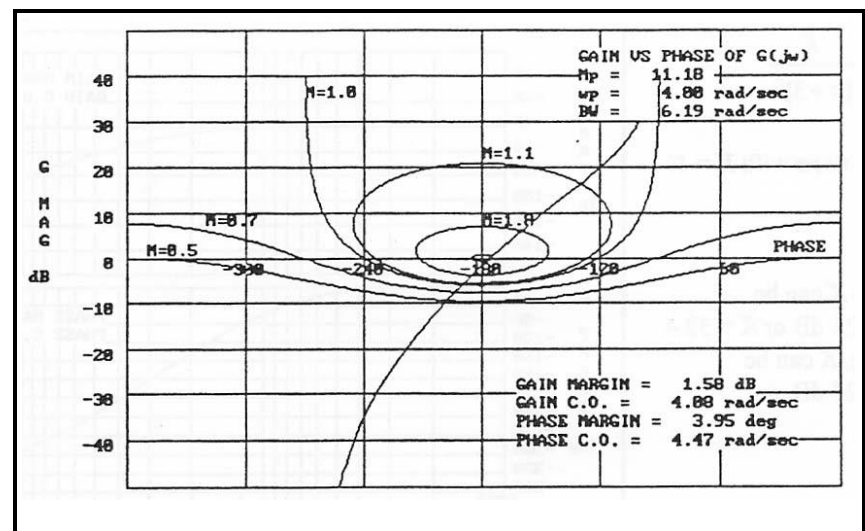
$$G(s) = \frac{10K}{s(1+0.1s)(1+0.5s)}$$

The gain-phase plot is done with

$K = 1$.

GM = 1.58 dB

PM = 3.95 deg.



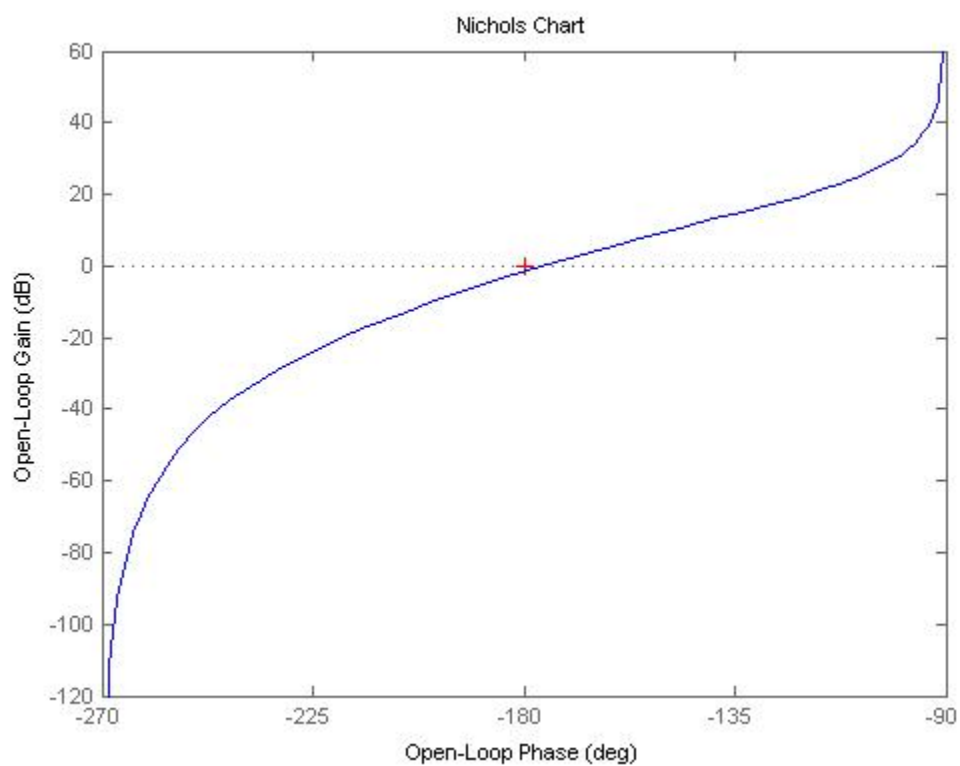
For GM = 10 dB, K must be decreased by -8.42 dB or $K = 0.38$.

For PM = 45 deg, K must be decreased by -14 dB, or $K = 0.2$.

For $M_p = 1.2$, K must be decreased to 0.16.

Sample MATLAB code:

```
s = tf('s')
num_G_a= 10;
den_G_a=s*(1+0.1*s)*(0.5*s+1);
G_a=num_G_a/den_G_a
nichols(G_a)
```

**(b)**

$$G(s) = \frac{5K(s+1)}{s(1+0.1s)(1+0.2s)(1+0.5s)}$$

The Gain-phase plot is done with

$K = 1$.

GM = 6 dB

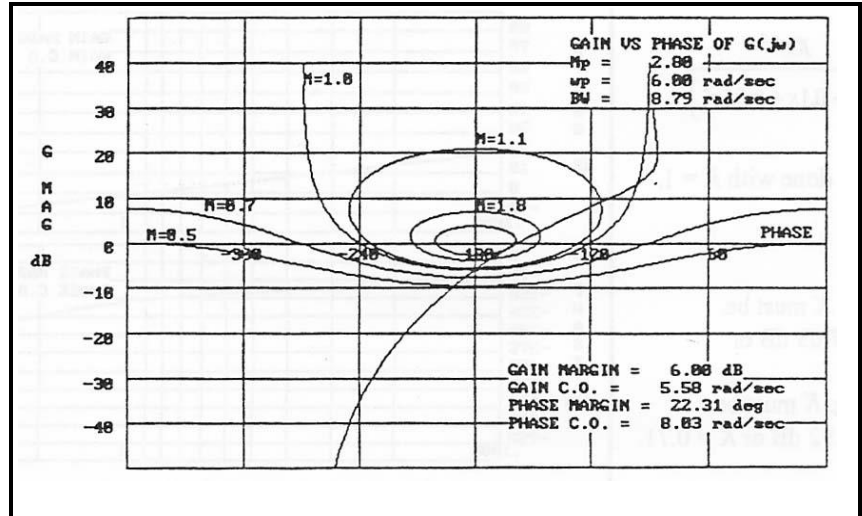
PM = 22.31 deg.

For GM = 10 dB, K must be decreased by -4 dB or $K = 0.631$.

For PM = 45 deg, K must be

decrease by -5 dB.

For $M_r = 1.2$, K must be decreased to 0.48.



10-40 (c)

$$G(s) = \frac{10K}{s(1+0.1s+0.01s^2)}$$

The gain-phase plot is done for

$K = 1$.

GM = 0 dB $M_r = \infty$

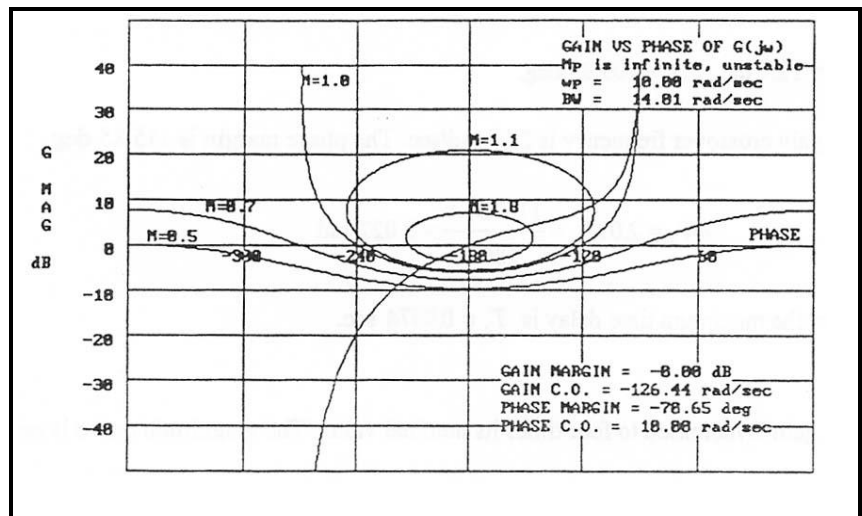
PM = 0 deg

For GM = 10 dB, K must be decreased by -10 dB or $K = 0.316$.

For PM = 45 deg, K must be decreased by -5.3 dB, or

$K = 0.543$.

For $M_r = 1.2$, K must be decreased to 0.2213.



(d)

$$G(s) = \frac{Ke^{-s}}{s(1+0.1s+0.01s^2)}$$

The gain-phase plot is done for

$K = 1$.

GM = 2.97 dB $M_r = 3.09$

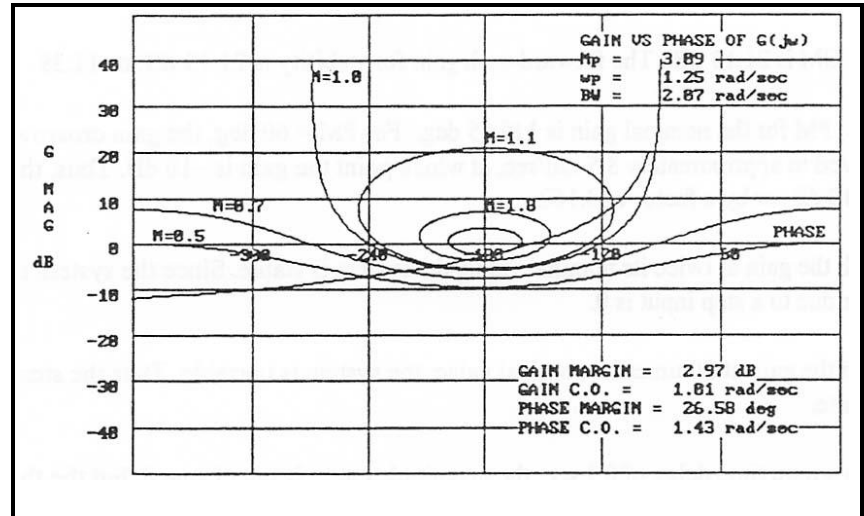
PM = 26.58 deg

For GM = 10 dB, K must be decreased by -7.03 dB, $K = 0.445$.

For PM = 45 deg, K must be decreased by -2.92 dB, or

$K = 0.71$.

For $M_r = 1.2$, $K = 0.61$.



10-41**MATLAB code:**

```
s = tf('s')
%a)
num_GH_a= 1*(s+1)*(s+2);
den_GH_a=s^2*(s+3)*(s^2+2*s+25);
GH_a=num_GH_a/den_GH_a;
CL_a = GH_a/(1+GH_a)
figure(1);
bode(CL_a)

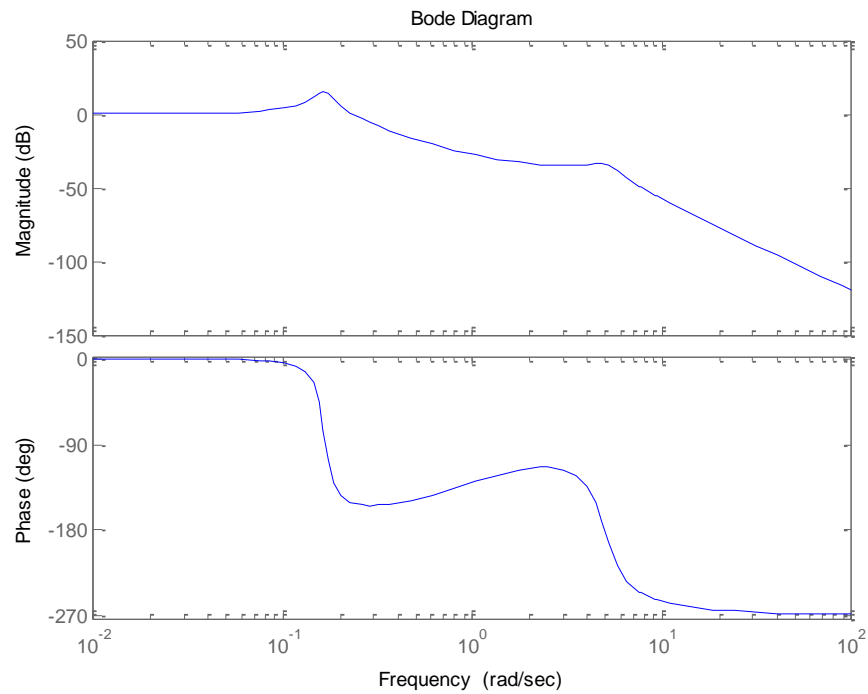
%b)
figure(2);
rlocus(GH_a)

%c)
num_GH_c= 53*(s+1)*(s+2);
den_GH_c=s^2*(s+3)*(s^2+2*s+25);
GH_c=num_GH_c/den_GH_c;
figure(3);
nyquist(GH_c)
xlim([-2 1])
ylim([-1.5 1.5])

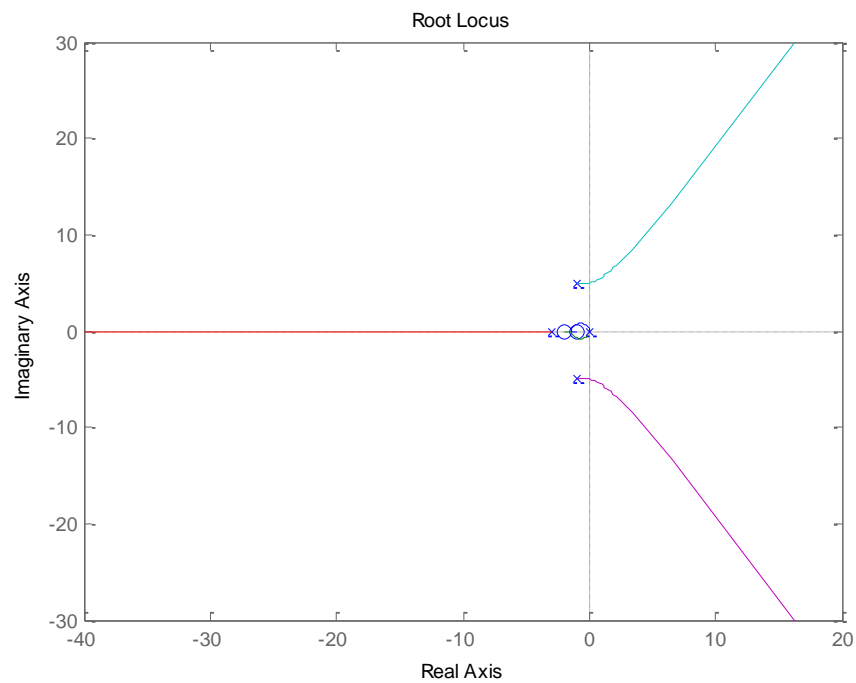
%d)
num_GH_d= (s+1)*(s+2);
den_GH_d=s^2*(s+3)*(s^2+2*s+25);
GH_d=num_GH_d/den_GH_d;
CL_d = GH_d/(1+GH_d)
figure(4);
margin(CL_d)

sisotool
```

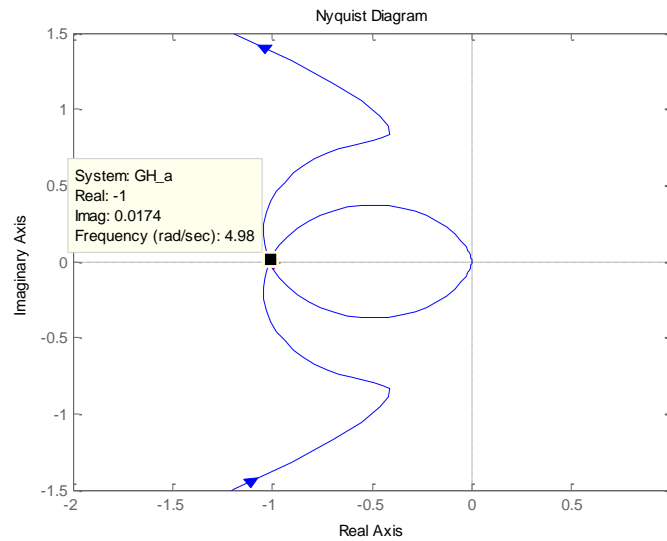
Part (a), Bode diagram:



Part (b), Root locus diagram:



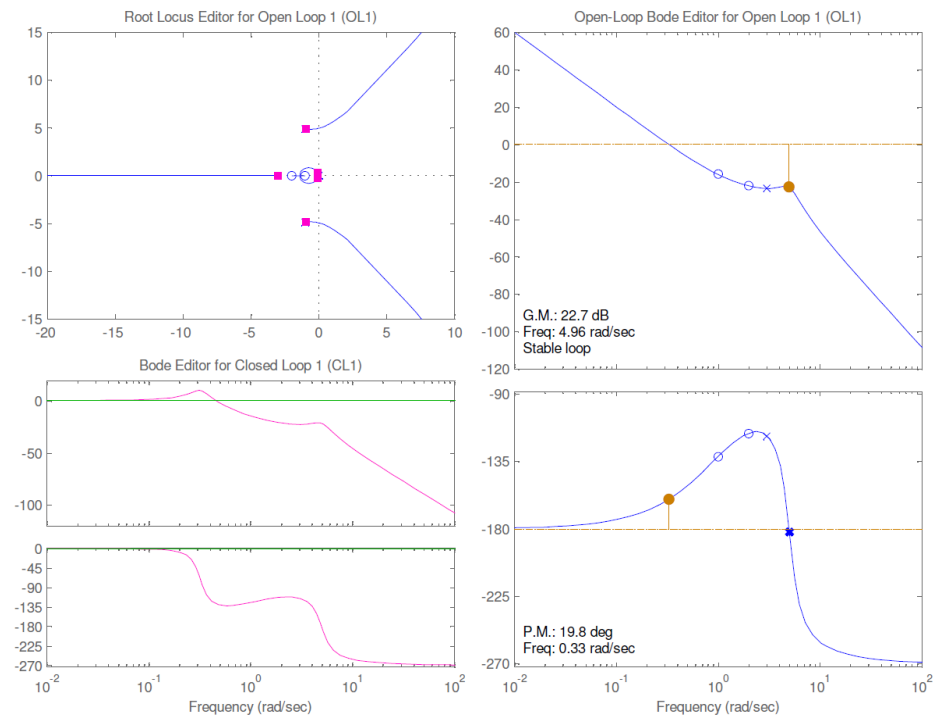
Part (c), Gain and frequency that instability occurs: Gain=53, Freq = 4.98 rad/sec, as seen in the data point in the figure:



Part (d), Gain and frequency that instability occurs:

Gain=0.127 at PM=20 deg:

By running sisotool command in MATLAB, the transfer functions are imported and the gain is iteratively changed until the phase margin of PM=20 deg is achieved. The corresponding Gain is $K=0.127$.



Part (e): The corresponding gain margin is $GM = 22.7$ dB is seen in the figure

10-42 (a) Gain crossover frequency = 2.09 rad/sec PM = 115.85 deg

Phase crossover frequency = 20.31 rad/sec GM = 21.13 dB

(b) Gain crossover frequency = 6.63 rad/sec PM = 72.08 deg

Phase crossover frequency = 20.31 rad/sec GM = 15.11 dB

(c) Gain crossover frequency = 19.1 rad/sec PM = 4.07 deg

Phase crossover frequency = 20.31 rad/sec GM = 1.13 dB

(d) For GM = 40 dB, reduce gain by $(40 - 21.13)$ dB = 18.7 dB, or gain = $0.116 \times$ nominal value.

(e) For PM = 45 deg, the magnitude curve reads -10 dB. This means that the loop gain can be increased by 10 dB from the nominal value. Or gain = $3.16 \times$ nominal value.

(f) The system is type 1, since the slope of $|G(j\omega)|$ is -20 dB/decade as $\omega \rightarrow 0$.

(g) GM = 12.7 dB. PM = 109.85 deg.

(h) The gain crossover frequency is 2.09 rad/sec. The phase margin is 115.85 deg.

Set

$$\omega T_d = 2.09 T_d = \frac{115.85^\circ \pi}{180^\circ} = 2.022 \text{ rad}$$

Thus, the maximum time delay is $T_d = 0.9674$ sec.

10-43 (a) The gain is increased to four times its nominal value. The magnitude curve is raised by 12.04 dB.

Gain crossover frequency = 10 rad/sec PM = 46 deg

Phase crossover frequency = 20.31 rad/sec GM = 9.09 dB

(b) The GM that corresponds to the nominal gain is 21.13 dB. To change the GM to 20 dB we need to increase the gain by 1.13 dB, or 1.139 times the nominal gain.

(c) The GM is 21.13 dB. The forward-path gain for stability is 21.13 dB, or 11.39.

(d) The PM for the nominal gain is 115.85 deg. For PM = 60 deg, the gain crossover frequency must be moved to approximately 8.5 rad/sec, at which point the gain is -10 dB. Thus, the gain must be increased by 10 dB, or by a factor of 3.162.

(e) With the gain at twice its nominal value, the system is stable. Since the system is type 1, the steady-state error due to a step input is 0.

(f) With the gain at 20 times its nominal value, the system is unstable. Thus the steady-state error would be infinite.

(g) With a pure time delay of 0.1 sec, the magnitude curve is not changed, but the the phase curve is subject to a negative phase of -0.1ω rad. The PM is

$$\text{PM} = 115.85 - 0.1 \times \text{gain crossover frequency} = 115.85 - 0.209 = 115.64 \text{ deg}$$

The new phase crossover frequency is approximately 9 rad/sec, where the original phase curve is reduced by -0.9 rad or -51.5 deg. The magnitude of the gain curve at this frequency is -10 dB.

Thus, the gain margin is 10 dB.

(h) When the gain is set at 10 times its nominal value, the magnitude curve is raised by 20 dB. The new gain crossover frequency is approximately 17 rad/sec. The phase at this frequency is -30 deg.

Thus, setting

$$\omega T_d = 17T_d = \frac{30^\circ \pi}{180^\circ} = 0.5236 \quad \text{Thus} \quad T_d = 0.0308 \text{ sec.}$$

10-44**MATLAB code:**

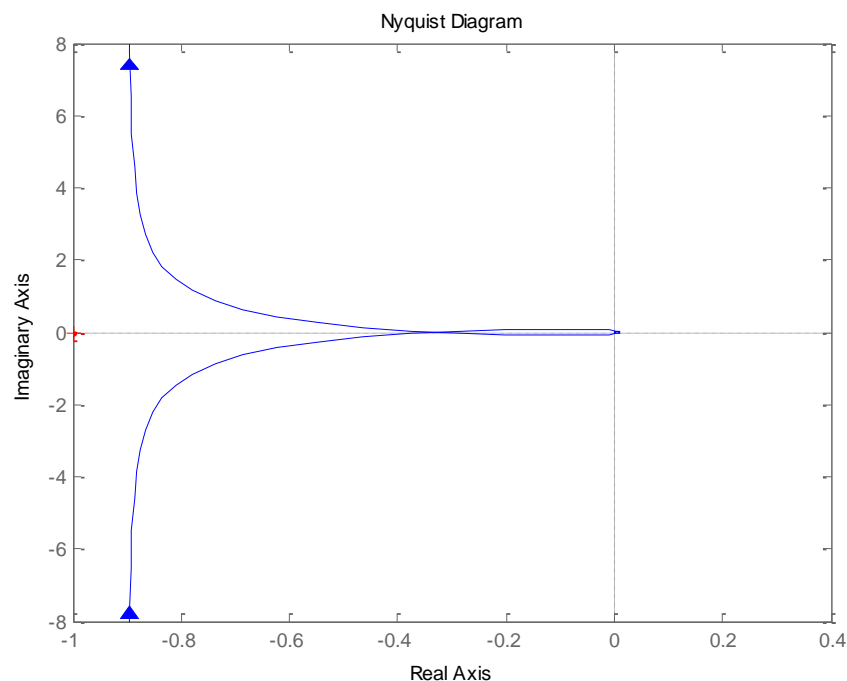
```

s = tf('s')
%using pade cammand for PADE approximation of exponential term
num_G_a= pade((80*exp(-0.1*s)),2);
den_G_a=s*(s+4)*(s+10);
G_a=num_G_a/den_G_a;
CL_a = G_a/(1+G_a)
OL_a = G_a*1;

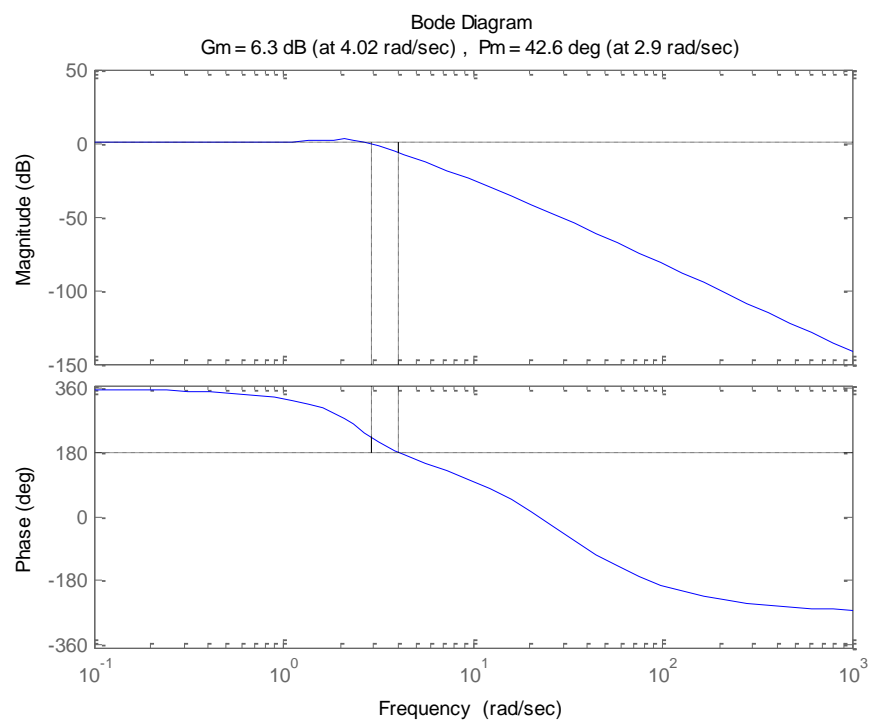
%(a)
figure(1)
nyquist(OL_a)

%(b) and (c)
figure(2);
margin(CL_a)

```

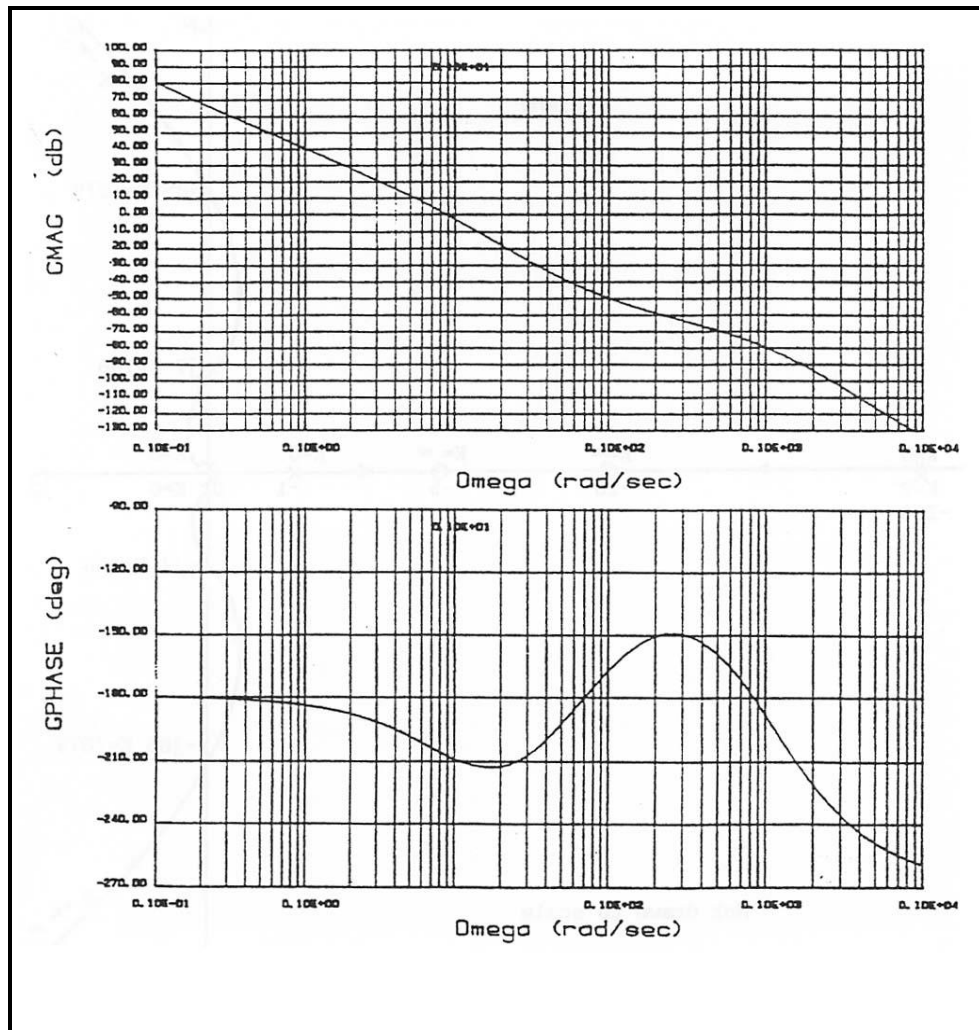
Part (a), Nyquist diagram:**Part (b) and (c), Bode diagram:**

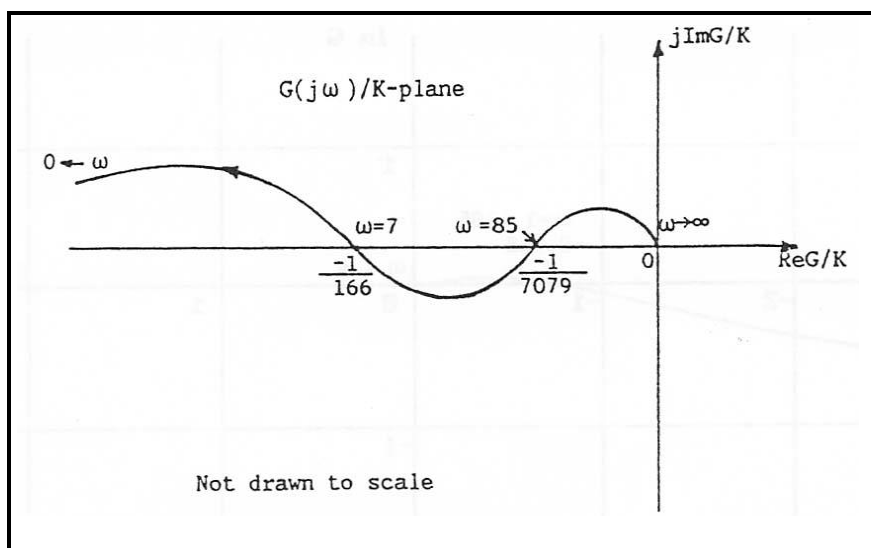
Using Margin command, the gain and phase margins are obtained as GM = 6.3 dB, PM = 42.6 deg:

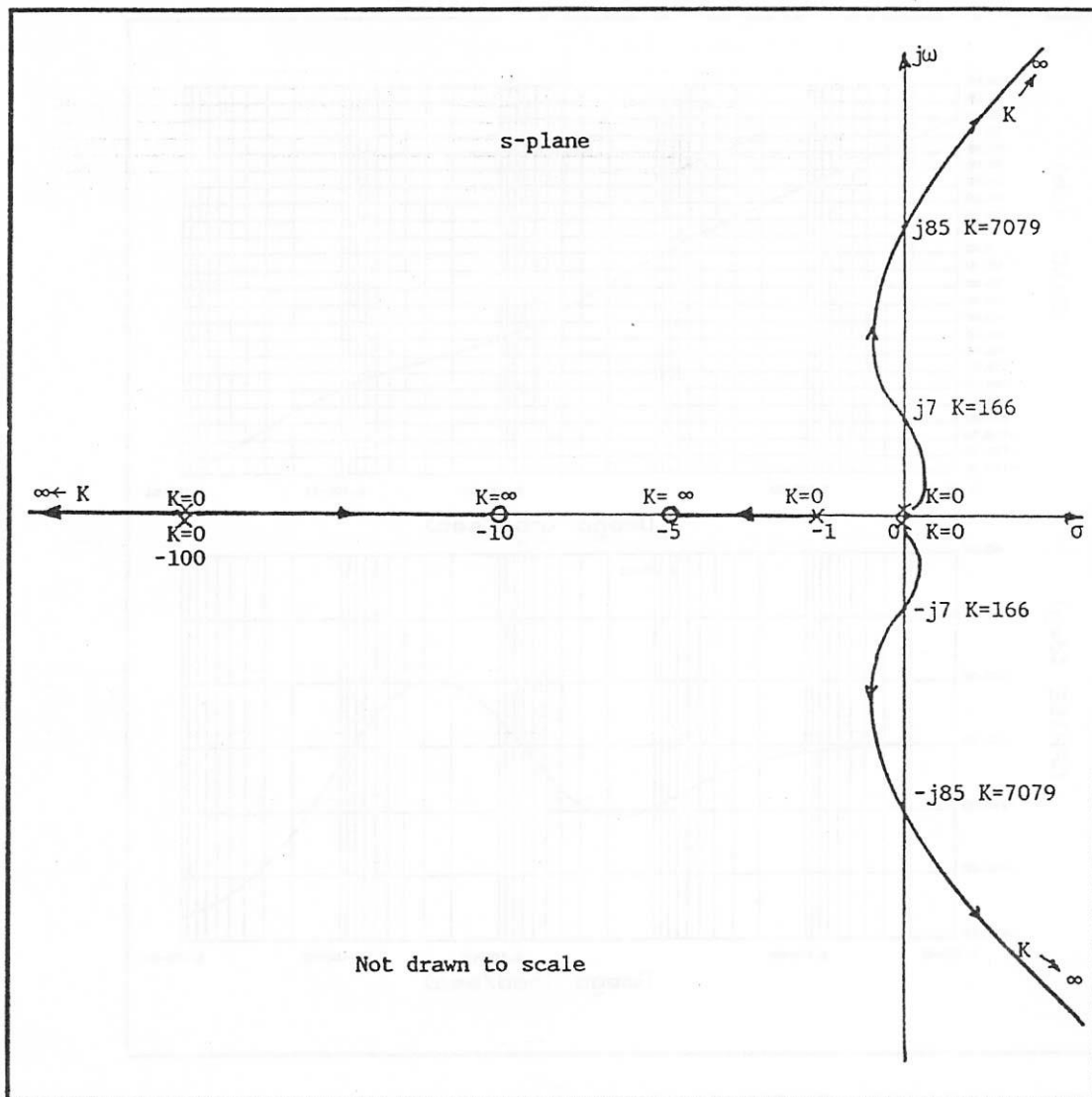


10-45 (a) Bode Plot:For stability: $166 (44.4 \text{ dB}) < K < 7079 (77 \text{ dB})$

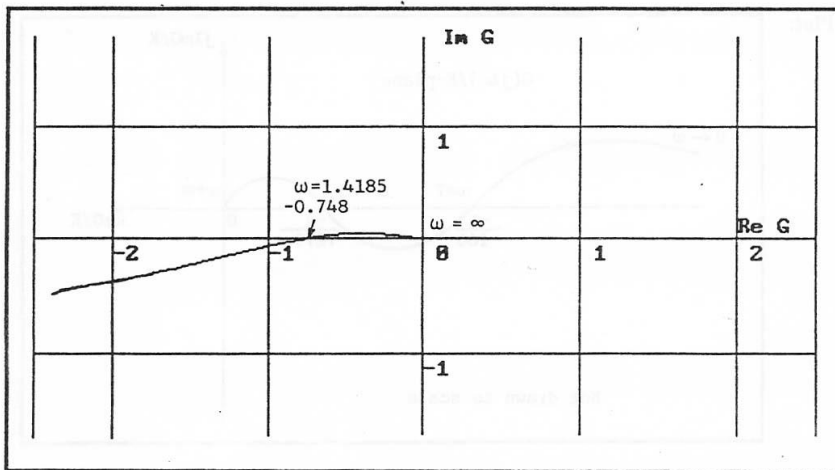
Phase crossover frequencies: 7 rad/sec and 85 rad/sec



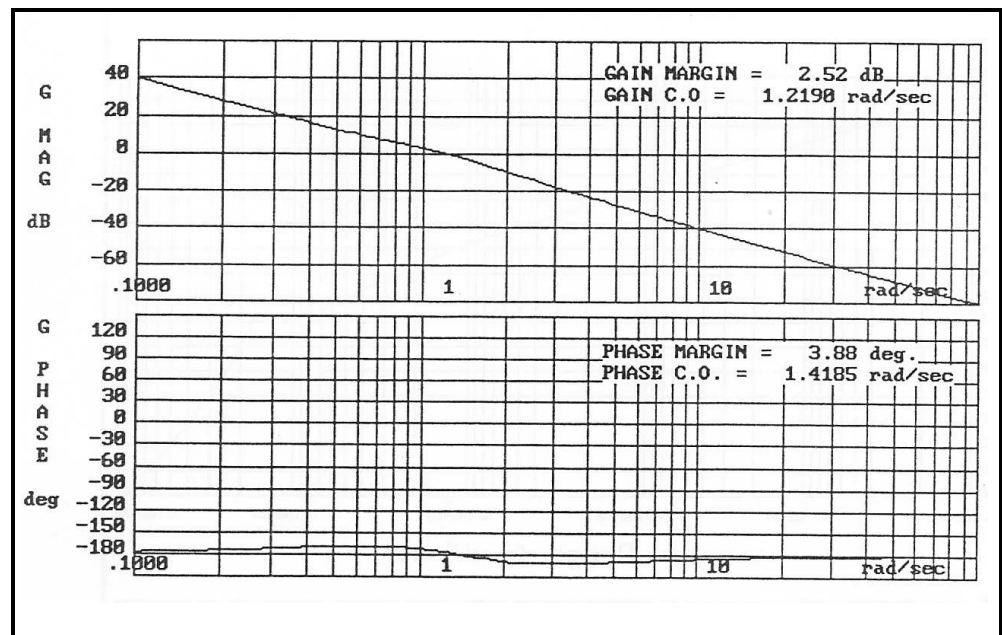
Nyquist Plot:

10-45 (b) Root Loci.

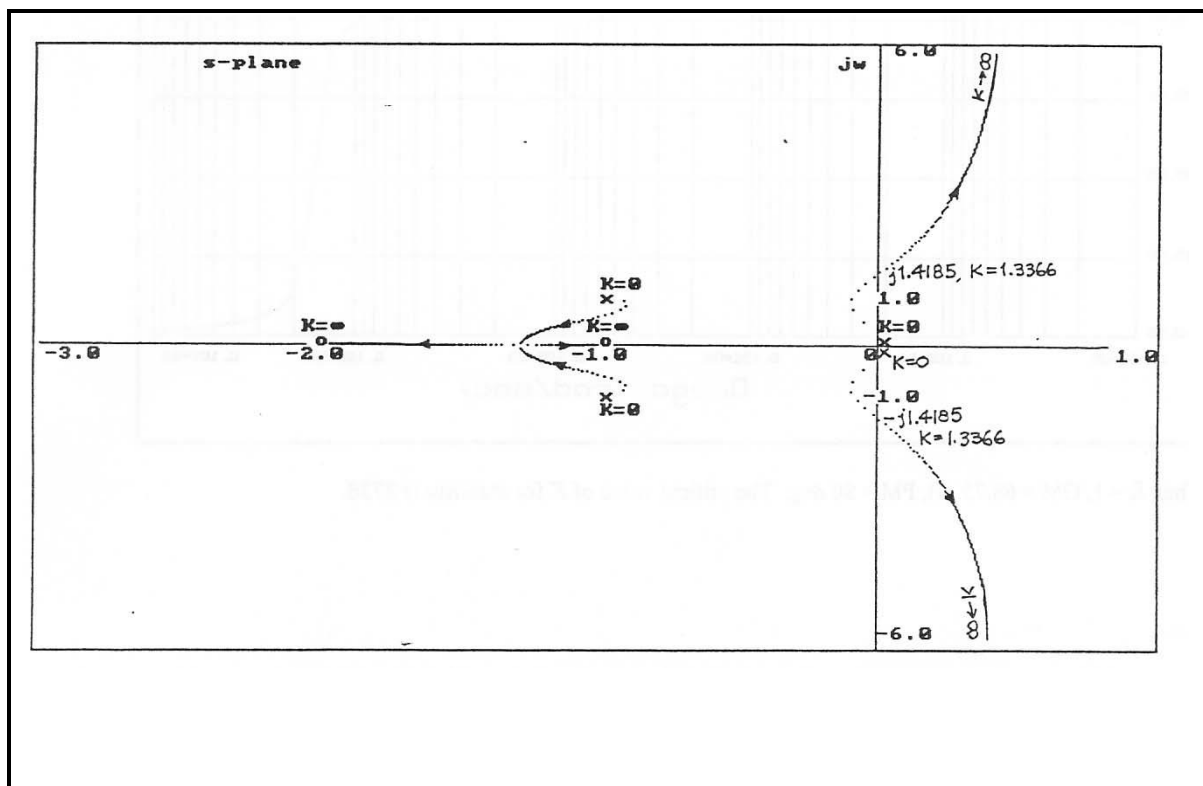
10-46 (a) Nyquist Plot



Bode Plot



Root Locus

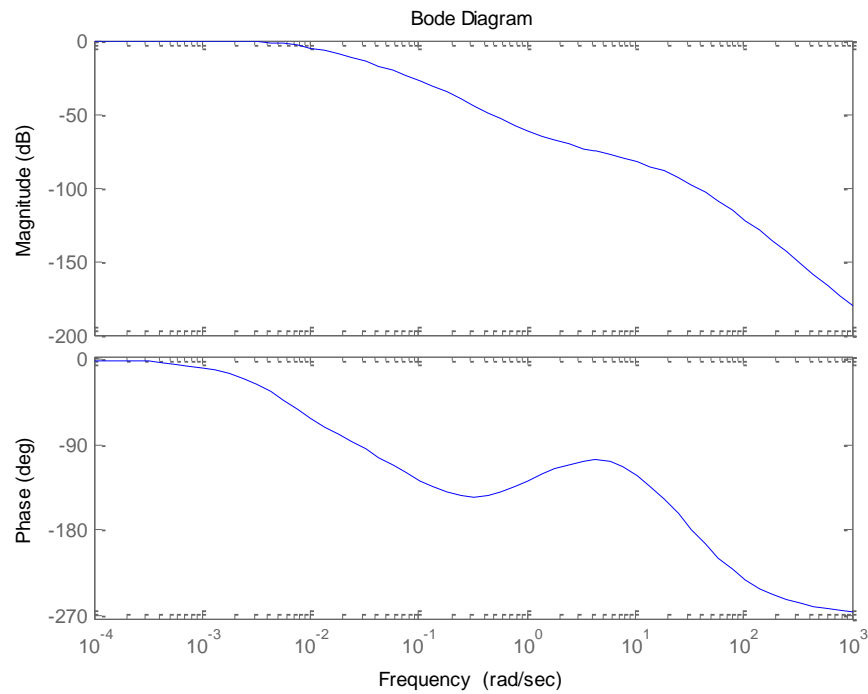


10-47**MATLAB code:**

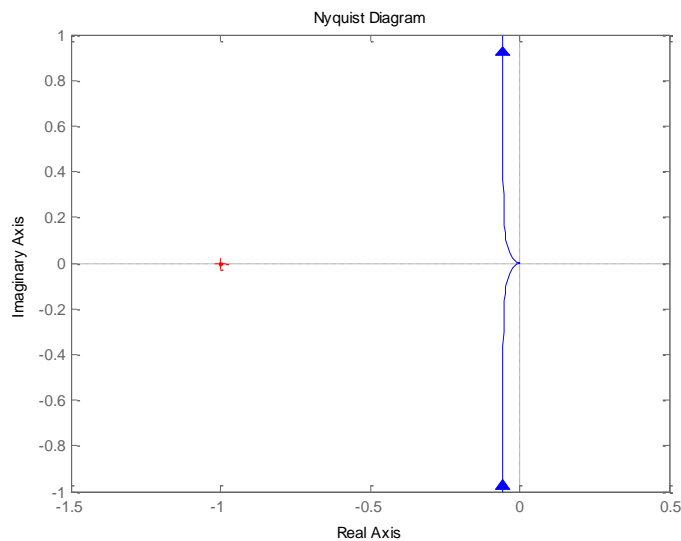
```
%a)
k=1
num_GH= k*(s+1)*(s+5);
den_GH=s*(s+0.1)*(s+8)*(s+20)*(s+50);
GH=num_GH/den_GH;
CL = GH/(1+GH)
figure(1);
bode(CL)
figure(2);
OL = GH;
nyquist(GH)
xlim([-1.5 0.5]);
ylim([-1 1]);

sisotool
```

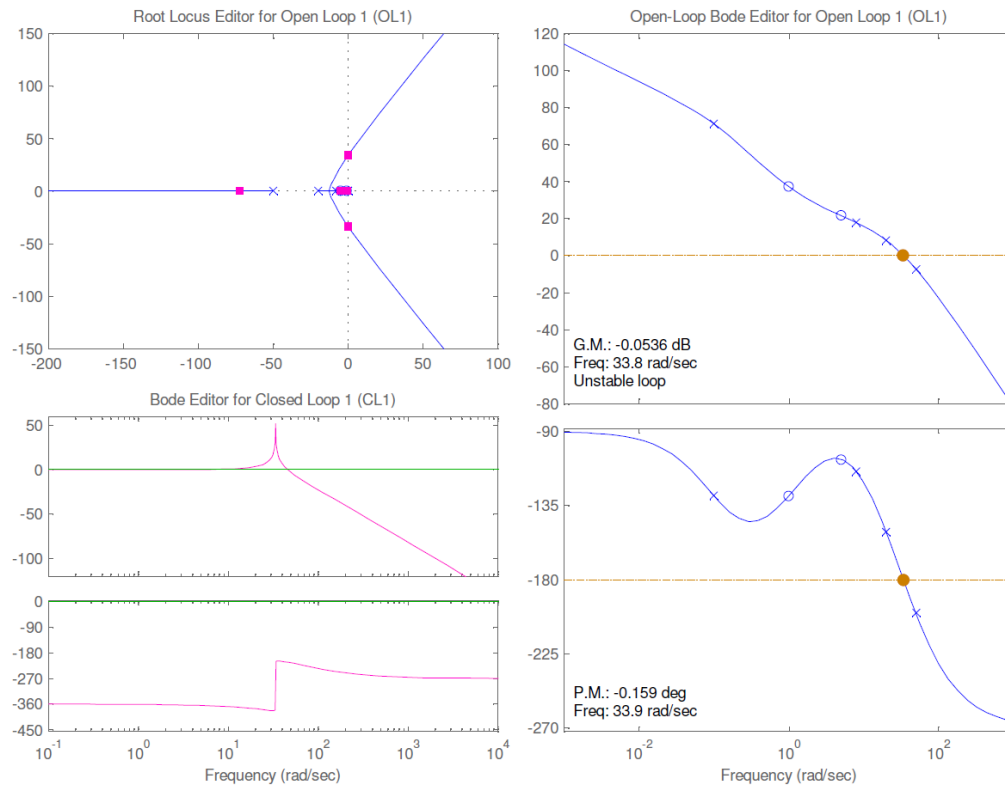
Part (a), Bode diagram:



Part (a), Nyquist diagram:

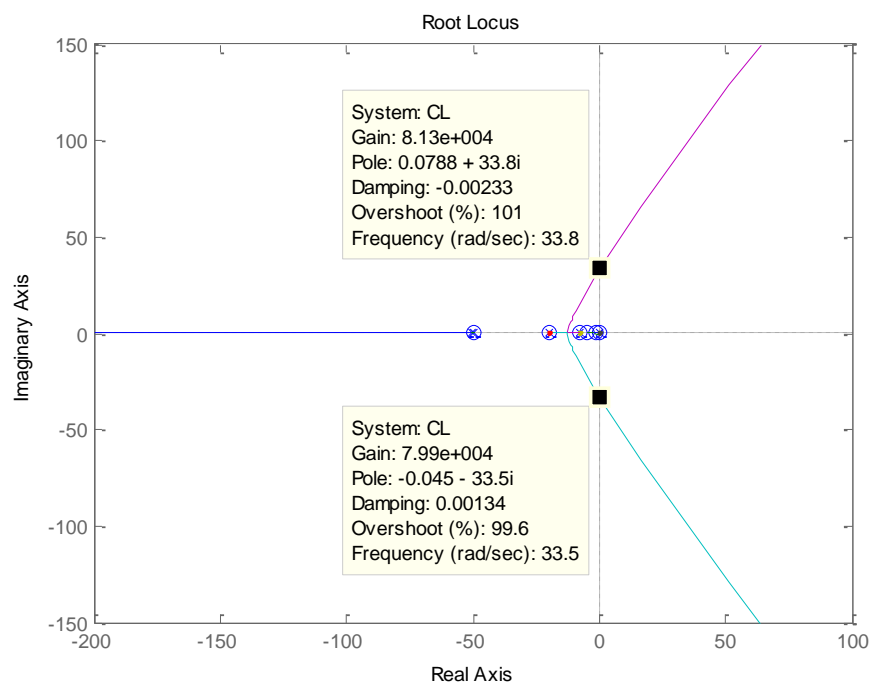


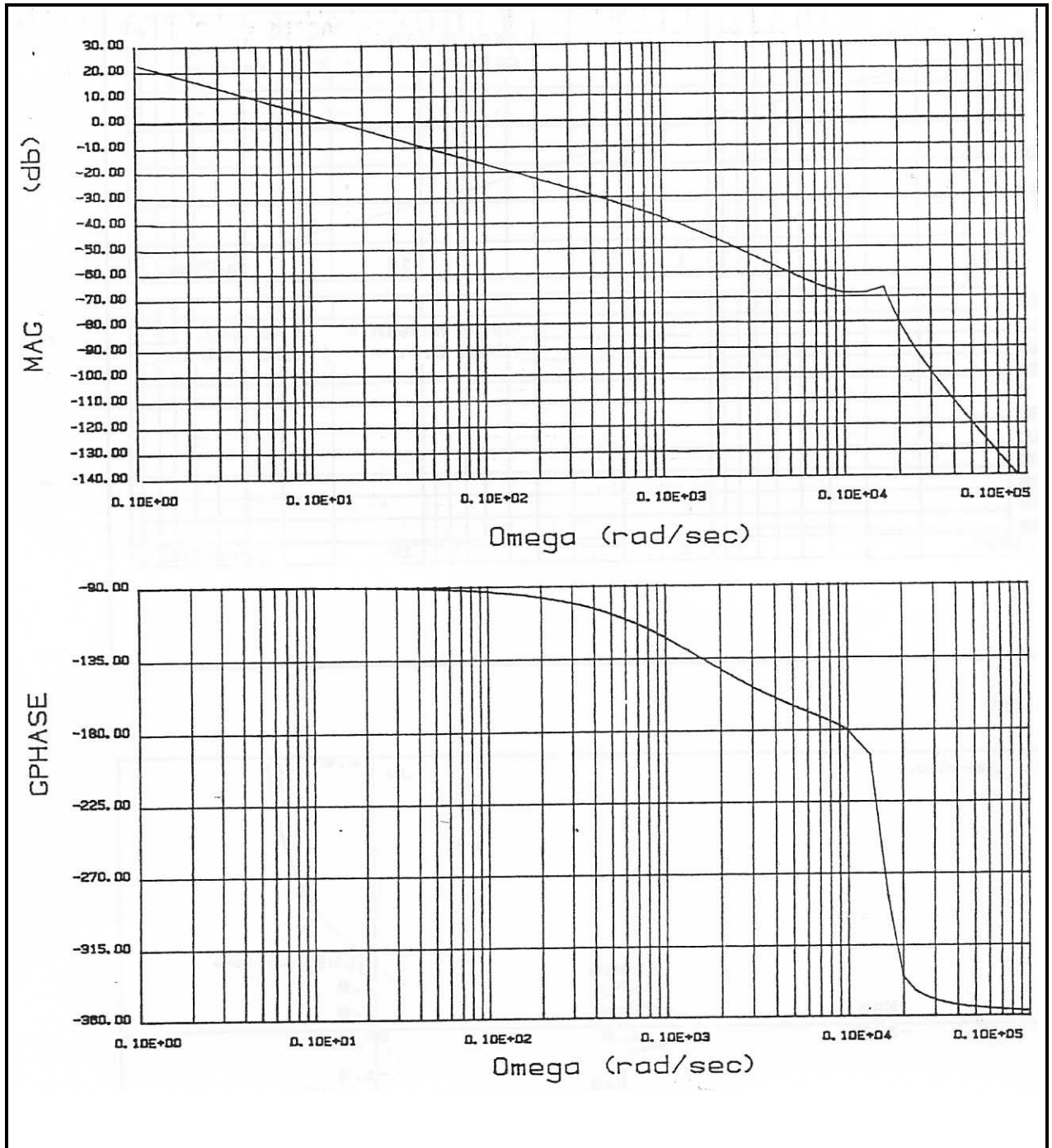
Part (a), range of K for stability: By running sisotool command in MATLAB, the transfer functions are imported and the gain is iteratively changed until the phase margin of $PM=0$ deg is achieved (where $K =$) which is the margin of stability. The stable rang for K is $K > 8.16 \times 10^4$:



Part (b), Root-locus diagram, K and ω at the points where the root loci cross the $j\omega$ -axis:

As can be seen in the figure at $K=8.13 \times 10^4$ and $\omega=33.8$ rad/sec, the poles cross the $j\omega$ axis. Both of these values are consistent with the results of part(a) from sisotool.



10-48 Bode Diagram

When $K = 1$, $GM = 68.75$ dB, $PM = 90$ deg. The critical value of K for stability is 2738.

10-49 (a) Forward-path transfer function:

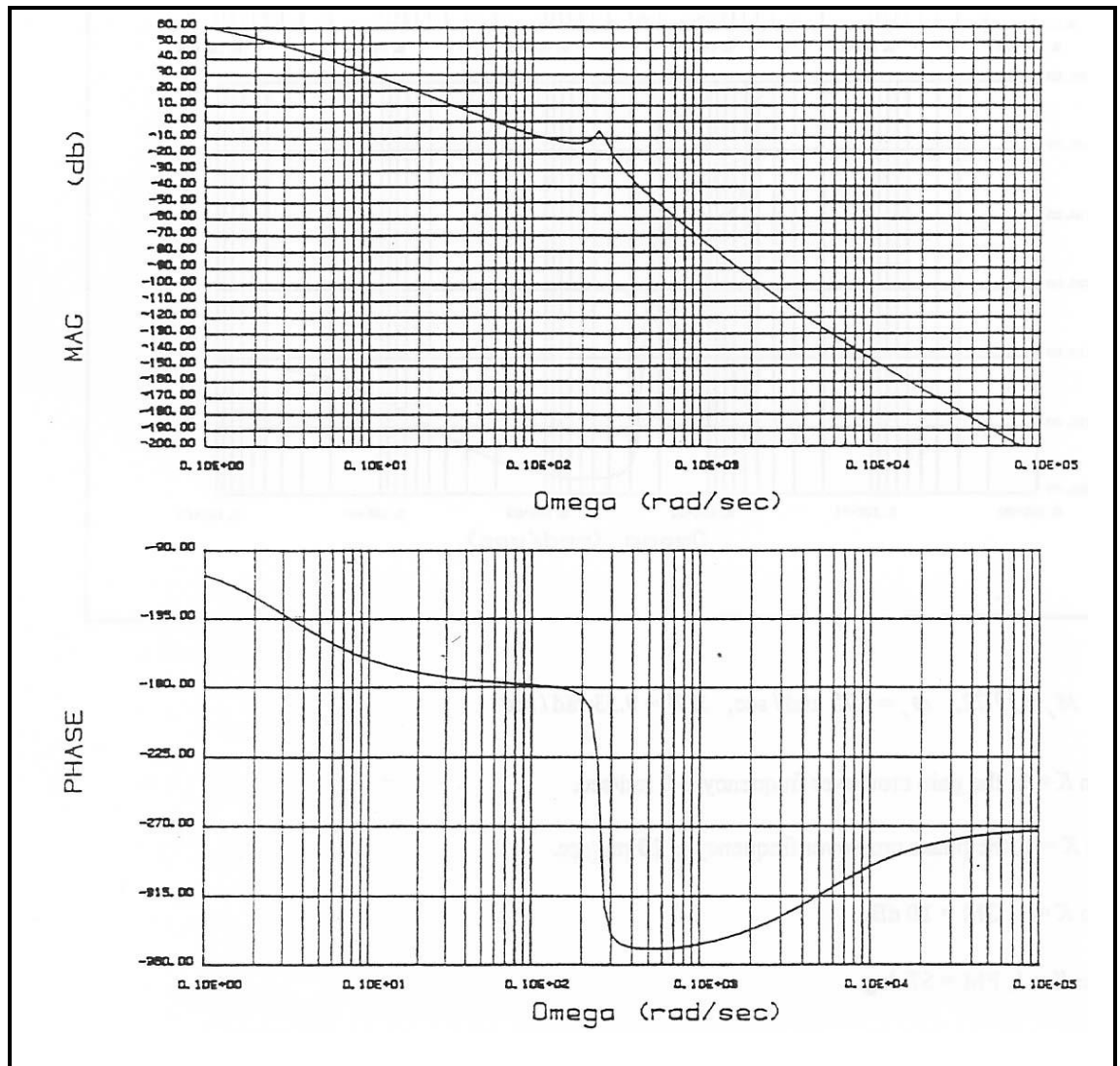
$$G(s) = \frac{\Theta_L(s)}{E(s)} = K_a G_p(s) = \frac{K_a K_i (Bs + K)}{\Delta_o}$$

where

$$\begin{aligned}\Delta_o &= 0.12s(s + 0.0325)(s^2 + 2.5675s + 6667) \\ &= s(0.12s^3 + 0.312s^2 + 80.05s + 26)\end{aligned}$$

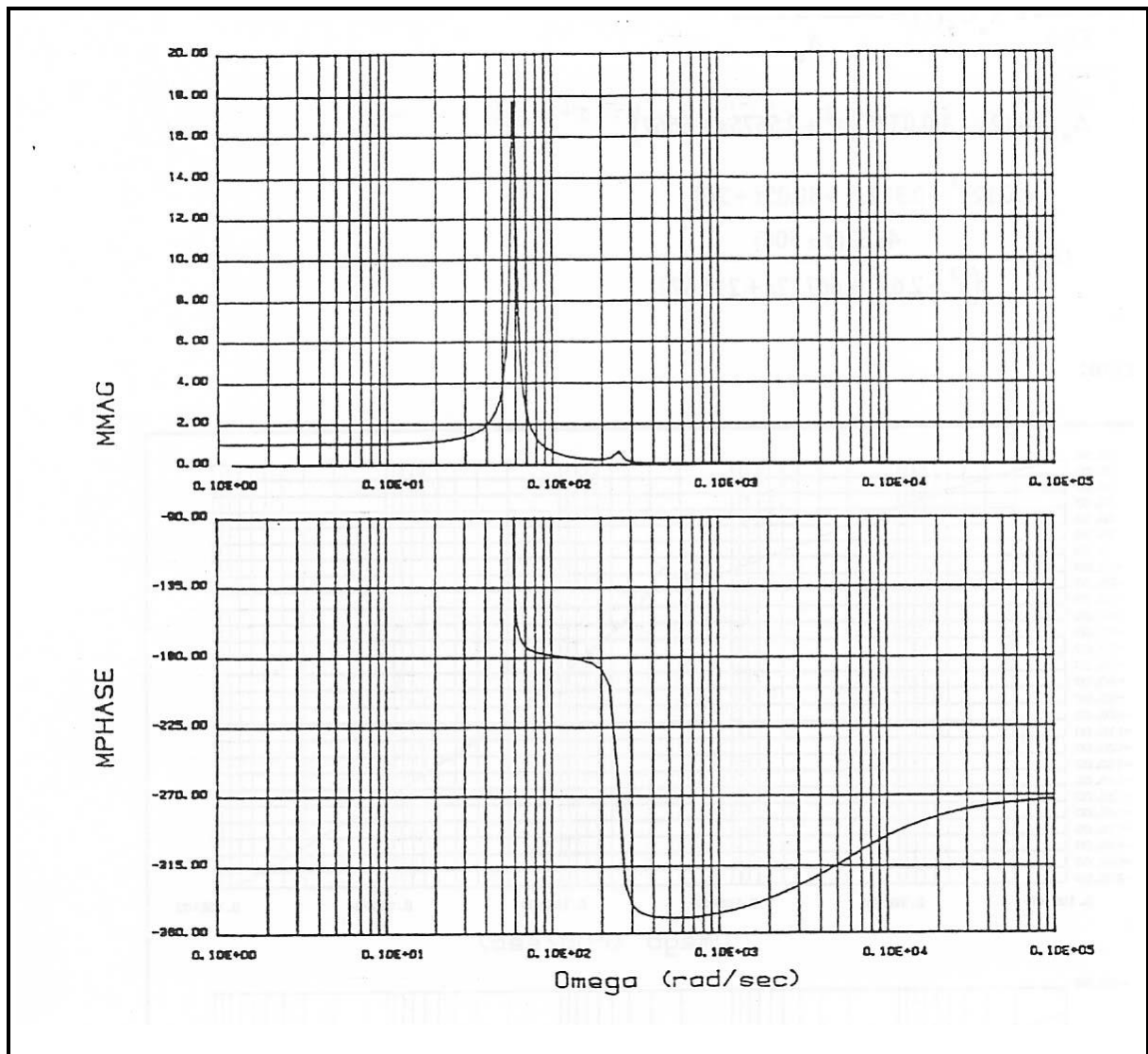
$$G(s) = \frac{43.33(s + 500)}{s(s^3 + 2.6s^2 + 667.12s + 216.67)}$$

(b) Bode Diagram:



Gain crossover frequency = 5.85 rad/sec PM = 2.65 deg.

Phase crossover frequency = 11.81 rad/sec GM = 10.51 dB

10-49 (c) Closed-loop Frequency Response:

$$M_r = 17.72, \quad \omega_r = 5.75 \text{ rad/sec}, \quad BW = 9.53 \text{ rad/sec}$$

10-50

$$(a) \quad G(s)H(s) = \frac{Kmgd}{L\left(\frac{J}{r^2} + m\right)s^2}$$

$$(b) \quad \frac{G(s)H(s)}{1+G(s)H(s)} = \frac{Kmgd}{L\left(\frac{J}{r^2} + m\right)s^2 + Kmgd}$$

(c) to (e)**MATLAB code:**

```

s = tf('s')

m = 0.11;
r = 0.015;
d = 0.03;
g = 9.8;
L = 1.0;
J = 9.99*10^-6

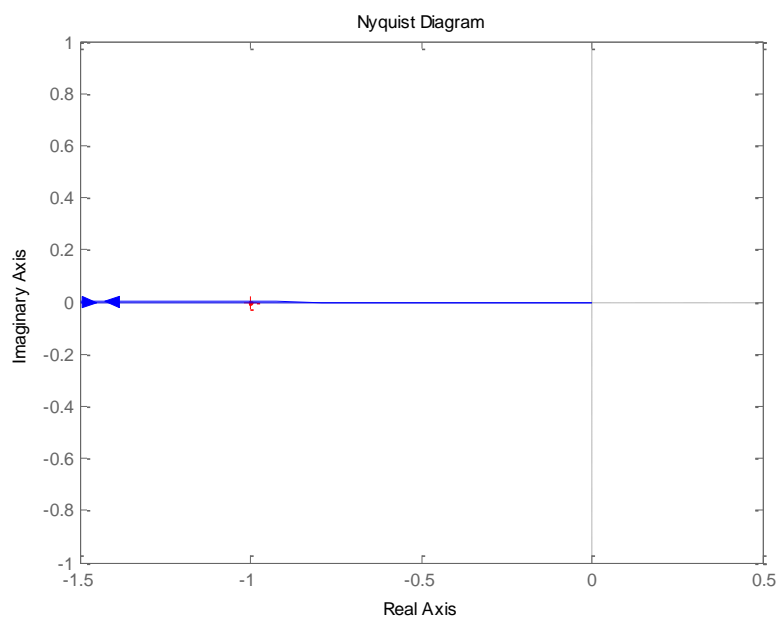
K=1
num_GH= K*m*g*d;
den_GH=L*(J/r^2+m)*s^2;
GH=num_GH/den_GH;
CL = GH/(1+GH)

%c)
figure(1);
nyquist(GH)
xlim([-1.5 0.5]);
ylim([-1 1]);

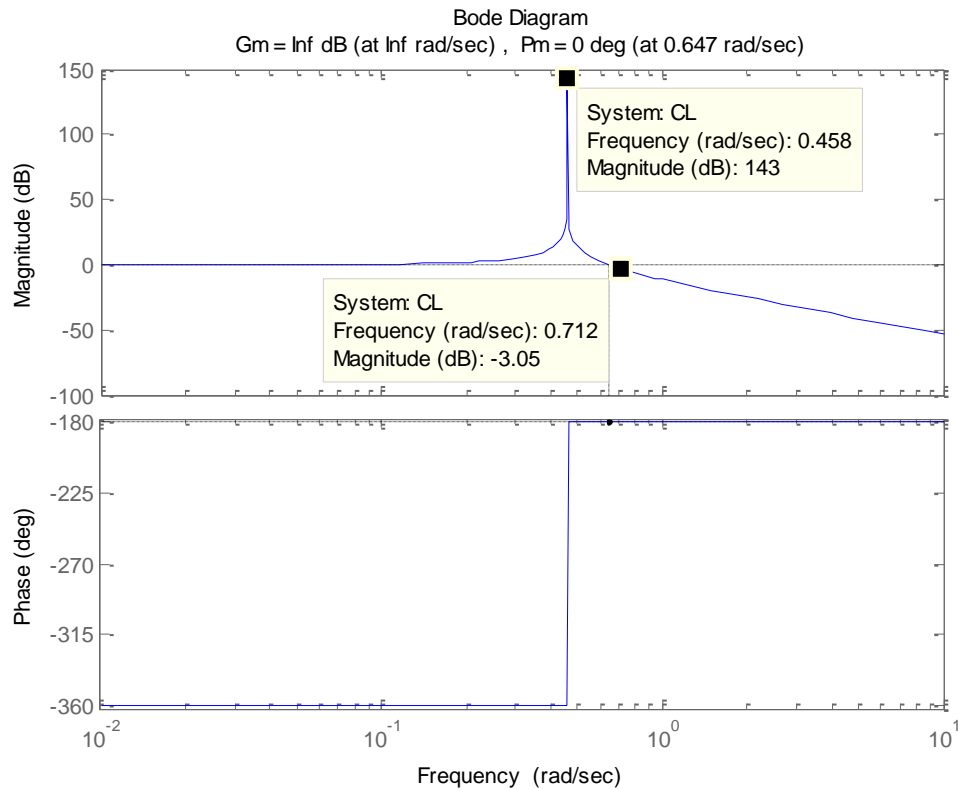
%d)
figure(2);
margin(CL)

```

Part (c): since the system is a double integrator ($1/s^2$), the phase is always -180 deg, and the system is always marginally stable for **any K**, leading to a complicated control problem.

**Part (d), Bode diagram:**

As explained in section (c), since the system is always marginally stable, $GM = \infty$ and $PM = 0$, as can be seen by MATLAB MARGIN command, resulting in the following figure:



Part (e), $M_r = 143$ dB, $\omega_r = 0.458$ rad/sec, and BW = 0.712 rad/sec as can be seen in the data points in the above figure.

10-51 (a) When $K = 1$, the gain crossover frequency is 8 rad/sec.

(b) When $K = 1$, the phase crossover frequency is 20 rad/sec.

(c) When $K = 1$, GM = 10 dB.

(d) When $K = 1$, PM = 57 deg.

(e) When $K = 1$, $M_r = 1.2$.

(f) When $K = 1$, $\omega_r = 3$ rad/sec.

(g) When $K = 1$, BW = 15 rad/sec.

(h) When $K = -10$ dB (0.316), GM = 20 dB

(i) When $K = 10$ dB (3.16), the system is marginally stable. The frequency of oscillation is 20 rad/sec.

(j) The system is type 1, since the gain-phase plot of $G(j\omega)$ approaches infinity at -90 deg. Thus, the steady-state error due to a unit-step input is zero.

10-52 When $K = 5$ dB, the gain-phase plot of $G(j\omega)$ is raised by 5 dB.

- (a) The gain crossover frequency is ~ 10 rad/sec.
- (b) The phase crossover frequency is ~ 20 rad/sec.
- (c) GM = 5 dB.
- (d) PM = ~ 34.5 deg.
- (e) When $K = 5$, $M_r = \sim 2$ (smallest circle tangent to an M circle).
- (f) $\omega_r = 15$ rad/sec
- (g) BW = 30 rad/sec
- (h) When $K = -30$ dB, the GM is 40 dB (shift the graph of $K=1$, 30 db down).

When $K = 10$ dB, the gain-phase plot of $G(j\omega)$ is raised by 10 dB.

- (a) The gain crossover frequency is 20 rad/sec.
- (b) The phase crossover frequency is 20 rad/sec.
- (c) GM = 0 dB.
- (d) PM = 0 deg.
- (e) When $K = 10$, $M_r = \sim 1.1$ (smallest circle tangent to an M circle).
- (f) $\omega_r = 5$ rad/sec
- (g) BW = ~ 40 rad/sec
- (h) When $K = -30$ dB, the GM is 40 dB (shift the graph of $K=1$, 30 db down).

10-53

Since the function has exponential term, PADE command has been used to obtain the transfer function.

$$G(s)H(s) = \frac{80e^{-0.1s}}{s(s+4)(s+10)}$$

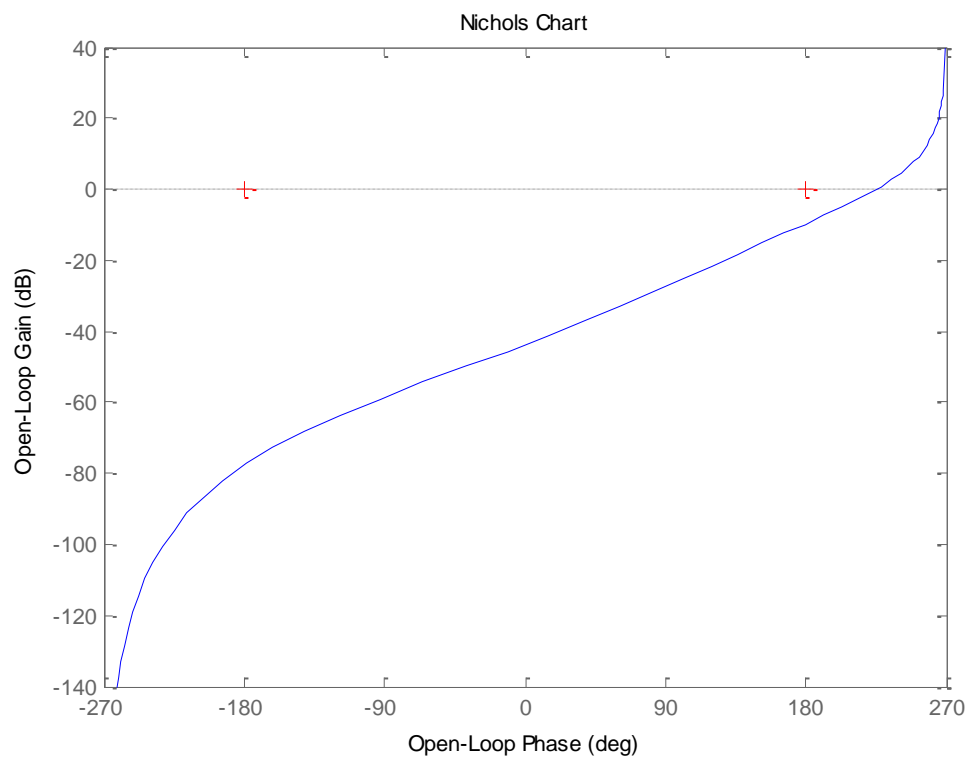
MATLAB code:

```
s = tf('s')
%a)
```

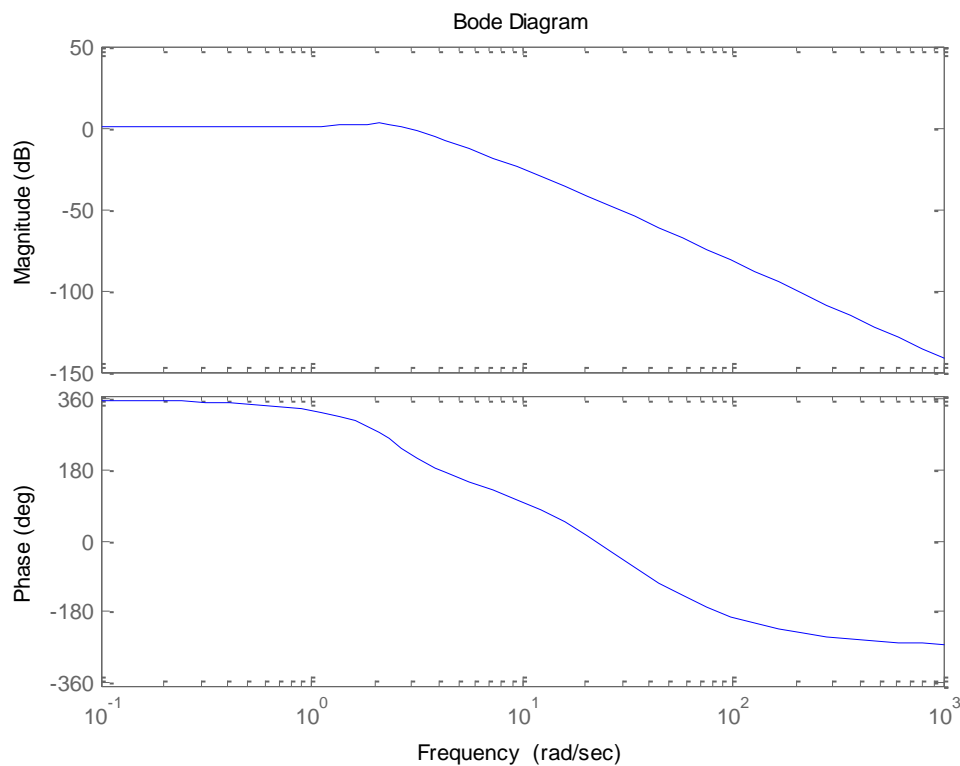
```
num_GH= pade(80*exp(-0.1*s),2);  
den_GH=s*(s+4)*(s+10);  
GH=num_GH/den_GH;  
CL = GH/(1+GH)  
BW = bandwidth(CL)  
bode(CL)
```

```
%b)  
figure(2);  
nichols(GH)
```

Part(a), Nicholas diagram:



Part(b), Bode diagram:



10- 54) Note: $G_{CL} = \frac{G}{1+G}$

To draw the Bode and polar plots use the closed loop transfer function, G_{CL} , and find BW. Use G to obtain the gain-phase plots and G_m and P_m . Use the Bode plot to graphically obtain M_r .

Sample MATLAB code:

```
s = tf('s')
%a)
num_G= 1+0.1*s;
den_G=s*(s+1)*(0.01*s+1);
G=num_G/den_G
figure(1)
nyquist(G)
figure(2)
margin(G)
GCL = G/(1+G)
BW = bandwidth(GCL)
figure(3)
bode(GCL)
```

Transfer function:

$$0.1 s + 1$$

$$\frac{0.01 s^3 + 1.01 s^2 + s}{0.0001 s^6 + 0.0202 s^5 + 1.041 s^4 + 2.131 s^3 + 2.11 s^2 + s}$$

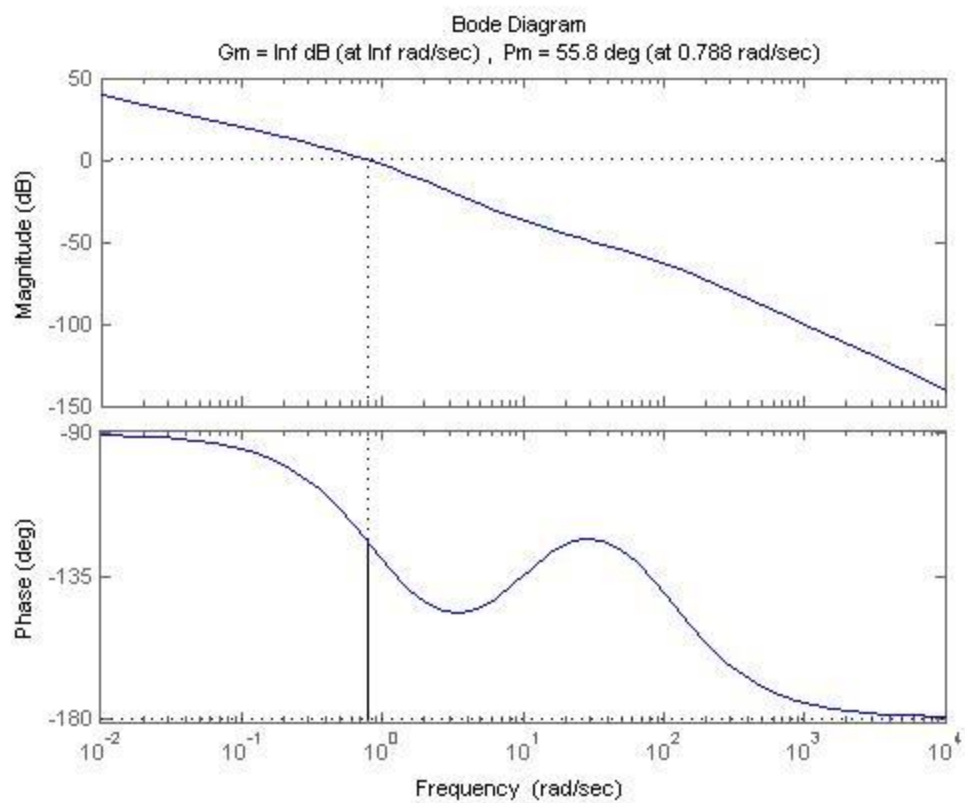
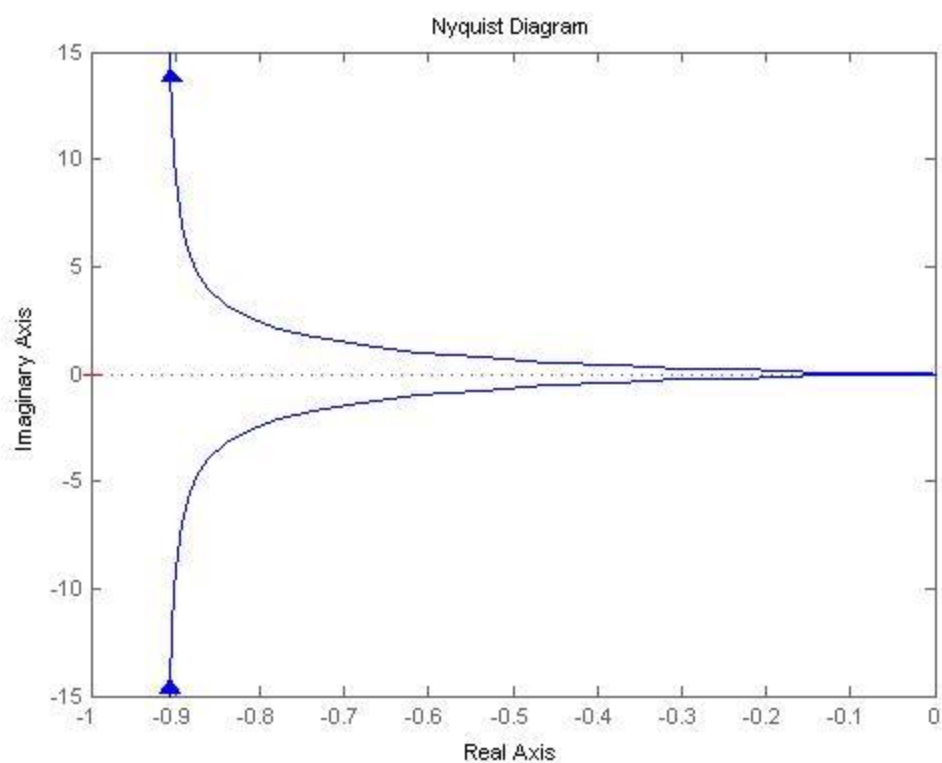
Transfer function:

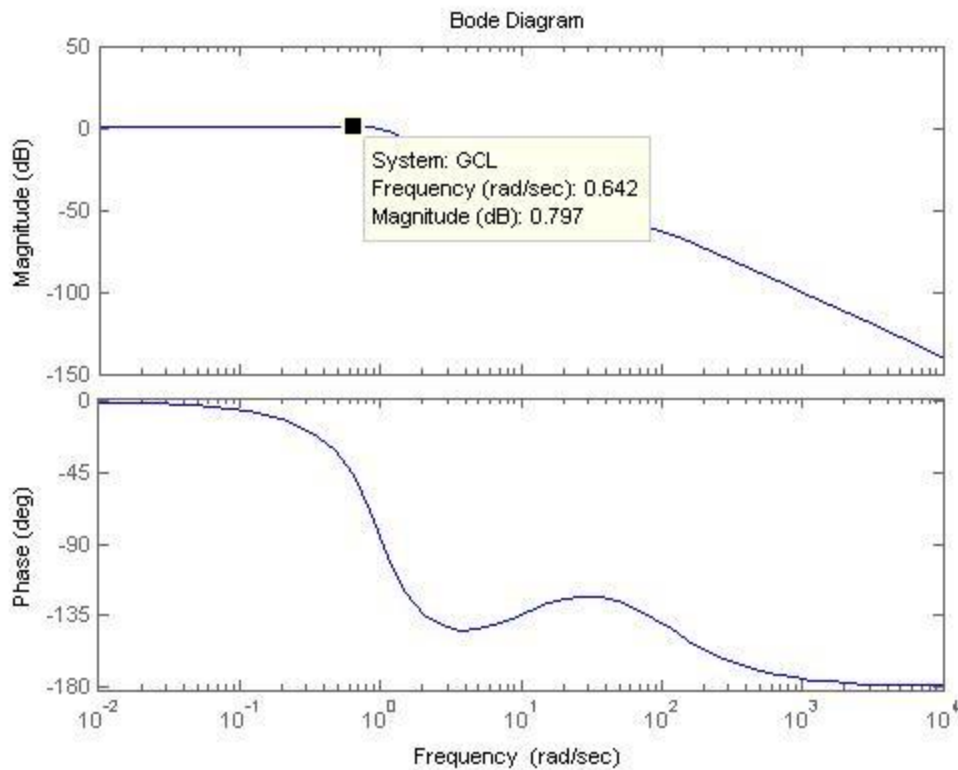
$$0.001 s^4 + 0.111 s^3 + 1.11 s^2 + s$$

$$\frac{0.001 s^4 + 0.111 s^3 + 1.11 s^2 + s}{0.0001 s^6 + 0.0202 s^5 + 1.041 s^4 + 2.131 s^3 + 2.11 s^2 + s}$$

BW =

$$1.2235$$





10-55 (a) The phase margin with $K = 1$ and $T_d = 0$ sec is approximately 57 deg. For a PM of 40 deg, the time delay produces a phase lag of -17 deg. The gain crossover frequency is 8 rad/sec.

Thus,

$$\omega T_d = 17^\circ = \frac{17^\circ \pi}{180^\circ} = 0.2967 \text{ rad/sec} \quad \text{Thus } \omega = 8 \text{ rad/sec}$$

$$T_d = \frac{0.2967}{8} = 0.0371 \text{ sec}$$

(b) With $K = 1$, for marginal stability, the time delay must produce a phase lag of -57 deg.

Thus, at $\omega = 8$ rad/sec,

$$\omega T_d = 57^\circ = \frac{57^\circ \pi}{180^\circ} = 0.9948 \text{ rad} \quad T_d = \frac{0.9948}{8} = 0.1244 \text{ sec}$$

10-56 (a) The phase margin with $K = 5$ dB and $T_d = 0$ is approximately 34.5 deg. For a PM of 30 deg, the time delay must produce a phase lag of -4.5 deg. The gain crossover frequency is 10 rad/sec. Thus,

$$\omega T_d = 4.5^\circ = \frac{4.5^\circ \pi}{180^\circ} = 0.0785 \text{ rad} \quad \text{Thus} \quad T_d = \frac{0.0785}{10} = 0.00785 \text{ sec}$$

(b) With $K = 5$ dB, for marginal stability, the time delay must produce a phase lag of -34.5 deg.

Thus at $\omega = 10$ rad/sec,

$$\omega T_d = 34.5^\circ = \frac{34.5^\circ \pi}{180^\circ} = 0.602 \text{ rad} \quad \text{Thus} \quad T_d = \frac{0.602}{10} = 0.0602 \text{ sec}$$

10-57) For a GM of 5 dB, the time delay must produce a phase lag of -34.5 deg at $\omega = 10$ rad/sec. Thus,

$$\omega T_d = 34.5^\circ = \frac{34.5^\circ \pi}{180^\circ} = 0.602 \text{ rad} \quad \text{Thus} \quad T_d = \frac{0.602}{10} = 0.0602 \text{ sec}$$

10-58 (a) Forward-path Transfer Function:

$$G(s) = \frac{Y(s)}{E(s)} = \frac{e^{-2s}}{(1+10s)(1+25s)}$$

From the Bode diagram, phase crossover frequency = 0.21 rad/sec GM = 21.55 dB

gain crossover frequency = 0 rad/sec

PM = infinite

(b)

$$G(s) = \frac{1}{(1+10s)(1+25s)(1+2s+2s^2)}$$

From the Bode diagram, phase crossover frequency = 0.26 rad/sec GM = 25 dB

gain crossover frequency = 0 rad/sec

PM = infinite

(c)

$$G(s) = \frac{1-s}{(1+s)(1+10s)(1+2s)}$$

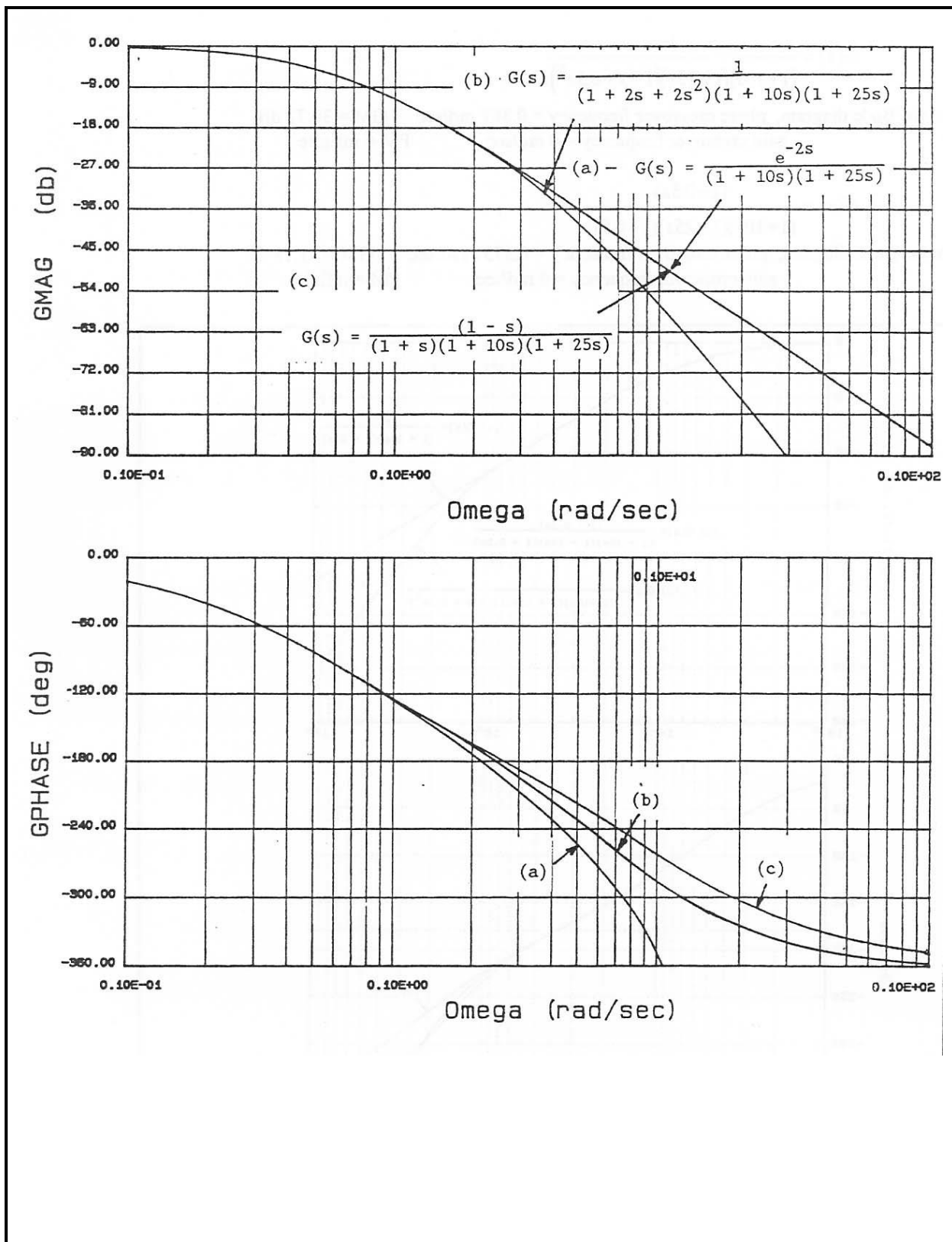
From the Bode diagram, phase crossover frequency = 0.26 rad/sec GM = 25.44 dB

gain crossover frequency = 0 rad/sec

PM = infinite

Sample MATLAB code

```
s = tf('s')
%a)
num_G=exp(-2*s);
den_G=(10*s+1)*(25*s+1);
G=num_G/den_G
figure(1)
margin(G)
```

10-58 (continued) Bode diagrams for all three parts.

10-59 (a) Forward-path Transfer Function:

$$G(s) = \frac{e^{-s}}{(1+10s)(1+25s)}$$

From the Bode diagram, phase crossover frequency = 0.37 rad/sec GM = 31.08 dB

gain crossover frequency = 0 rad/sec PM = infinite

(b)

$$G(s) = \frac{1}{(1+10s)(1+25s)(1+s+0.5s^2)}$$

From the Bode diagram, phase crossover frequency = 0.367 rad/sec GM = 30.72 dB

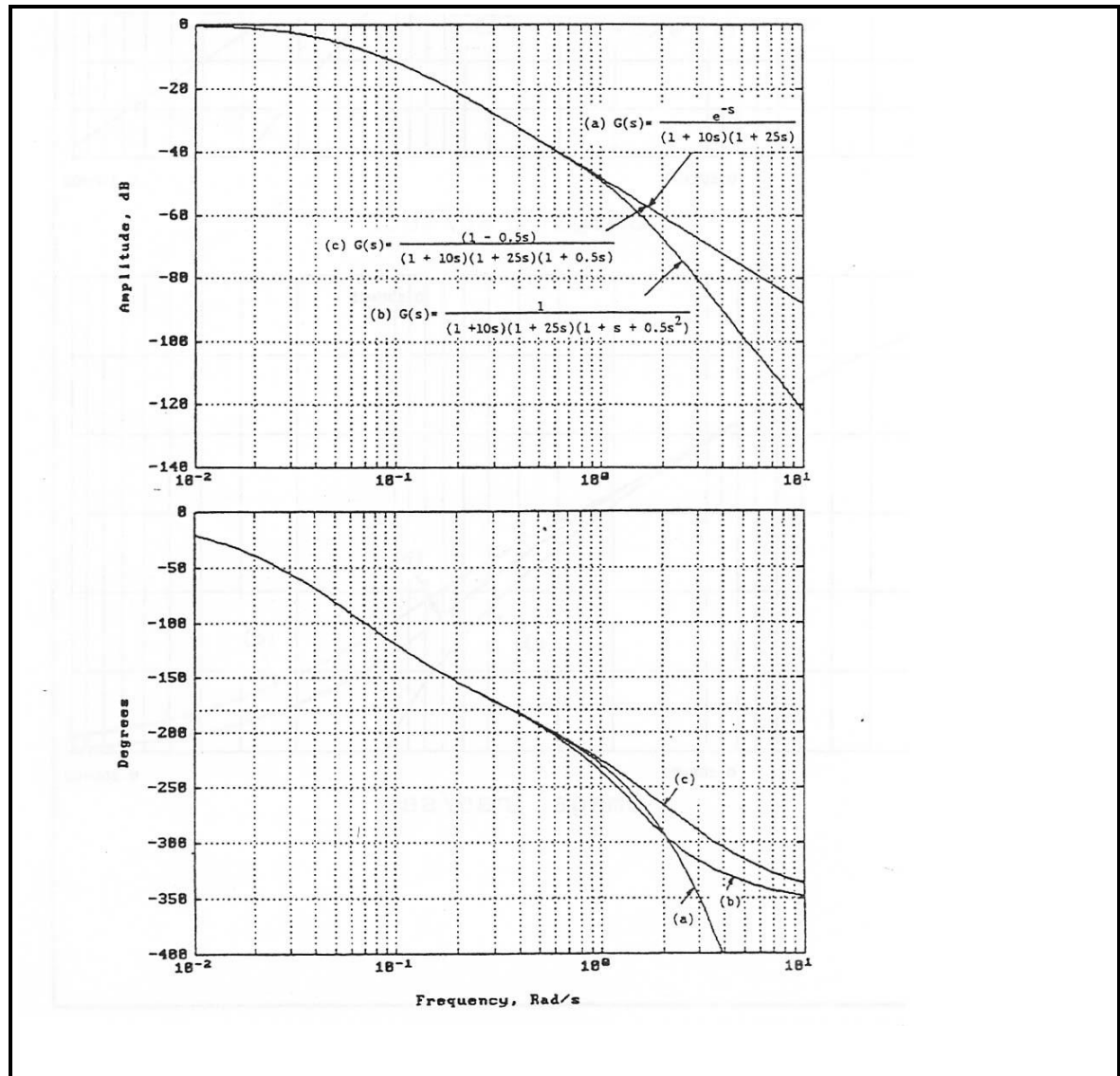
gain crossover frequency = 0 rad/sec PM = infinite

(c)

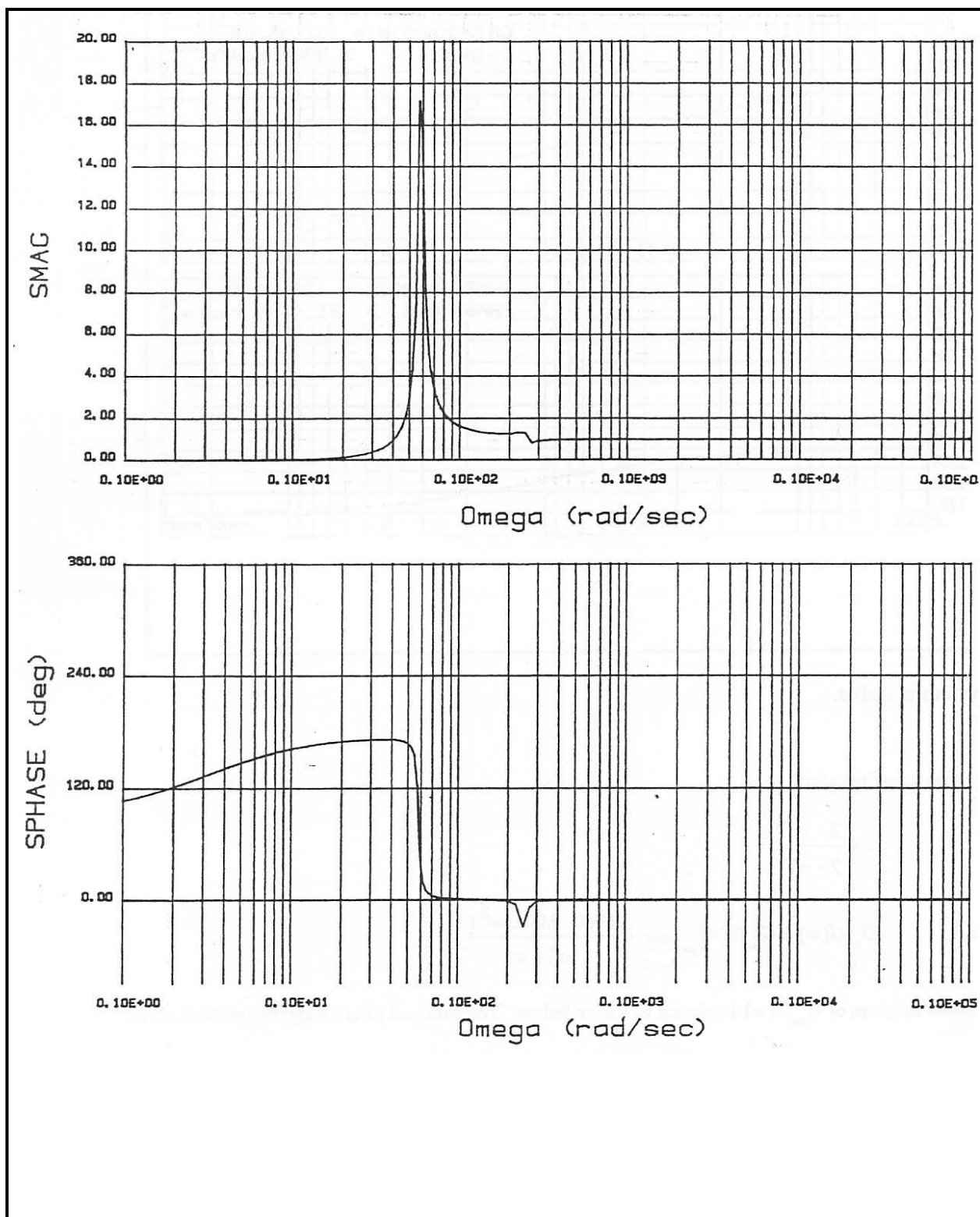
$$G(s) = \frac{(1-0.5s)}{(1+10s)(1+25s)(1+0.5s)}$$

From the Bode diagram, phase crossover frequency = 0.3731 rad/sec GM = 31.18 dB

gain crossover frequency = 0 rad/sec PM = infinite



Plots 10-59 (a-c)

10-60 Sensitivity Plot:

$$\left| S_G^M \right|_{\max} = 17.15 \quad \omega_{\max} = 5.75 \text{ rad/sec}$$

10-61)

$$(a) \quad G(s)H(s) = \frac{K(1.151s+0.1774)}{s^3+0.739s^2+0.921s}$$

$$(b) \quad \frac{G(s)H(s)}{1+G(s)H(s)} = \frac{K(1.151s+0.1774)}{s^3+0.739s^2+(0.921+1.151K)s+0.1774K}$$

(c)&(d)**MATLAB code:**

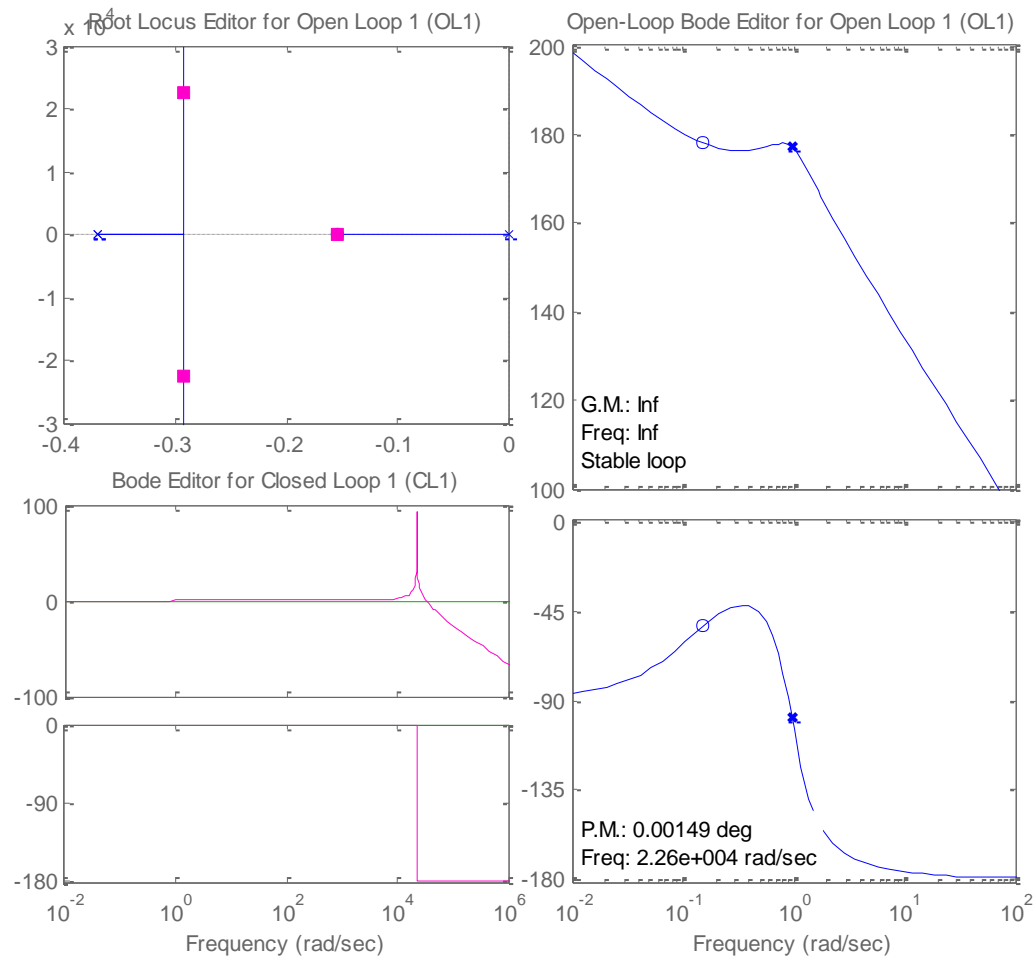
```

s = tf('s')
% c)
K = 1
num_GH= K*(1.151*s+0.1774);
den_GH=(s^3+0.739*s^2+0.921*s);
GH=num_GH/den_GH;
CL = GH/(1+GH)
sisotool
% d)
figure(1)
margin(CL)

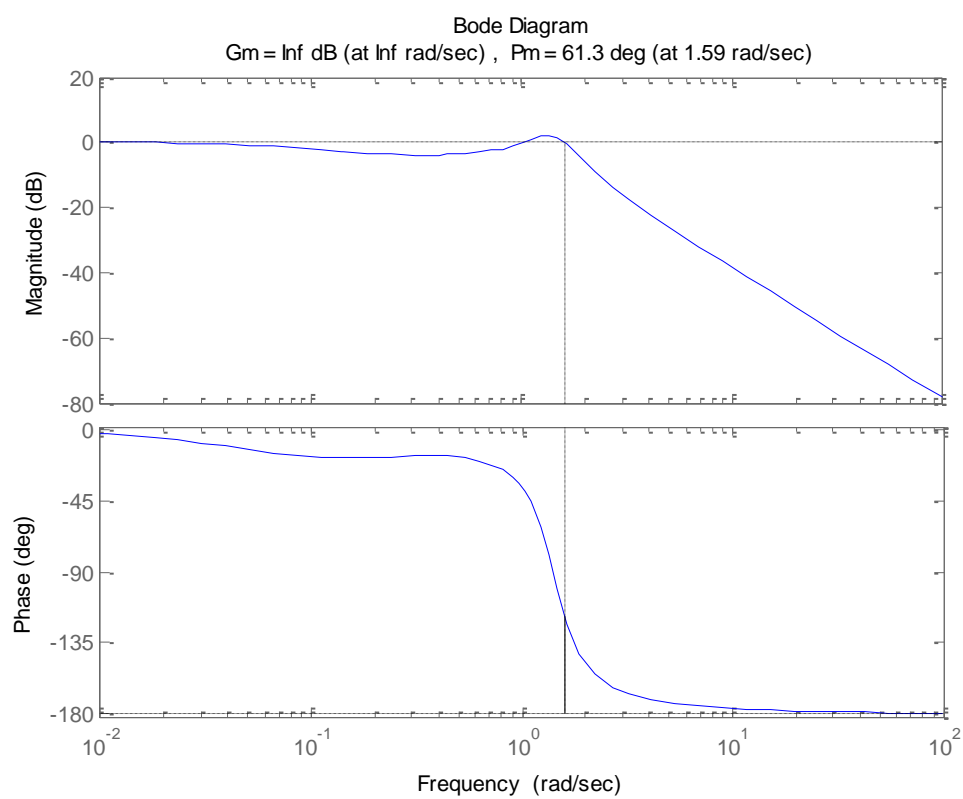
```

Part (c), range of K for stability:

Sisotool Result shows that by changing K between 0 and inf., all the roots of closed loop system remain in the left hand side plane and PM remains positive. Therefore, the system is stable for all positive K.



Part (d), Bode, GM & PM for K=1:



Chapter 11

11-1 Forward-path Transfer Function:

$$G(s) = \frac{M(s)}{1 - M(s)} = \frac{K}{s^3 + (20 + a)s^2 + (200 + 20a)s + 200a - K}$$

For type 1 system, $200a - K = 0$ Thus $K = 200a$

Ramp-error constant:

$$K_v = \lim_{s \rightarrow 0} sG(s) = \frac{K}{200 + 20a} = \frac{200a}{200 + 20a} = 5 \quad \text{Thus} \quad a = 10 \quad K = 2000$$

The forward-path transfer function is

The controller transfer function is

$$G(s) = \frac{2000}{s(s^2 + 30s + 400)}$$

$$G_c(s) = \frac{G(s)}{G_p(s)} = \frac{20(s^2 + 10s + 100)}{(s^2 + 30s + 400)}$$

The maximum overshoot of the unit-step response is 0 percent.

11-2 Forward-path Transfer Function:

$$G(s) = \frac{M(s)}{1 - M(s)} = \frac{K}{s^3 + (20 + a)s^2 + (200 + 20a)s + 200a - K}$$

For type 1 system, $200a - K = 0$ Thus $K = 200a$

Ramp-error constant:

$$K_v = \lim_{s \rightarrow 0} sG(s) = \frac{K}{200 + 20a} = \frac{200a}{200 + 20a} = 9 \quad \text{Thus} \quad a = 90 \quad K = 18000$$

The forward-path transfer function is

$$G(s) = \frac{18000}{s(s^2 + 110s + 2000)}$$

The controller transfer function is

$$G_c(s) = \frac{G(s)}{G_p(s)} = \frac{180(s^2 + 10s + 100)}{(s^2 + 110s + 2000)}$$

The maximum overshoot of the unit-step response is 4.3 percent.

From the expression for the ramp-error constant, we see that as a or K goes to infinity, K_v approaches 10.

Thus the maximum value of K_v that can be realized is 10. The difficulties with very large values of K and

a are that a high-gain amplifier is needed and unrealistic circuit parameters are needed for the controller.

11-3) The close loop transfer function is:

$$\frac{Y(s)}{X(s)} = \frac{K}{s^2 + s + K} = \frac{K}{s^2 + \frac{1}{\tau}s + \frac{K}{\tau}}$$

Comparing with second order system:

$$\omega_n = \sqrt{\frac{K}{\tau}} \text{ and } 2\xi\omega_n = \frac{1}{\tau}$$

$$M_p = \exp\left(-\frac{\pi\xi}{\sqrt{1-\xi^2}}\right) = 0.254 \Rightarrow \xi = 0.4$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = 3 \Rightarrow \omega_n = 1.14$$

$$\tau = \frac{1}{2\xi\omega_n} \Rightarrow \tau = 1.09$$

$$K = \tau\omega_n^2 \Rightarrow K = 1.42$$

11-4) The forward path transfer function of the system is:

$$G(s)H(s) = \frac{24}{s(s+1)(s+6)}$$

1. The steady state error is less than to $\pi/10$ when the input is a ramp with a slope of 2π rad/sec

$$e_{ss} = \frac{\pi}{10} = \lim_{s \rightarrow 0} \frac{R}{sG_c(s)G(s)} = \lim_{s \rightarrow 0} \frac{2\pi}{s(K_p + K_d s) \frac{24}{s(s+1)(s+6)}} = \frac{2\pi}{4K_p}$$

As a result $K_p > 0.2$

2. The phase margin is between 40 to 50 degrees

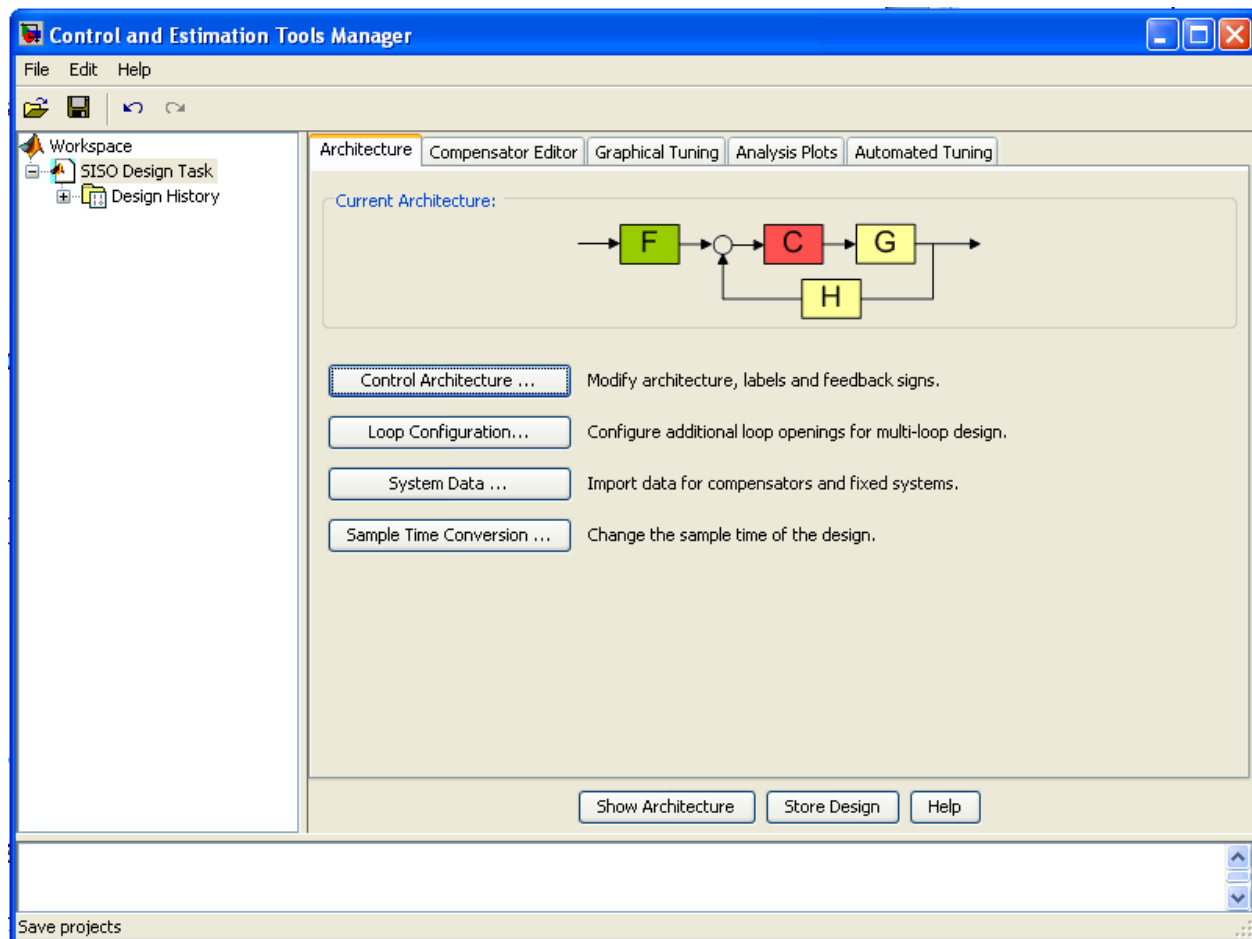
In this part of the solution, MATLAB sisotool can be very helpful. More detailed instructions on using MATLAB sisotool is presented in the solutions for this particular problem. Similar guidelines could be used for similar questions of this chapter.

SISOTOOL quick instructions:

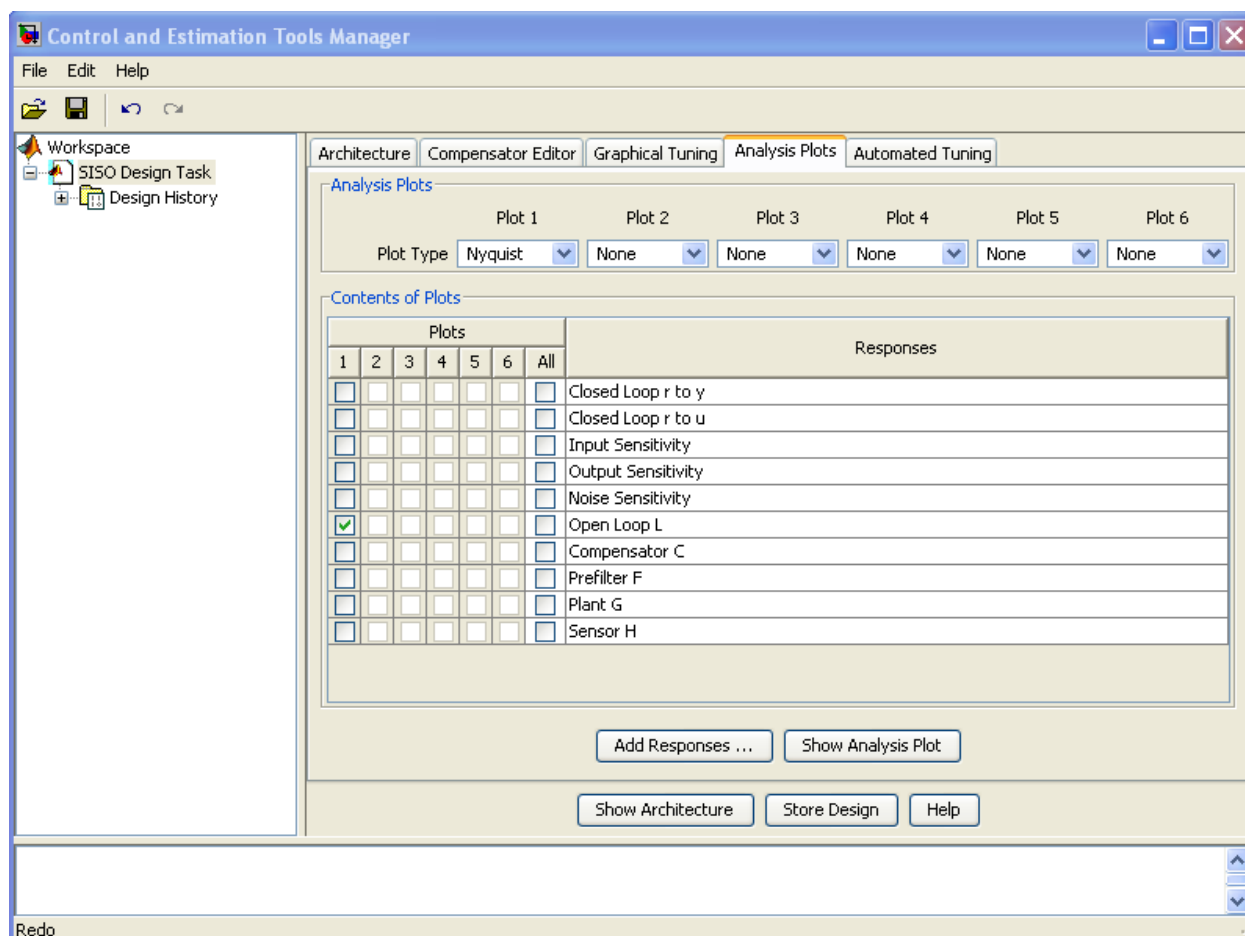
Once opening the sisotool by typing sisotool in MATLAB the command window (or in the “m” code), the following window pops up:

Where you can insert transfer functions for C, G, H, and F, or you can leave some of them as default value (1), by clicking on “System Data”. Once you substitute transfer functions, you will see a graph including the root-locus diagram, a closed loop Bode diagram, and an open loop Bode diagram indicating the Gain Margin and Phase Margin as well.

** You can drag the open-loop bode magnitude diagram up and down to see the effect of gain change on all of the graphs. Sisotool updates all these graph instantly. You can also drag the poles and zeros on the root locus diagram to observe the effect on the other diagrams.

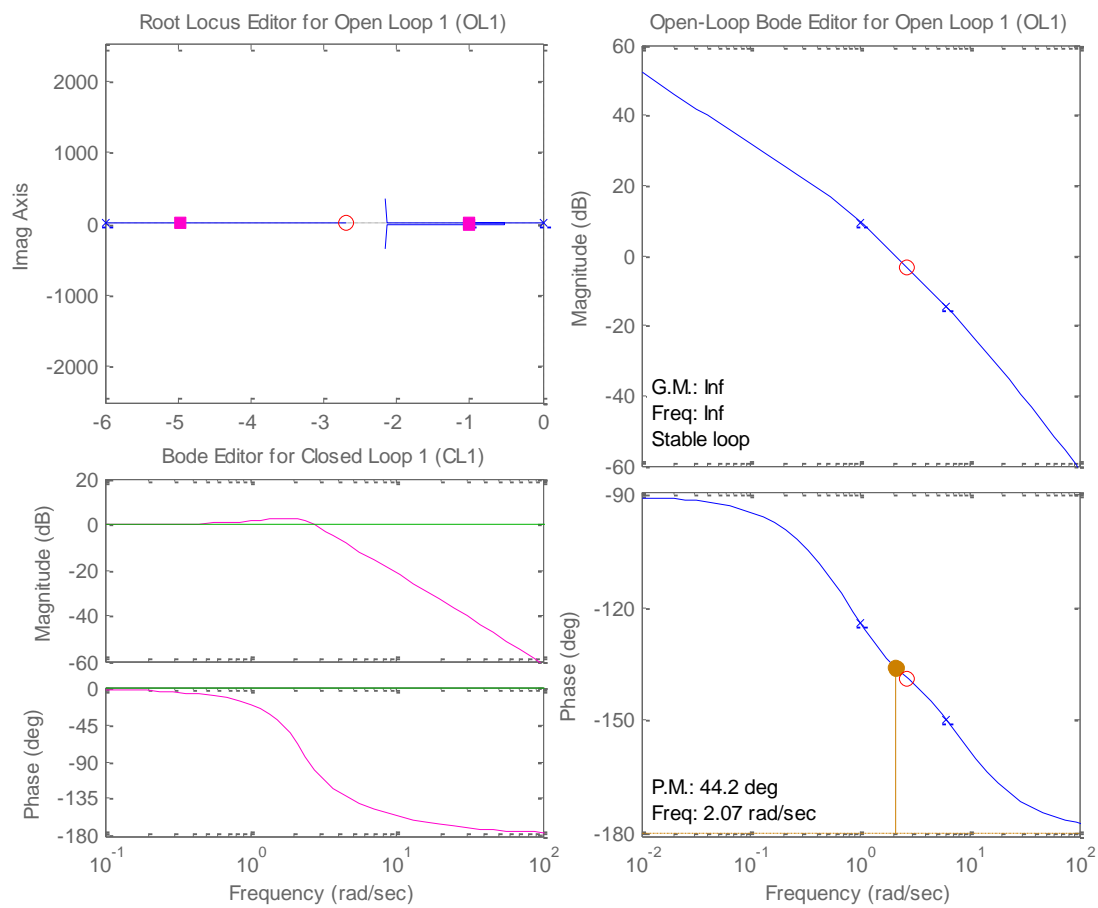
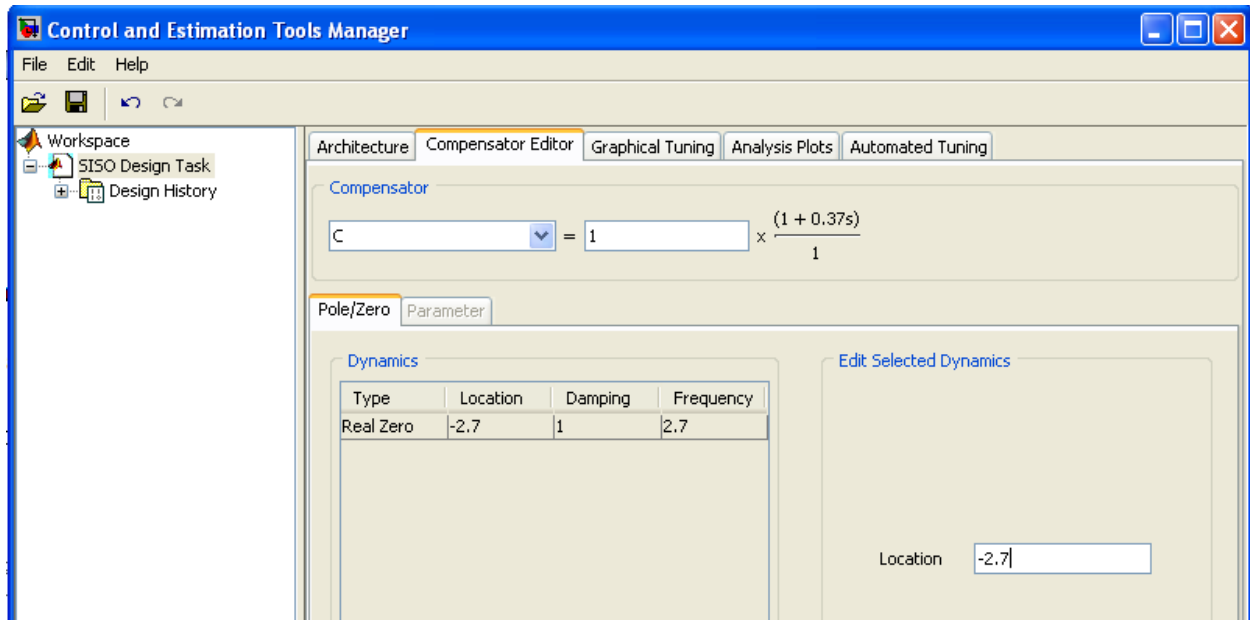


You can also open “Analysis Plots” tab to add other graphs such as Nyquist diagram as shown in the following figure:



In this particular question, you need to add a zero to include the effect of K_d . You can add a zero by using the “Compensator Editor” tab, as shown in the following graph. The last thing you need to do for this problem, is to drag the location of zero and gain in the following diagram (or edit these locations by assigning C gain and Zero location in the “Compensator Editor” tab), so it satisfies the PM of 40 to 50 deg; while gain of C is kept above 0.2 ($K_p \geq 0.2$, from part (a)).

In the following snapshot of “Compensator Editor”, C gain or K_p is set to 1, and the zero location is set to -2.7, resulting in 44.2 [deg] phase margin, presented in the following figure.



Final answers: $K_p = 1$, $K_d = 0.3704$

Preliminary MATLAB code for 11-4:

```
%solving for k:

syms kc

omega=1.5

sol=eval(solve('0.25*kc^2=0.7079^2*(-0.25*omega^3+omega)^2+(-0.375*omega^2+0.5*kc)^2',kc))

%plotting bode with K=1.0370

s = tf('s')

K=1.0370;

num_G_a= 0.5*K;

den_G_a=s*(0.25*s^2+0.375*s+1);

G_a=num_G_a/den_G_a;

CL_a = G_a/(1+G_a)

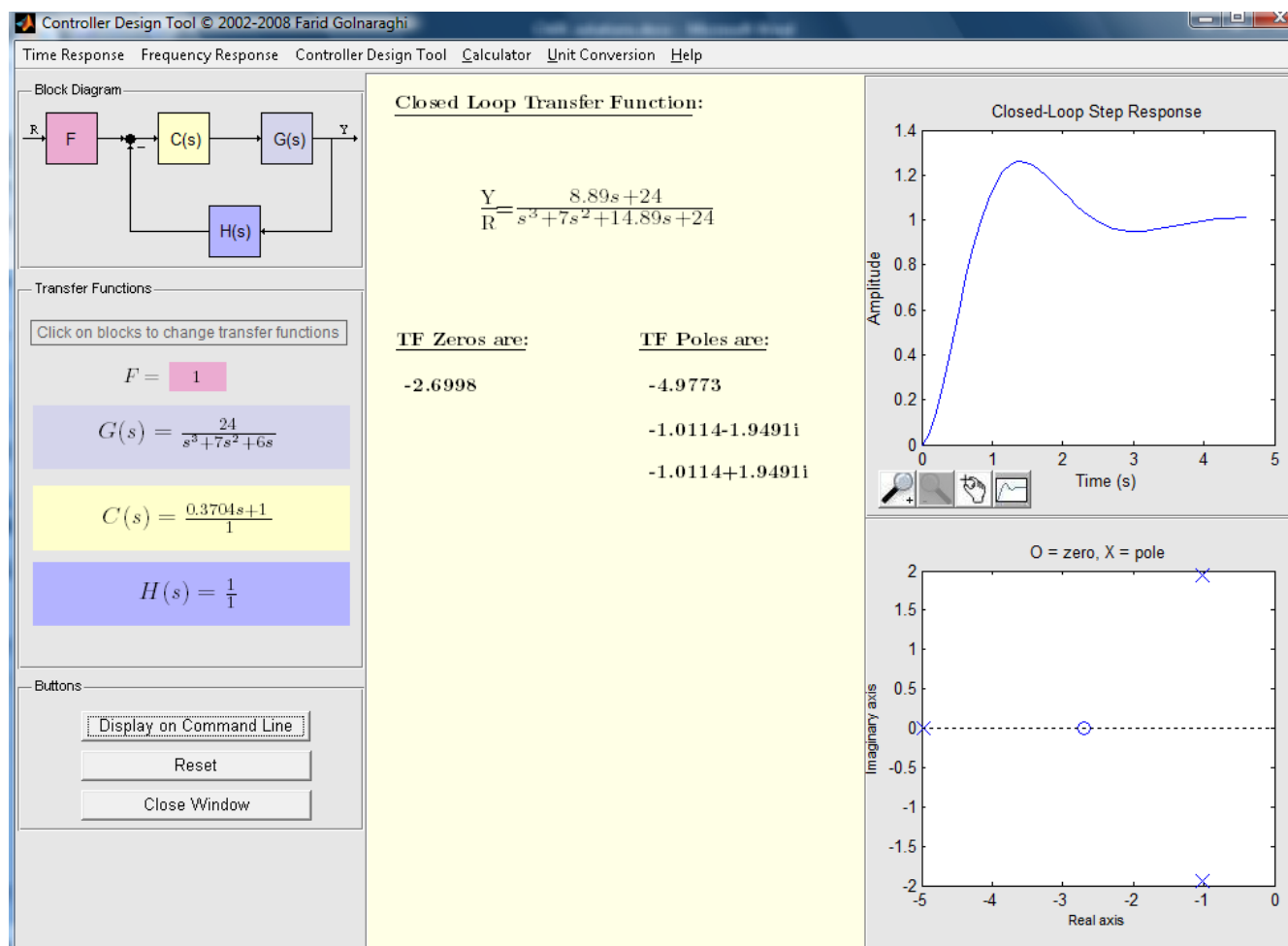
BW = bandwidth(CL_a)

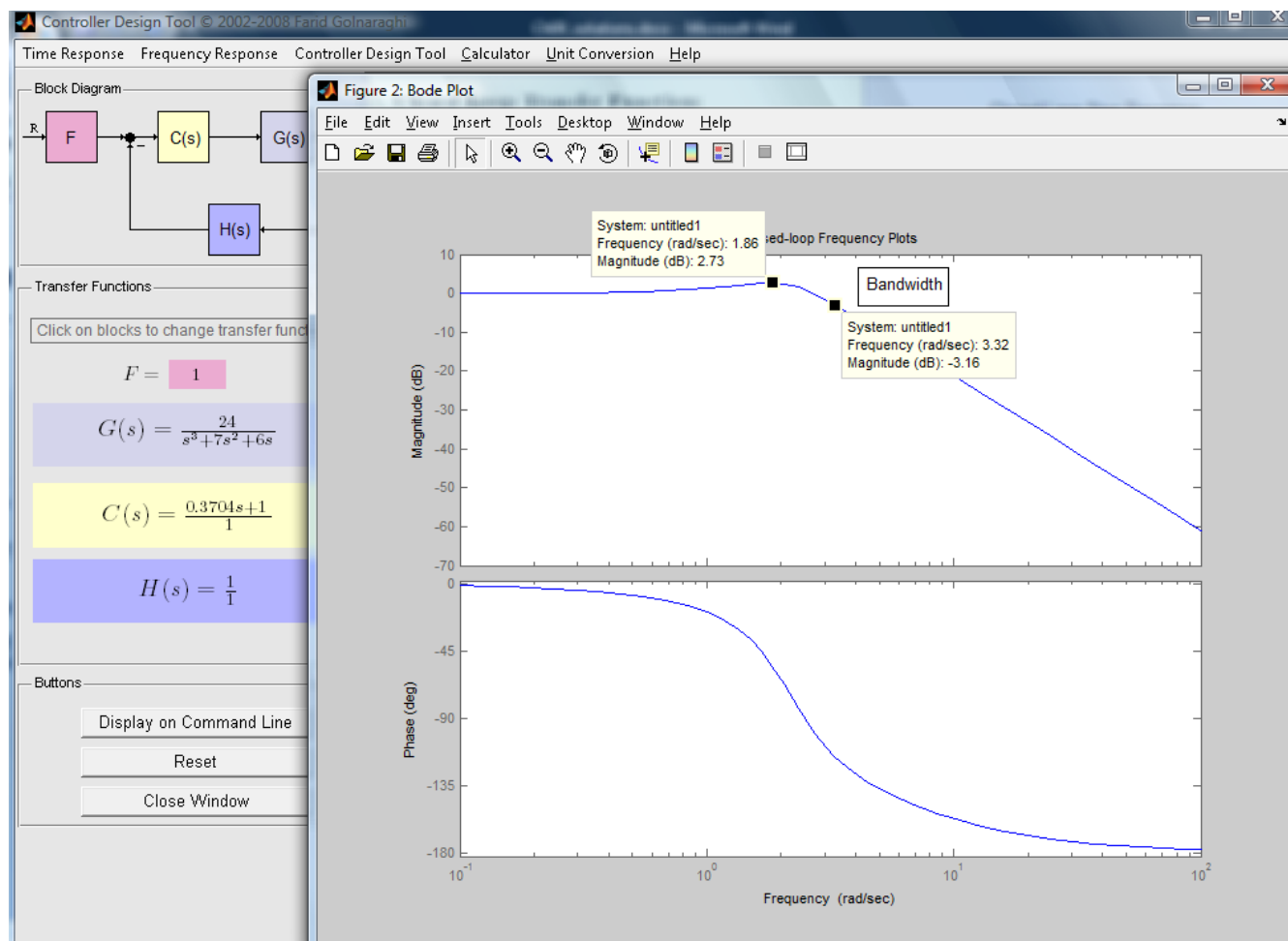
bode(CL_a);

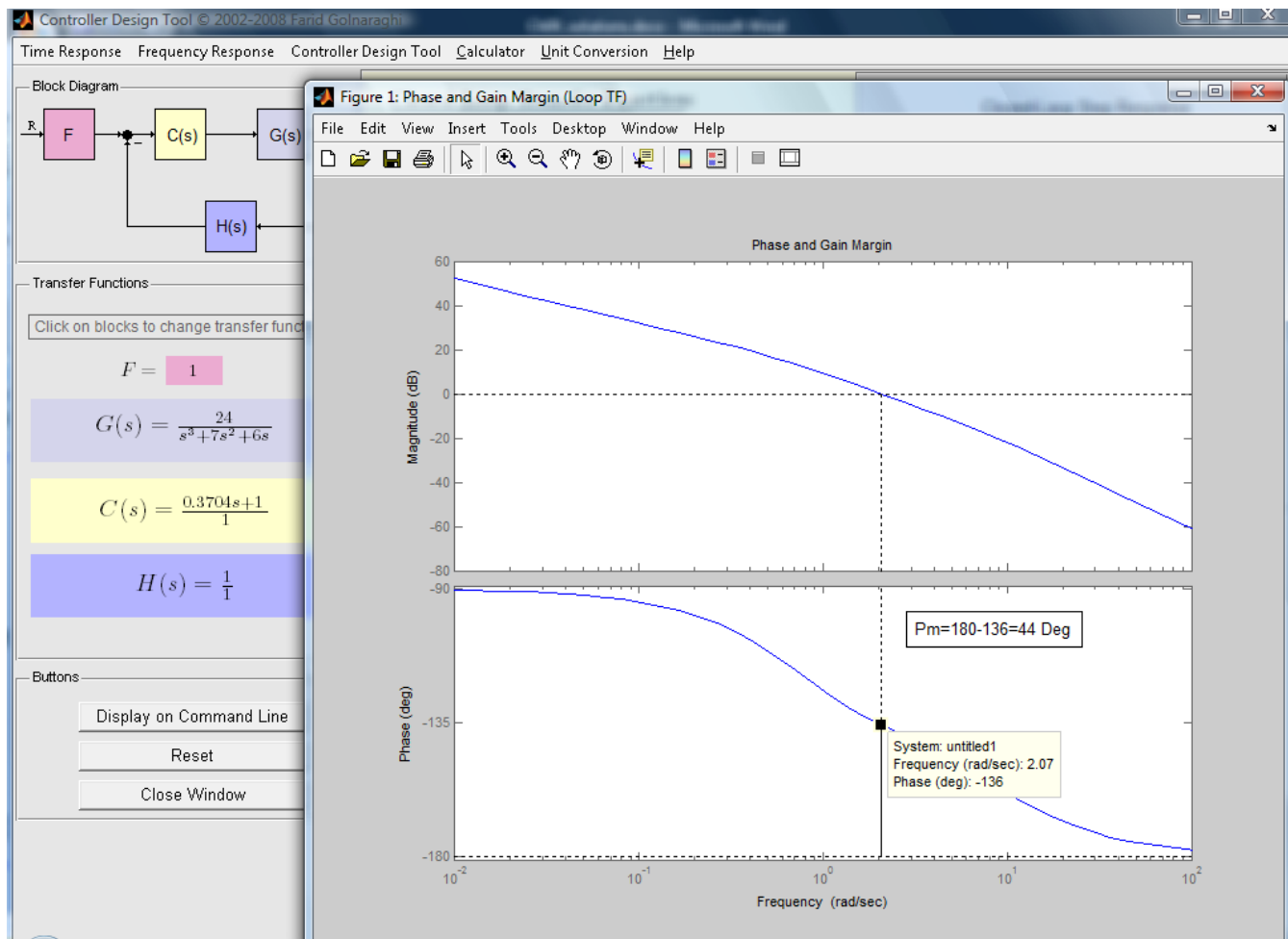
sisotool;
```

Alternatively we can use ACSYS.

$$K_p = 1, K_d = 0.3704$$







11-5) (a) Ramp-error Constant:

$$K_v = \lim_{s \rightarrow 0} s \frac{1000(K_p + K_D s)}{s(s+10)} = \frac{1000K_p}{10} = 100K_p = 1000 \quad \text{Thus} \quad K_p = 10$$

Characteristic Equation: $s^2 + (10 + 1000K_D)s + 1000K_p = 0$

$$\omega_n = \sqrt{1000K_p} = \sqrt{10000} = 100 \text{ rad/sec} \quad 2\zeta\omega_n = 10 + 1000K_D = 2 \times 0.5 \times 100 = 100$$

Thus $K_D = \frac{90}{1000} = 0.09$

11-5 (b) For $K_v = 1000$ and $\zeta = 0.707$, and from part (a), $\omega_n = 100$ rad/sec,

$$2\zeta\omega_n = 10 + 1000K_D = 2 \times 0.707 \times 100 = 141.4 \quad \text{Thus} \quad K_D = \frac{131.4}{1000} = 0.1314$$

(c) For $K_v = 1000$ and $\zeta = 1.0$, and from part (a), $\omega_n = 100$ rad/sec,

$$2\zeta\omega_n = 10 + 1000K_D = 2 \times 1 \times 100 = 200 \quad \text{Thus} \quad K_D = \frac{190}{1000} = 0.19$$

11-6) The ramp-error constant:

$$K_v = \lim_{s \rightarrow 0} s \frac{1000(K_p + K_D s)}{s(s+10)} = 100K_p = 10,000 \quad \text{Thus} \quad K_p = 100$$

The forward-path transfer function is:
$$G(s) = \frac{1000(100 + K_D s)}{s(s+10)}$$

K_D	PM (deg)	GM	M_r	BW (rad/sec)	Max overshoot (%)
0	1.814	∞	13.5	493	46.6
0.2	36.58	∞	1.817	525	41.1
0.4	62.52	∞	1.291	615	22
0.6	75.9	∞	1.226	753	13.3
0.8	81.92	∞	1.092	916	8.8
1.0	84.88	∞	1.06	1090	6.2

The phase margin increases and the maximum overshoot decreases monotonically as K_D increases.

Sample MATLAB CODE for time frequency responses

```

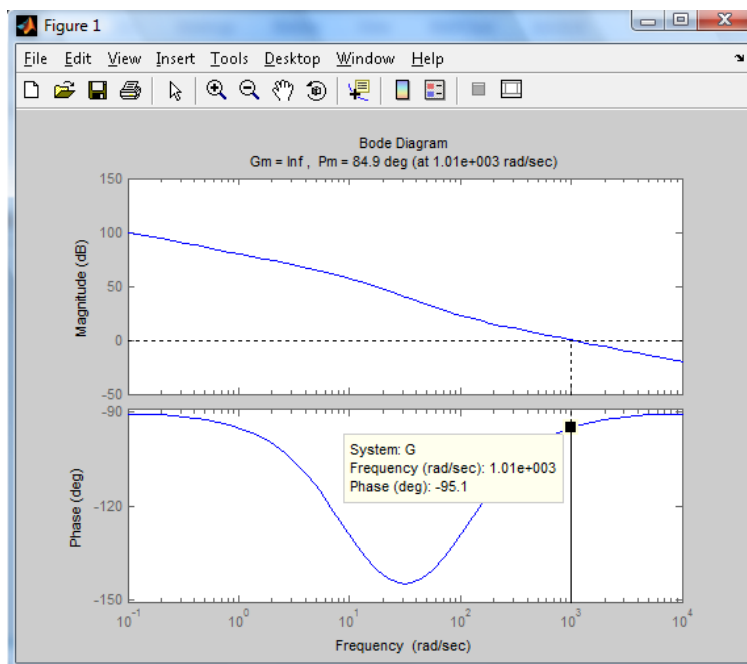
clear all
KD=1.0;
num = [-100/KD];
den = [0 -10];
G=zpk(num,den,1000);
figure(1)
margin(G)
M=feedback(G,1)
figure(2)
step(M);
figure(3)
bode(M)

```

Zero/pole/gain:

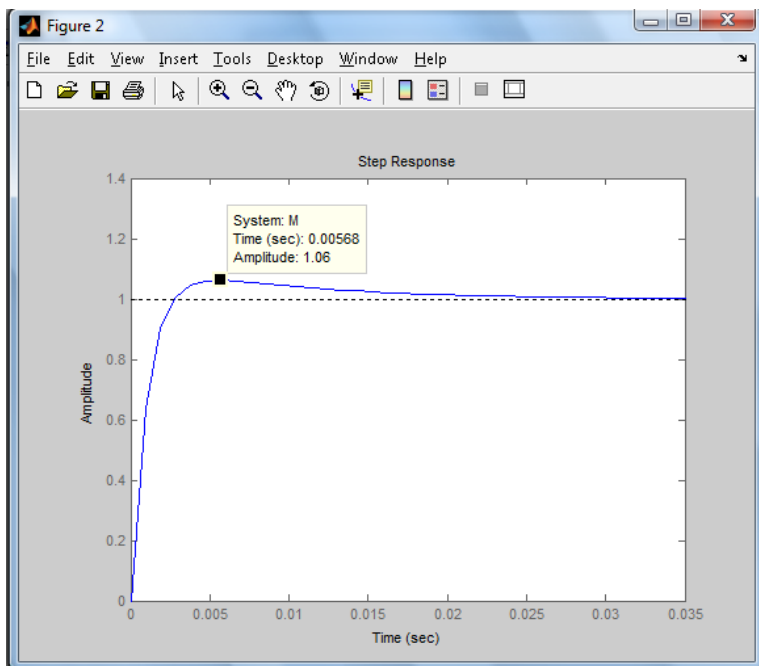
1000 (s+100)

(s+111.3) (s+898.7)



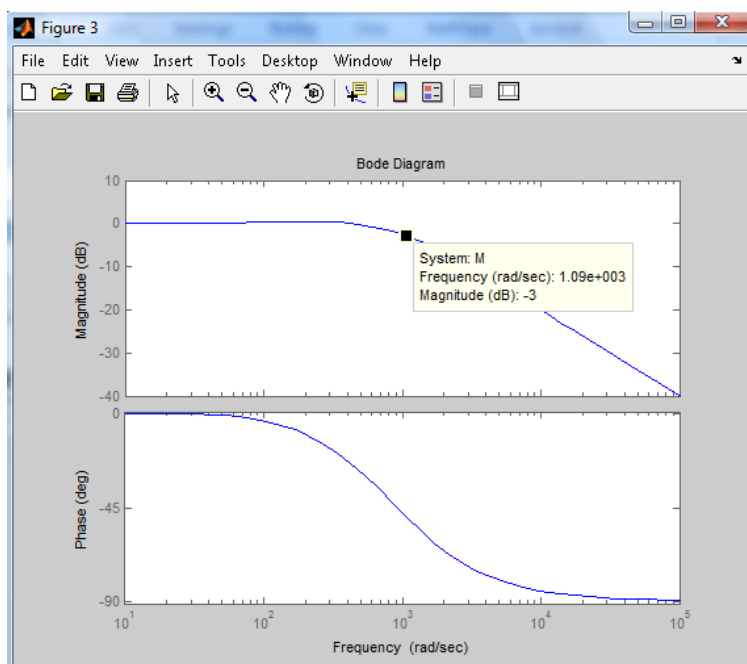
Phase margin is $Pm=180-95=85$ Deg

$Gm=\infty$



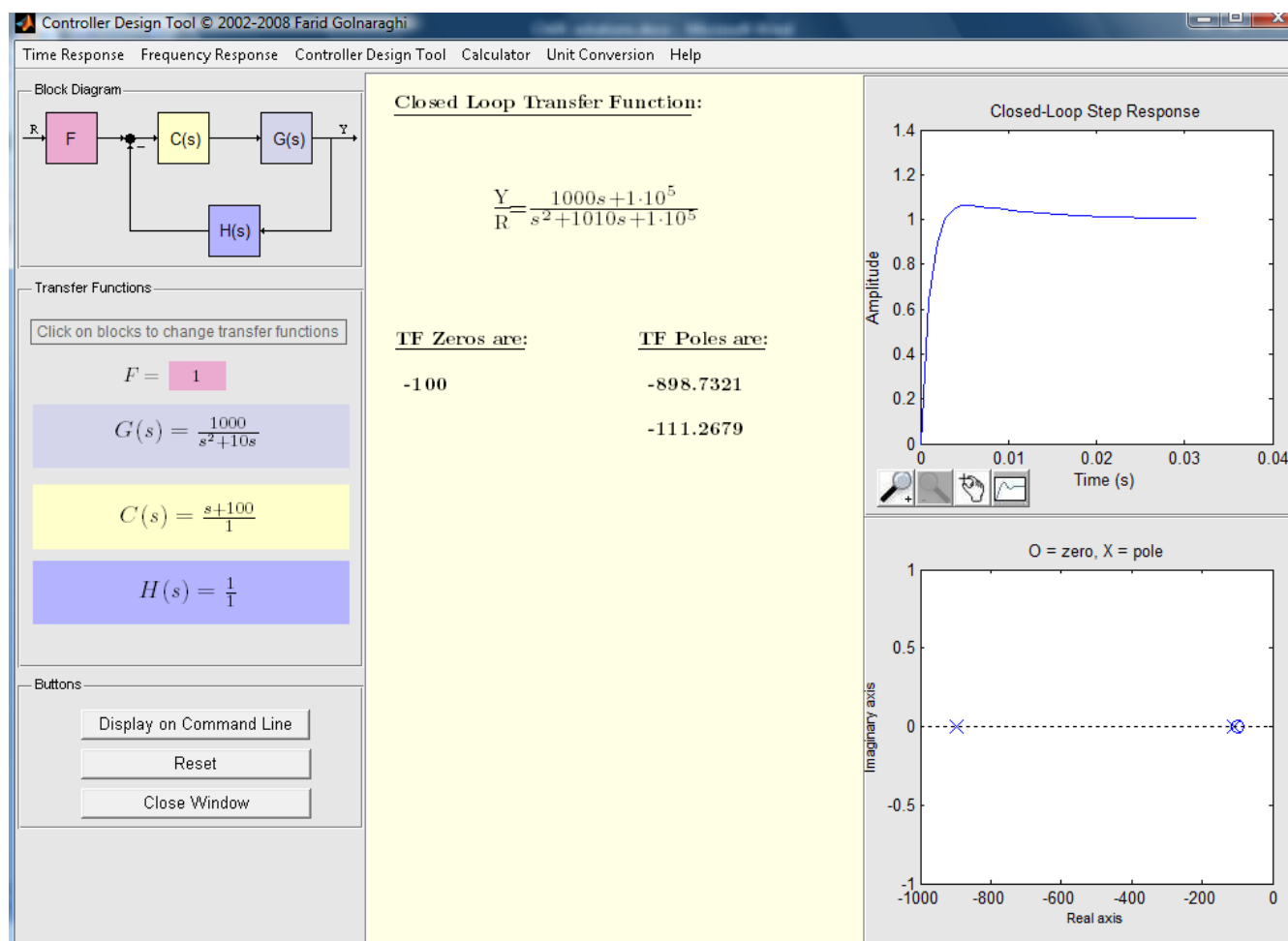
Use the cursor to obtain the PO and tr values.

KD increase results in the minimum overshoot.



Bandwidth is 1090 rads/s.

Use ACSYS to find the same results:

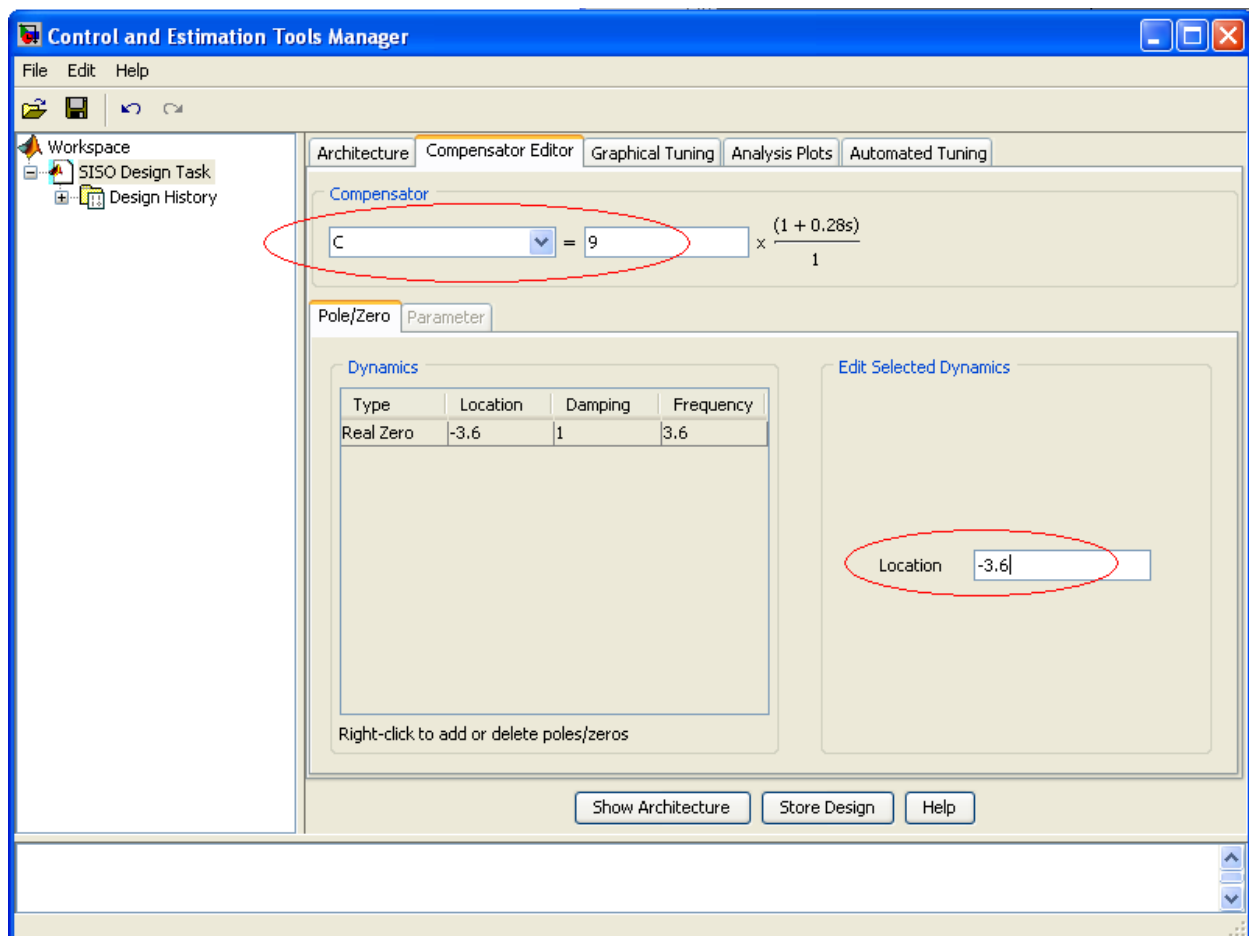


11-7)**PD controller design**

The open-loop transfer function of a system is:

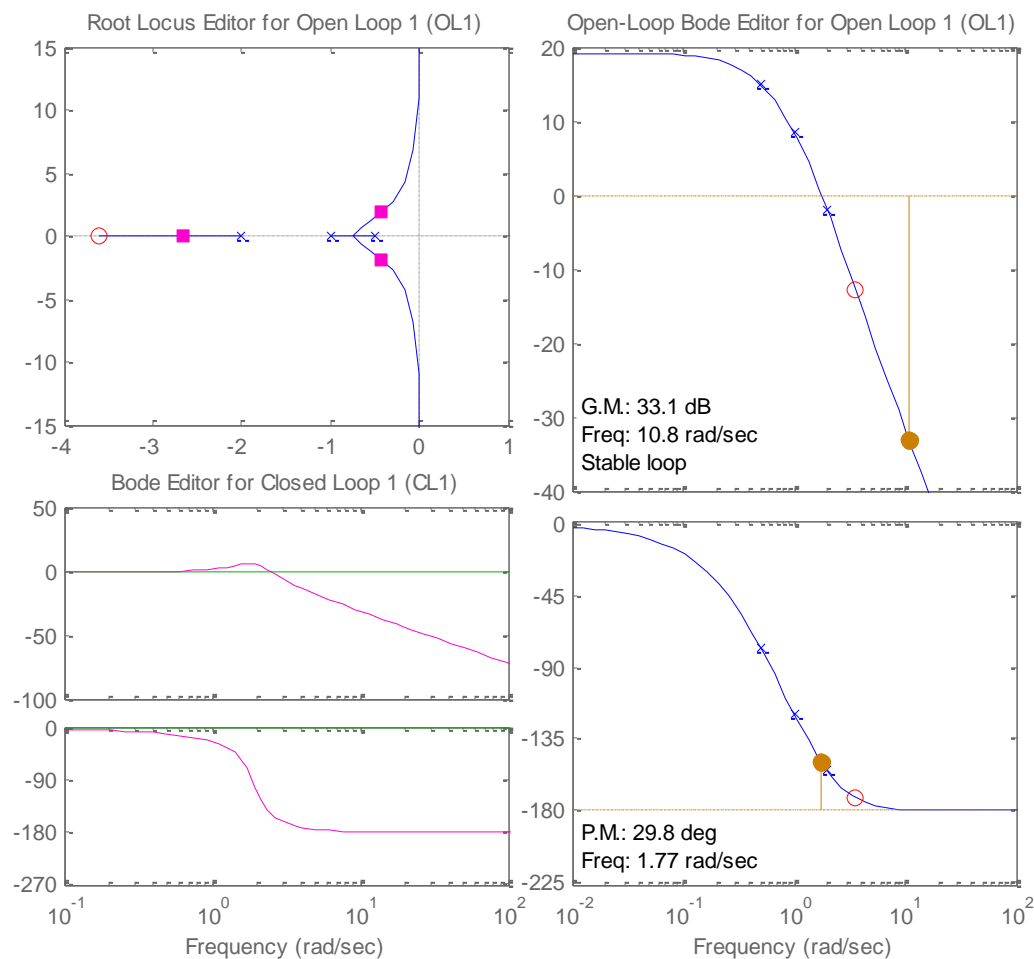
$$G(s)H(s) = \frac{1}{(2s+1)(s+1)(0.5s+1)}$$

The solution is very similar to 11-4. The transfer functions are inserted into sisotool, where another real zero is added to represent the effect of K_d . That is $C(s) = K_p + K_d s = K_p(1 + K_d s / K_p)$, which is called the compensator transfer function in sisotool. The place of real zero is $Z = -K_p / K_d$, and the gain of the compensator is equal to K_p , as noted in the following sisotool window:



By fixing the gain to 9, and starting to change the zero location, PM can be adjusted to above 25 [deg] as required by the question. The current setting has a zero at -3.5, which resulted in 30 [deg] phase margin and 33.1 dB gain margin as seen in the following diagrams.

The design requires $K_p = 9$ and $K_d = -K_p / Z = -9 / -3.6 = 2.5$



Preliminary MATLAB code for 11-7:

```
s = tf('s')
Kp = 1
num_GH= Kp*1;
den_GH=(2*s+1)*(s+1)*(0.5*s+1);
```

GH=num_GH/den_GH;

CL = GH/(1+GH)

Sisotool

11-8: PD controller design: The open-loop transfer function of a system is:

$$G(s)H(s) = \frac{60}{s(0.4s + 1)(s + 1)(s + 6)}$$

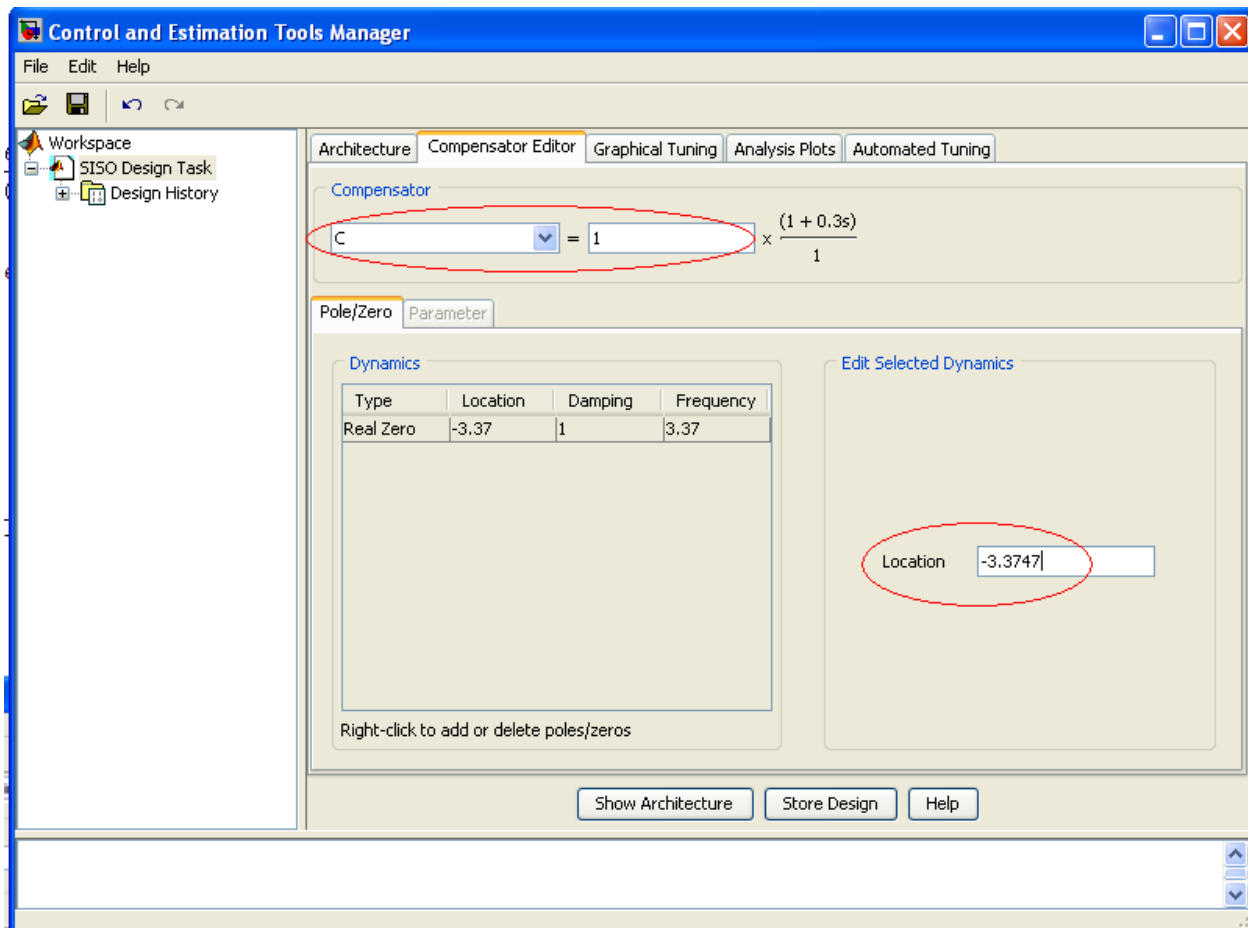
(a) Design a PD controller to satisfy the following specifications:

- (i) $K_v = 10$
- (ii) the phase margin is 45 degrees.

$$K_v = \lim_{s \rightarrow 0} sG(s) = \lim_{s \rightarrow 0} \frac{60(K_D s + K_p)}{(0.4s + 1)(s + 1)(s + 6)} = 10K_p = 10$$

As a result: $K_p = 1$

The rest of the procedure is similar to 11-7:

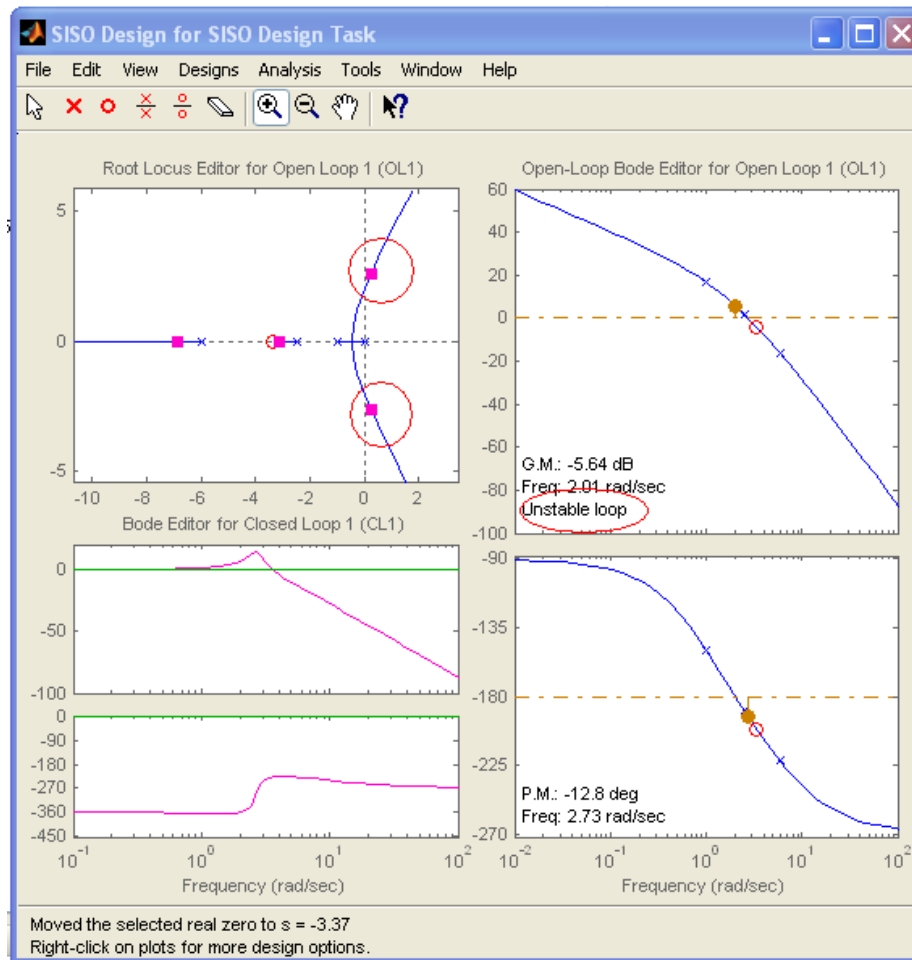


The transfer functions are inserted into sisotool, where another real zero is added to represent the effect of K_d . That is $C(s) = K_p + K_d s = K_p (1 + K_d s / K_p)$, which is called the compensator transfer function in sisotool.

The place of real zero is $Z = -K_p / K_d$, and the gain of the compensator is equal to K_p , as noted in the sisotool window:

$K_p = 1$ fixed to 1, and zero location was changed in the entire real axis. However, 2 of the closed loop poles remained in the right hand side of S plane in the root locus diagram, indicating instability for all K_d values.

Solution for $K_v = 10$ with PD controller and $PM=45$ [deg] does not exist. Unstable close loop poles are indicated in the root locus diagram of the following figure:

**11-8)****Preliminary MATLAB code for 11-8:**

```

s = tf('s')
Kp = 1
num_GH= Kp*60;
den_GH=s*(0.4*s+1)*(s+1)*(s+6);
GH=num_GH/den_GH;
CL = GH/(1+GH)

```

```

sisotool

```

11-9) (a) Forward-path Transfer Function:

$$G(s) = G_c(s)G_p(s) = \frac{4500K(K_D + K_P s)}{s(s + 361.2)}$$

Ramp Error Constant: $K_v = \lim_{s \rightarrow 0} sG(s) = \frac{4500KK_P}{361.2} = 12.458KK_P$

$$e_{ss} = \frac{1}{K_v} = \frac{0.0802}{KK_P} \leq 0.001 \quad \text{Thus} \quad KK_P \geq 80.2 \quad \text{Let} \quad K_P = 1 \quad \text{and} \quad K = 80.2$$

Attributes of Unit-step Response:

K_D	t_r (sec)	t_s (sec)	Max Overshoot (%)
0	0.00221	0.0166	37.1
0.0005	0.00242	0.00812	21.5
0.0010	0.00245	0.00775	12.2
0.0015	0.0024	0.0065	6.4
0.0016	0.00239	0.00597	5.6
0.0017	0.00238	0.00287	4.8
0.0018	0.00236	0.0029	4.0
0.0020	0.00233	0.00283	2.8

Select $K_D \geq 0.0017$

(b) BW must be less than 850 rad/sec.

K_D	GM	PM (deg)	M_r	BW (rad/sec)
0.0005	∞	48.45	1.276	827
0.0010	∞	62.04	1.105	812
0.0015	∞	73.5	1.033	827
0.0016	∞	75.46	1.025	834
0.0017	∞	77.33	1.018	842
0.00175	∞	78.22	1.015	847
0.0018	∞	79.07	1.012	852

Select $K_D \cong 0.00175$. A larger K_D would yield a BW larger than 850 rad/sec.

11-10)**The forward-path Transfer Function: $N = 20$**

$$G(s) = \frac{200(K_p + K_D s)}{s(s+1)(s+10)}$$

To stabilize the system, we can reduce the forward-path gain. Since the system is type 1, reducing the gain does not affect the steady-state liquid level to a step input. Let $K_p = 0.05$

$$G(s) = \frac{200(0.05 + K_D s)}{s(s+1)(s+10)}$$

Unit-step Response Attributes:

K_D	t_s (sec)	Max Overshoot (%)
0.01	5.159	12.7
0.02	4.57	7.1
0.03	2.35	3.2
0.04	2.526	0.8
0.05	2.721	0
0.06	3.039	0
0.10	4.317	0

When $K_D = 0.05$ the rise time is 2.721 sec, and the step response has no overshoot

11-11)**(a)** For $e_{ss} = 1$,

$$K_v = \lim_{s \rightarrow 0} sG(s) = \lim_{s \rightarrow 0} s \frac{200(K_p + K_D s)}{s(s+1)(s+10)} = 20K_p = 1 \quad \text{Thus } K_p = 0.05$$

Forward-path Transfer Function:

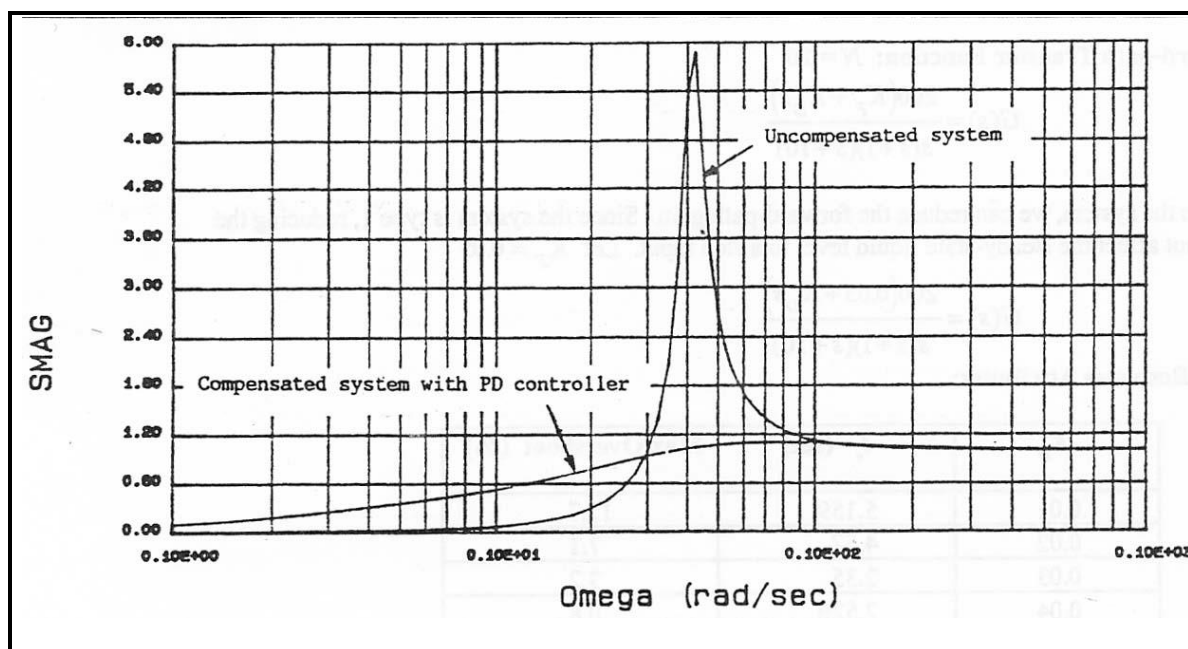
$$G(s) = \frac{200(0.05 + K_D s)}{s(s+1)(s+10)}$$

Attributes of Frequency Response:

K_D	PM (deg)	GM (deg)	M_r	BW (rad/sec)
0	47.4	20.83	1.24	1.32
0.01	56.11	∞	1.09	1.24
0.02	64.25	∞	1.02	1.18
0.05	84.32	∞	1.00	1.12
0.09	93.80	∞	1.00	1.42
0.10	93.49	∞	1.00	1.59
0.11	92.71	∞	1.00	1.80
0.20	81.49	∞	1.00	4.66
0.30	71.42	∞	1.00	7.79
0.50	58.55	∞	1.03	12.36

For maximum phase margin, the value of K_D is 0.09. PM = 93.80 deg. GM = ∞ , $M_r = 1$,

and BW = 1.42 rad/sec.

(b) Sensitivity Plots:

The PD control reduces the peak value of the sensitivity function $|S_G^M(j\omega)|$

11-12)**PD controller design:** The open loop transfer function of a system is:

$$G(s)H(s) = \frac{100}{s(0.1s + 1)(0.02s + 1)}$$

Design the PD controller so that the phase margin is greater than 50 degrees and the BW is greater than 20 rad/sec.

The transfer functions are generated and imported in sisotool as in 11-4:

MATLAB code:

```
s = tf('s')

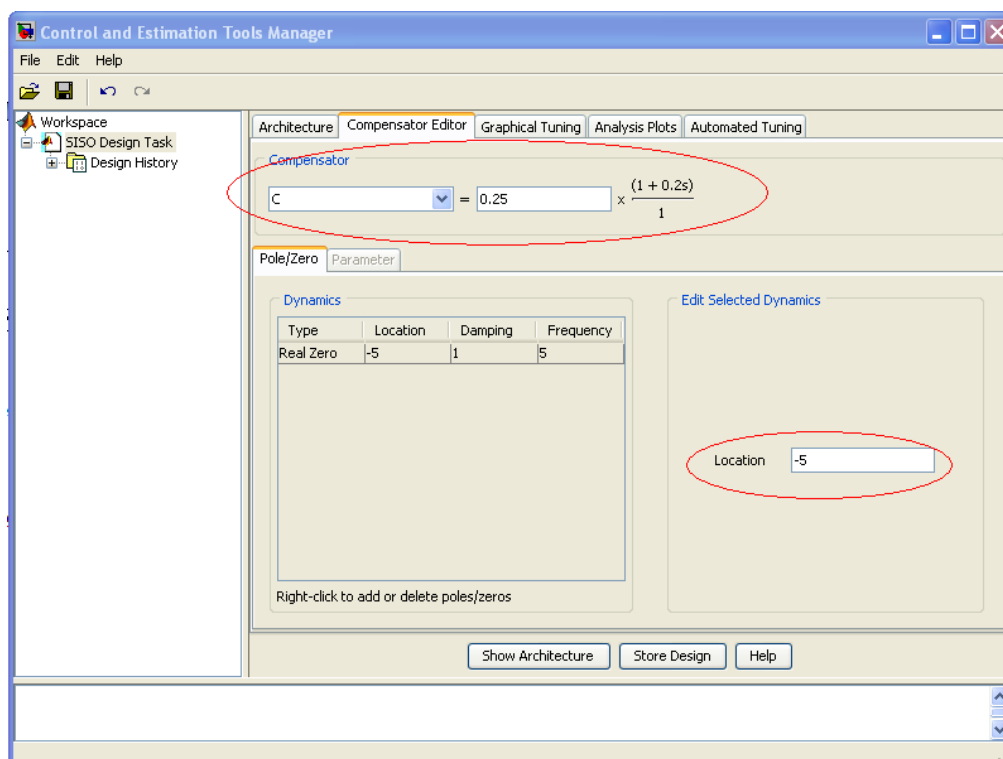
Kp = 1
num_GH= Kp*100;
den_GH=s*(0.1*s+1)*(0.02*s+1);
GH=num_GH/den_GH;
CL = GH/(1+GH)

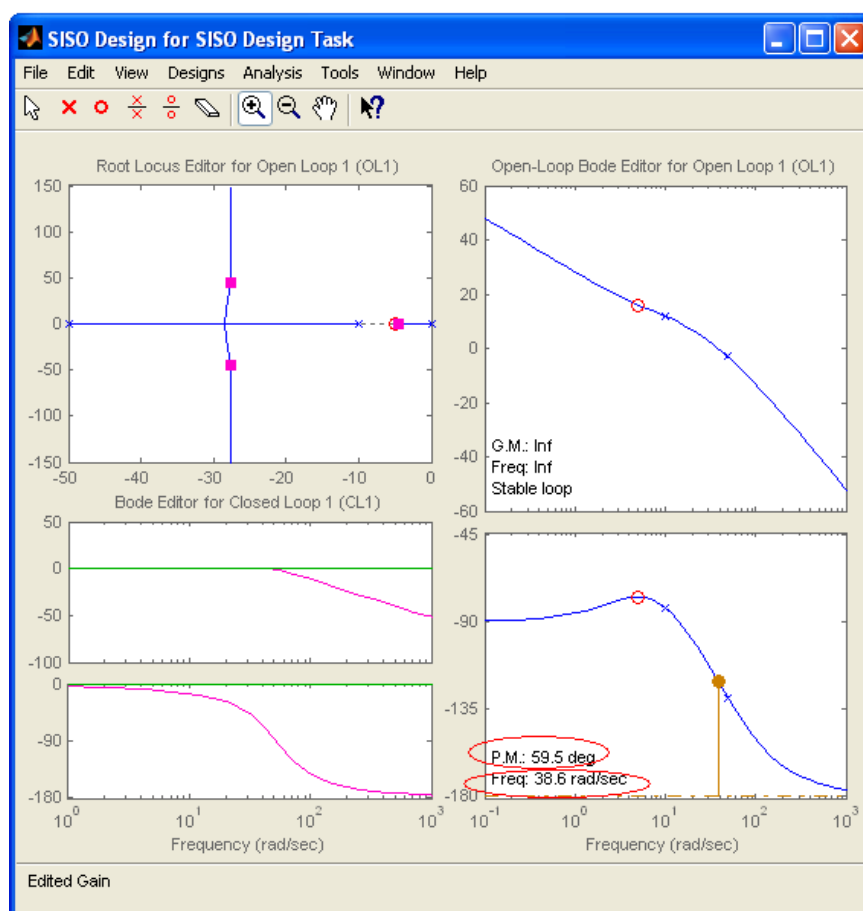
sisotool
```

Following similar steps in 11-4, the loop transfer functions are inserted into sisotool. Another real zero is added to represent the effect of K_d . That is $C(s) = K_p + K_d s = K_p(1 + K_d s / K_p)$, which is called the compensator transfer function in sisotool. The place of real zero is $Z = -K_p / K_d$, and the gain of the compensator is equal to K_p . The zero location and K_p gain were changed interactively in sisotool until the desired PM (59.5 [deg]) and BW is achieved. Following figures shows this PM at cross over frequency of 38.6

rad/sec, which insures BW of higher than 38.6 rad/sec (as the bandwidth is @ -3dB rather than 0 DB, i.e. bandwidth occurs at higher frequency compared to cross over frequency).

Final possible answer: $K_p = 0.25$ and $K_d = -K_p / Z = -0.25 / -5 = 0.05$





11-13)**Lead compensator design:**

$$G(s)H(s) = \frac{1000K}{s(0.2s + 1)(0.005s + 1)}$$

Design a compensator such that the steady state error to the unit step input is less than 0.01 and the closed loop damping ratio $\zeta > 0.4$.

The transfer functions are generated and imported in sisotool as in 11-4:

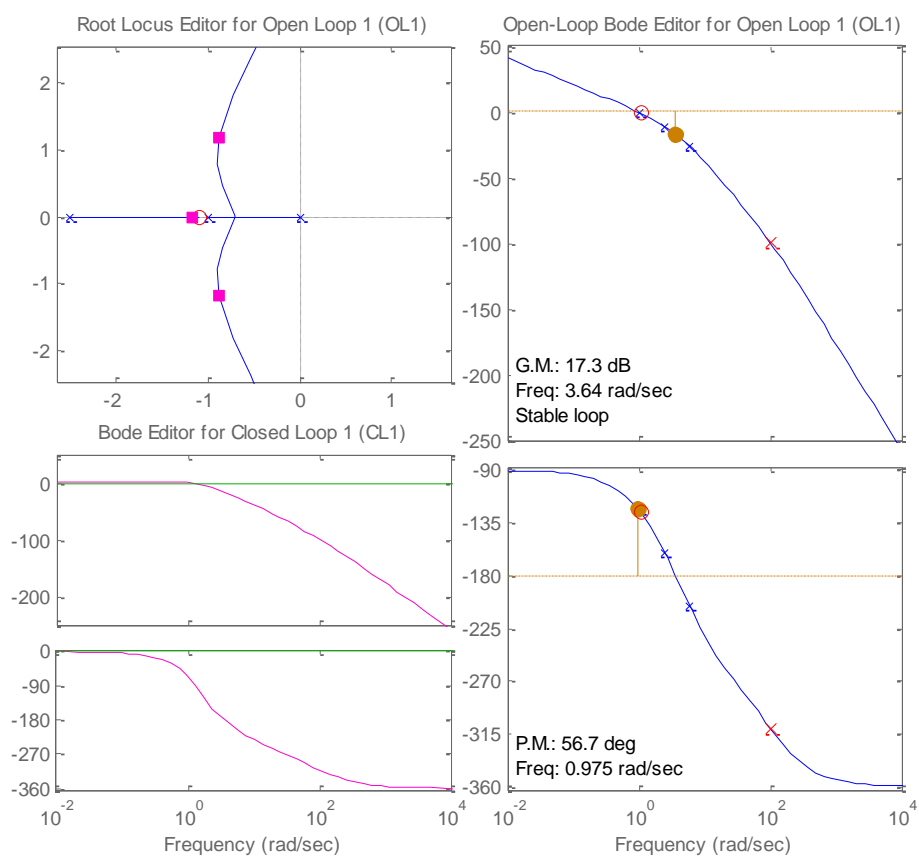
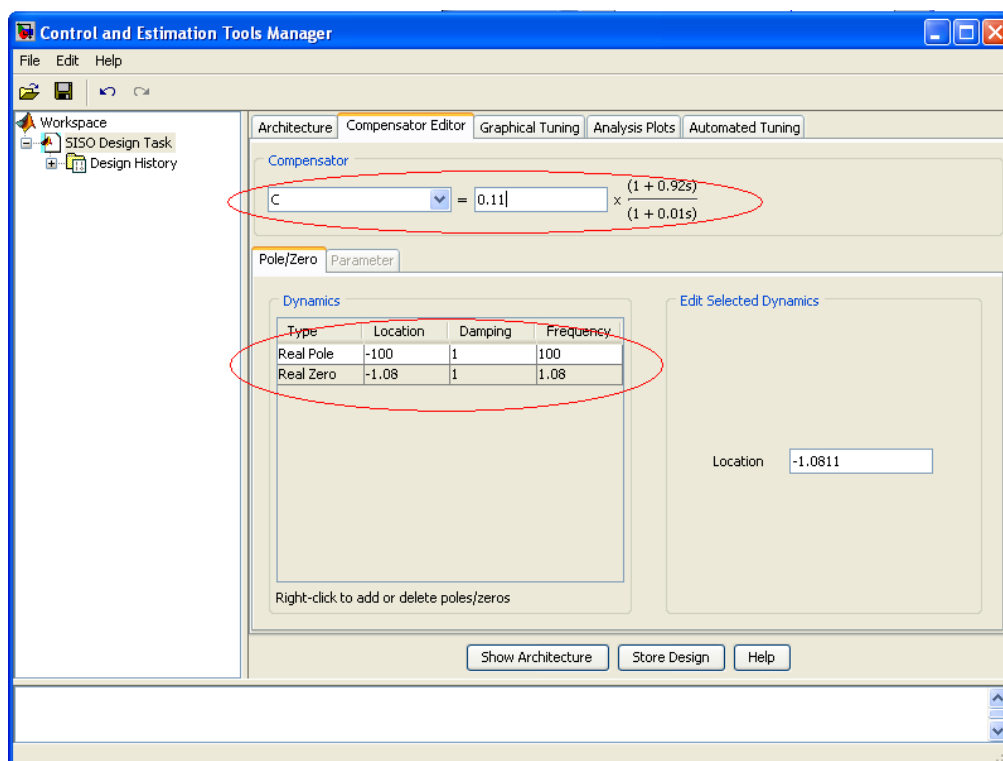
$$e_{ss} = \frac{1}{1 + 1000K_p} < 0.01 \Rightarrow 1000K_p > 101 \Rightarrow \text{Therefore, } K_p \text{ is selected as 150: } (K_p = 0.11)$$

To achieve the required damping ratio, the poles of the closed loop system are placed with an angle of less than $\text{ArcCos}(\zeta=0.4)$, in the root locus diagram of sisotool. This is done by iteratively change the location of poles and zeros of a lead compensator and setting $K_p = 0.11$. The pole and zero (which perform as a lead compensator when the pole is further away from zero to the left) are inserted in sisotool as explained in 11-4. The lead compensator will introduce some phase lead at lower frequencies about the zero location which improves the closed loop response in terms of damping and phase margin. Following is the chosen location for lead compensator pole and zero:

Pole @ -100 rad/sec

Zero @ -1.08 rad/sec

Which resulted in smallest angles of dominant pole locations (the ones closer to imaginary axis) with the real axis. This small angle means higher damping of the poles as $\zeta = \text{ArcCos}(\text{pole's angle with real axis})$.



In this particular case, closed loop complex poles can be observed in the shown root locus diagram at about $-0.8 \pm 1.2j$. This corresponds to damping of about:

$$\cos(\tan^{-1}(1.2/0.8)) = 0.554 \rightarrow \zeta \approx 0.55$$

Preliminary MATLAB code for 11-13:

```
s = tf('s')
Kp = 1
num_GH= Kp*60;
den_GH=s*(0.4*s+1)*(s+1)*(s+6);
GH=num_GH/den_GH;
CL = GH/(1+GH)

figure(1)
margin(CL)

sisotool
```

11-14)

PD controller designed for a maximum overshoot and a maximum steady state error

$$e_{ss} = \frac{1}{\lim_{s \rightarrow 0} s G_c(s) G(s)} \leq -0.005$$

Therefore:

$$\frac{1}{250K_p} \leq 0.005 \Rightarrow K_p > \frac{1}{0.005 * 250} \Rightarrow K_p > \frac{4}{5}$$

Let $K_p = 1$, then:

$$M_p = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) < 0.20 \Rightarrow \zeta > 0.45$$

Let $\zeta = 0.6$; then:

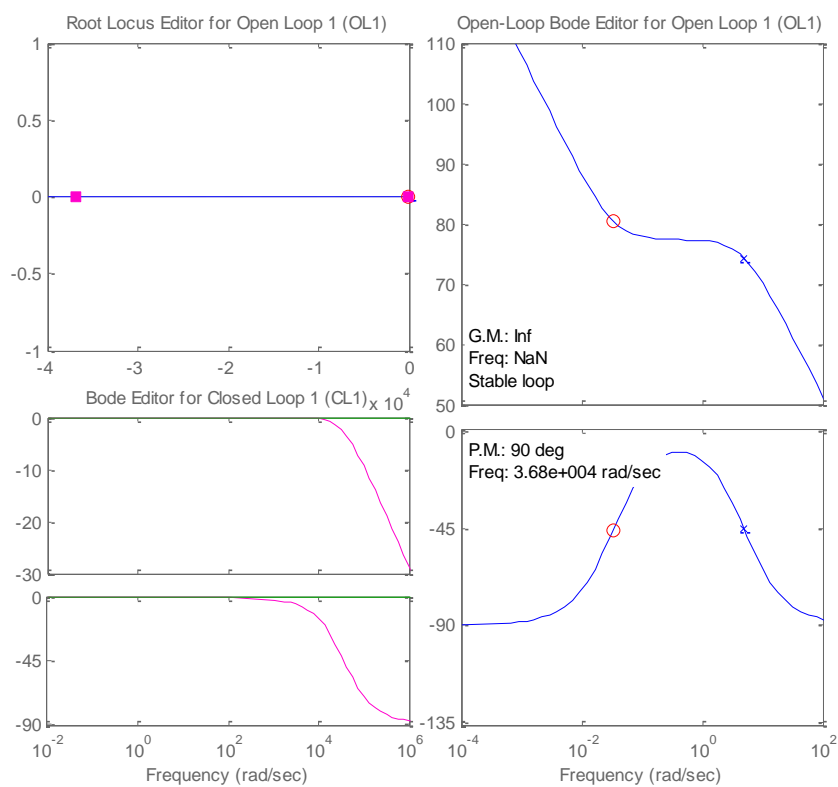
$$\frac{Y(s)}{X(s)} = \frac{250(K_D s + 1)}{0.2s^2 + (250K_D + 1)s + 250} = \frac{1250(K_D s + 1)}{s^2 + (1250K_D + 5)s + 1250}$$

Accordingly, $\omega_n^2 = 1250$, or $\omega_n = 35.35$. therefore,

$$(1250K_D + 5) = 2\zeta\omega_n \Rightarrow 1250K_D = (2)(0.6)(0.35)$$

which gives: $K_D \approx 0.034$

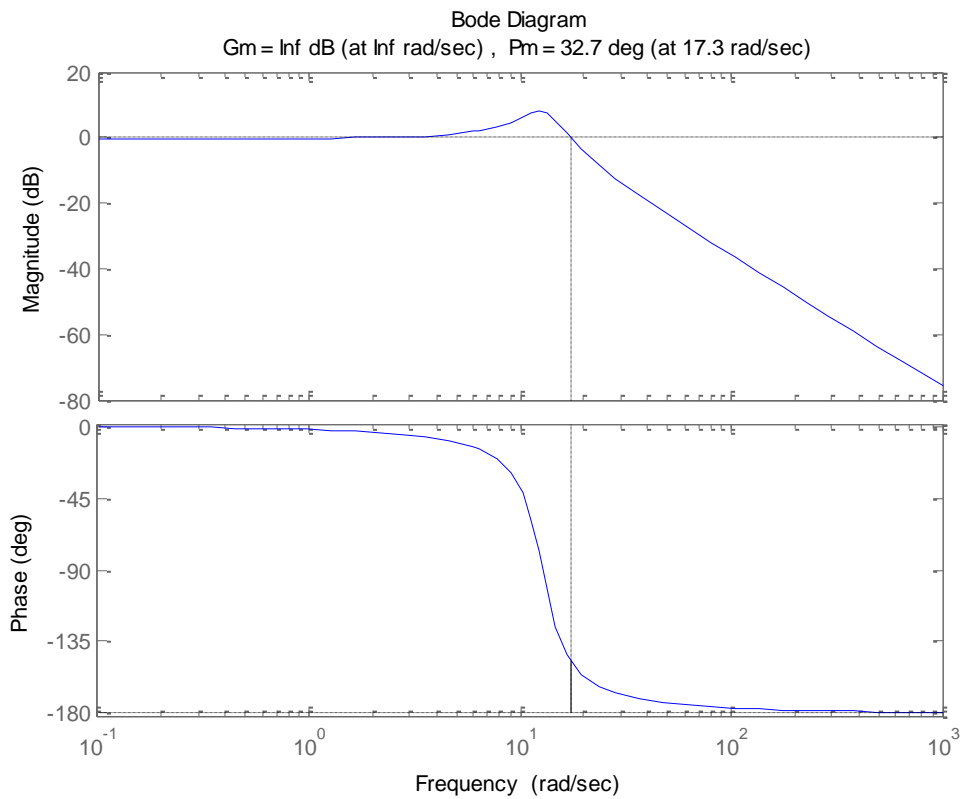
The design characteristics can be observed in the diagram below:



11-15)

Lead compensator controller design

The bode diagram of the system without lead compensator is shown below:



Indicating a PM of 32.7 [deg]. To reach 45 [deg] phase margin, additional 12.3 [deg] phase lead is needed. At $\omega = 17.3$ rad/s crossover frequency, if $\Phi_{in} = 12.3$ then

$$r = \frac{1 - \sin \Phi_{in}}{1 + \sin \Phi_{in}} = 0.6488$$

As $\omega = \frac{1}{\tau\sqrt{r}} = 17.3$, then $\tau = 0.0718$

Since the gain is lowered by $\left| \frac{r(1+j\omega\tau)}{1+r\tau j\omega} \right|_{\omega=17.3} = 1.2838$

A gain compensator with gain of 0.7789 is required, where,

$$G_c(s) = \frac{Kr(\tau s + 1)}{r\tau s + 1}$$

11-16) If $r = 0.1 \Rightarrow \Phi_m = \sin^{-1} \frac{1-r}{1+r} \approx 55$

$$\Phi_m(\omega) = \tan^{-1} \omega\tau - \tan^{-1} r\omega\tau \Rightarrow \Phi_m(17.3) = \tan^{-1} 17.3\tau - \tan^{-1} 1.73\tau = 12.3$$

then from trial and error we found $\tau = 0.014088$ and required gain would be 9.7185

11-17) $K_v = \lim_{s \rightarrow 0} sG_i(s)G(s) = 100K \geq 100 \Rightarrow K \geq 1$

First plot the bode diagram of uncompensated system when $K = 1$

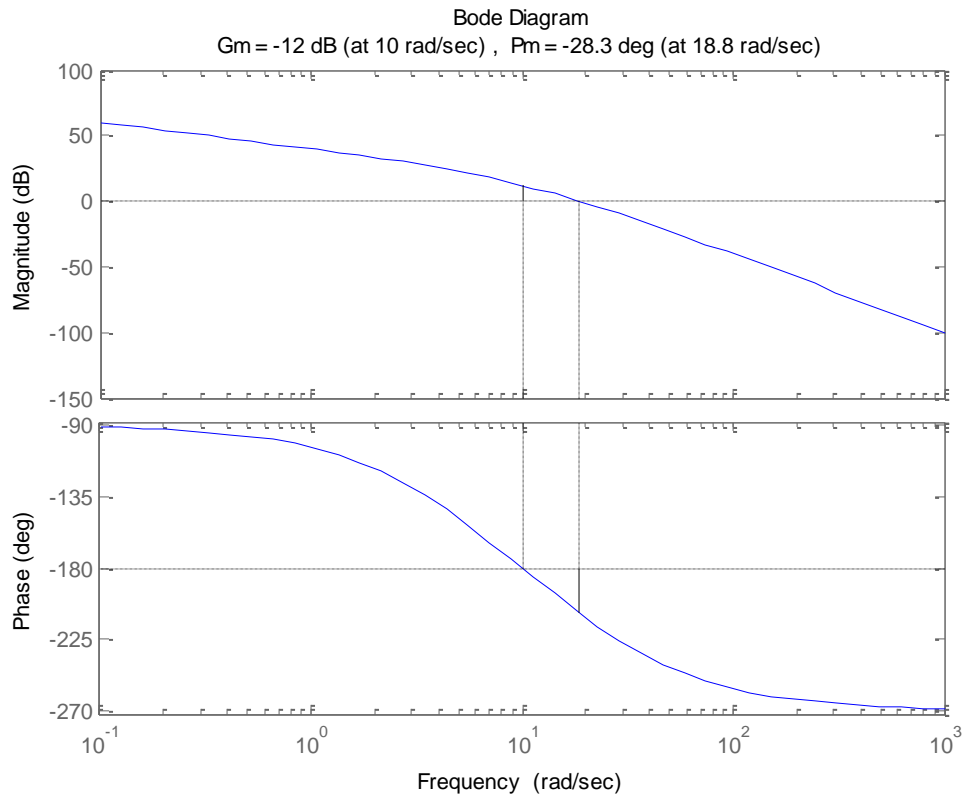
Bode diagram for Loop T.F. is included for $K=1$

MATLAB CODE:

```
s = tf('s')

Kp = 1
num_GH= Kp*100;
den_GH=s*(0.2*s+1)*(0.05*s+1);
GH=num_GH/den_GH
lag_tf=(s/2+1)/(s/0.2+1)
lead_tf=(s/4+1)/(s/50+1)
LL=lag_tf*lead_tf
OL=GH*LL
CL =OL/(1+OL);

figure(1)
margin(GH)
figure(2)
margin(OL)
figure(3)
bode(CL)
grid on;
```



The bode diagram with $K=1$ shows -28 deg PM at 18.8 rad/sec.

According to the requirements the gain must be greater than $\frac{1}{0.004}$ or 250 for $\omega_1 \leq 0.2 \text{ rad/s}$ and must be less than $\frac{1}{100}$ or 0.01 for $\omega_2 \geq 200 \text{ rad/s}$

In order to achieve above requirements, a lead-lag compensator will be appropriate.

Using a lag compensator will allow lower gain at frequencies less than ω_1 and using a lead compensator will allow to increase phase margin

For the lag compensator, $\alpha = 1/10$ is chosen to boost the low frequency amplitude

$$Lag = \frac{1 + \alpha Ts}{1 + Ts} = \frac{s/2 + 1}{s/0.2 + 1}$$

In order to introduce some phase lead to obtain the require PM, a lead compensator is also designed as:

$$T = \frac{1}{\sqrt{10}\omega} = \frac{1}{28.3\sqrt{10}} = 0.0112$$

Where ω is overlaid with the crossover frequency (28.3 rad/sec) for applying the maximum phase lead at this frequency. The Lead compensator T.F. will be as follows:

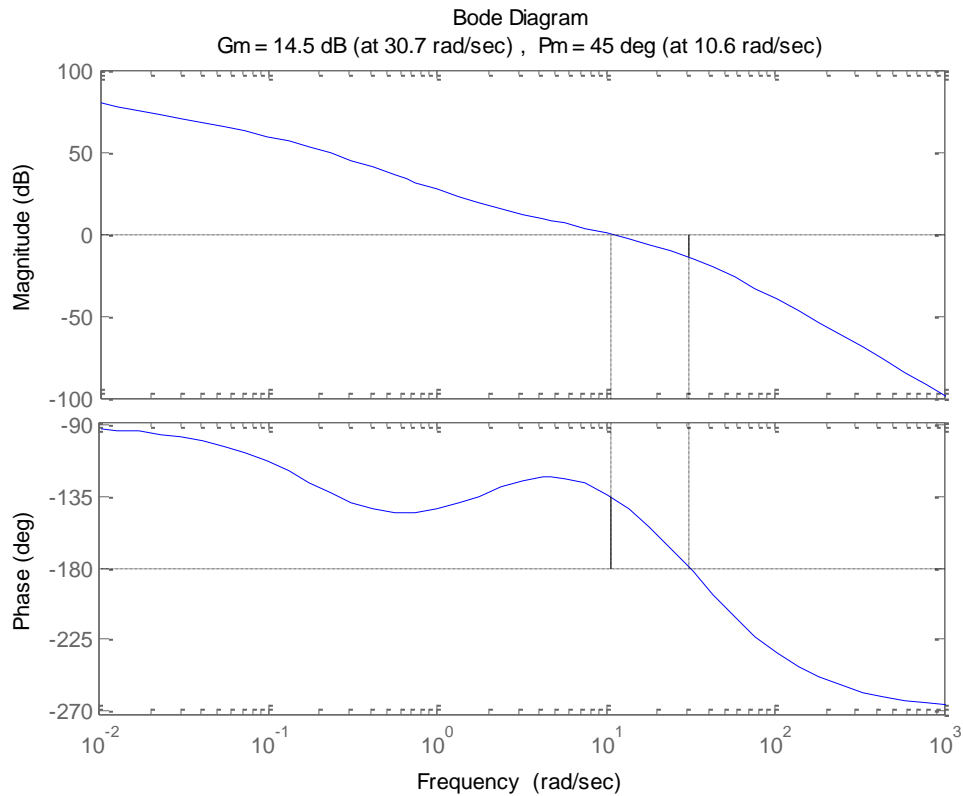
$$Lead = \frac{1 + \alpha Ts}{1 + Ts} = \frac{s/5 + 1}{s/50 + 1}$$

Resulting in the following Bode diagram for the compensated system, showing 44 deg PM:

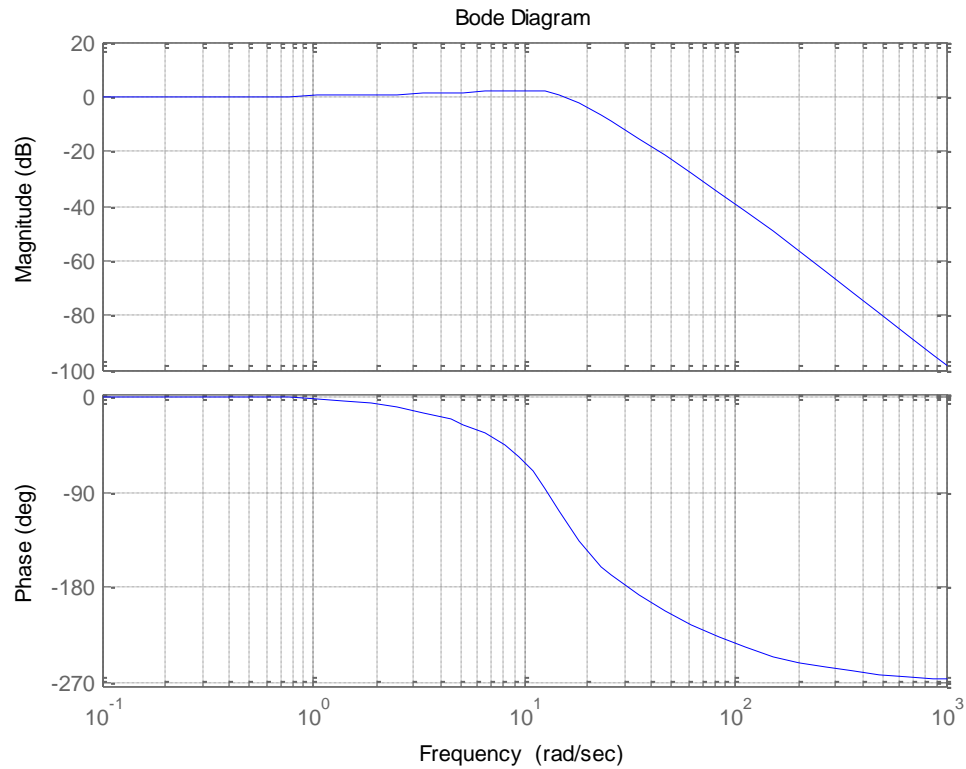
To obtain a slightly higher PM, lead compensator zero was re-tuned, where the zero is pulled closer to imaginary axis from -5 to -4:

$$Lead = \frac{1 + \alpha Ts}{1 + Ts} = \frac{s/4 + 1}{s/50 + 1}$$

This resulted in a higher PM as shown in the following bode diagram of loop transfer function:



Correspondingly, the Bode diagram of closed loop system can be shown as:



11-18) See Chapter 5 solutions for MATLAB codes for this problem.

(a) Forward-path Transfer Function:

$$G(s) = \frac{100 \left(K_p + \frac{K_I}{s} \right)}{s^2 + 10s + 100} \quad \text{For } K_v = 10, \quad K_v = \lim_{s \rightarrow 0} sG(s) = \lim_{s \rightarrow 0} s \frac{100(K_p s + K_I)}{s(s^2 + 10s + 100)} = K_I = 10$$

Thus $K_I = 10$.

(b) Let the complex roots of the characteristic equation be written as $s = -\sigma + j15$ and $s = -\sigma - j15$.

The quadratic portion of the characteristic equation is $s^2 + 2\sigma s + (\sigma^2 + 225) = 0$

The characteristic equation of the system is $s^3 + 10s^2 + (100 + 100K_p)s + 1000 = 0$

The quadratic equation must satisfy the characteristic equation. Using long division and solve for zero remainder condition.

$$\begin{array}{r}
 s + (10 - 2\sigma) \\
 s^2 + 2\sigma s + \sigma^2 + 225 \overline{) s^3 + 10s^2 + (100 + 100K_p)s + 1000} \\
 \underline{s^3 + 2\sigma s^2 + (\sigma^2 + 225)s} \\
 (10 - 2\sigma)s^2 + (100K_p - \sigma^2 - 125)s + 1000 \\
 \underline{(10 - 2\sigma)s^2 + (20\sigma - 4\sigma^2)s + (10 - 2\sigma)(s^2 + 225)} \\
 (100K_p + 3\sigma^2 - 20\sigma - 125)s + 2\sigma^3 - 10\sigma^2 + 450\sigma - 1250
 \end{array}$$

For zero remainder, $2\sigma^3 - 10\sigma^2 + 450\sigma - 1250 = 0$ (1)

and $100K_p + 3\sigma^2 - 20\sigma - 125 = 0$ (2)

The real solution of Eq. (1) is $\sigma = 2.8555$. From Eq. (2),

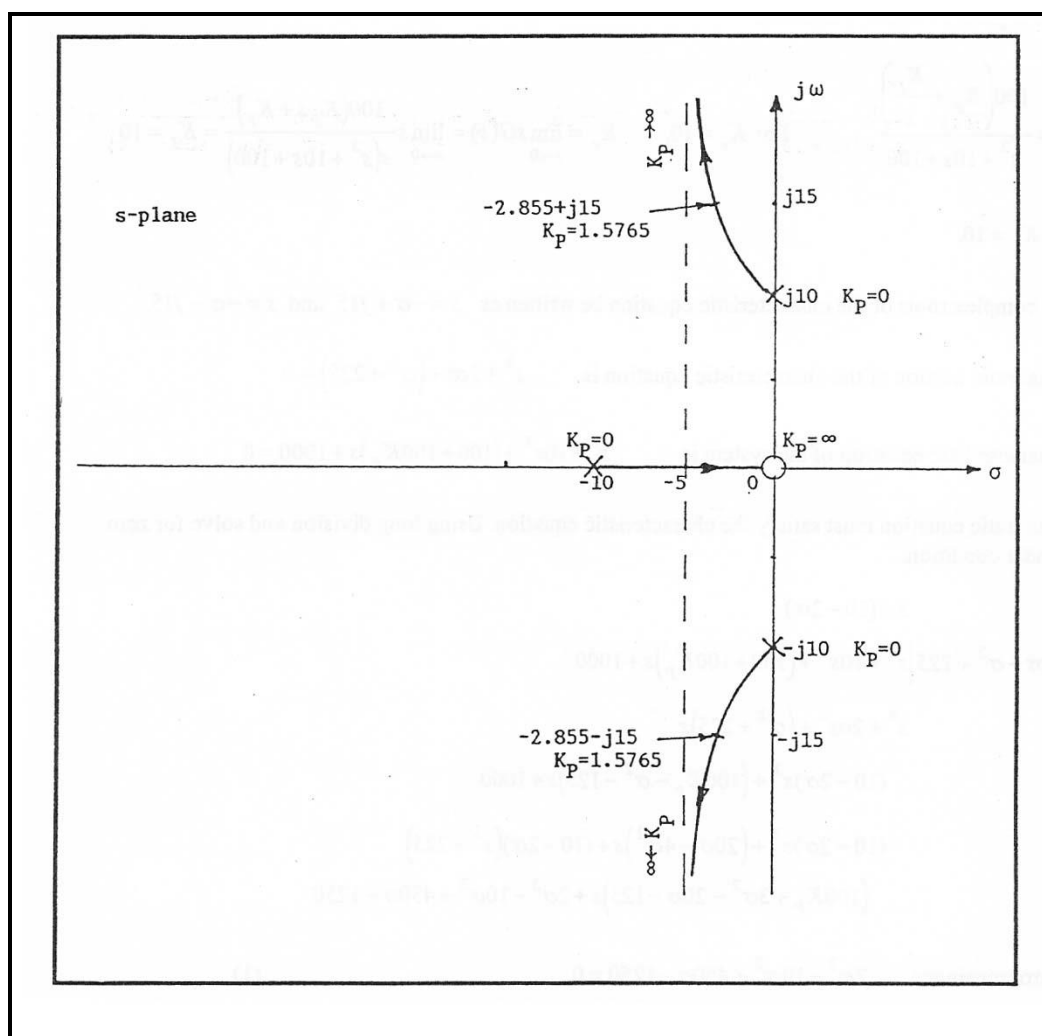
$$K_p = \frac{125 + 20\sigma - 3\sigma^2}{100} = 1.5765$$

The characteristic equation roots are: $s = -2.8555 + j15$, $-2.8555 - j15$, and $s = -10 + 2\sigma = -4.289$

(c) Root Contours:

$$G_{eq}(s) = \frac{100K_p s}{s^3 + 10s^2 + 100s + 1000} = \frac{100K_p s}{(s+10)(s^2+100)}$$

Root Contours:



11-19)**(a) Forward-path Transfer Function:**

$$G(s) = \frac{100 \left(K_p + \frac{K_I}{s} \right)}{s^2 + 10s + 100} \quad \text{For } K_v = 10, \quad K_v = \lim_{s \rightarrow 0} sG(s) = \lim_{s \rightarrow 0} s \frac{100(K_p s + K_I)}{s(s^2 + 10s + 100)} = K_I = 10$$

Thus the forward-path transfer function becomes

$$G(s) = \frac{10(1 + 0.1K_p s)}{s(1 + 0.1s + 0.01s^2)}$$

Attributes of the Frequency Response:

K_p	PM (deg)	GM (dB)	M_r	BW (rad/sec)
0.1	5.51	1.21	10.05	14.19
0.5	22.59	6.38	2.24	15.81
0.6	25.44	8.25	1.94	16.11
0.7	27.70	10.77	1.74	16.38
0.8	29.40	14.15	1.88	16.62

0.9	30.56	20.10	1.97	17.33
1.0	31.25	∞	2.00	18.01
1.5	31.19	∞	1.81	20.43
1.1	31.51	∞	2.00	18.59
1.2	31.40	∞	1.97	19.08

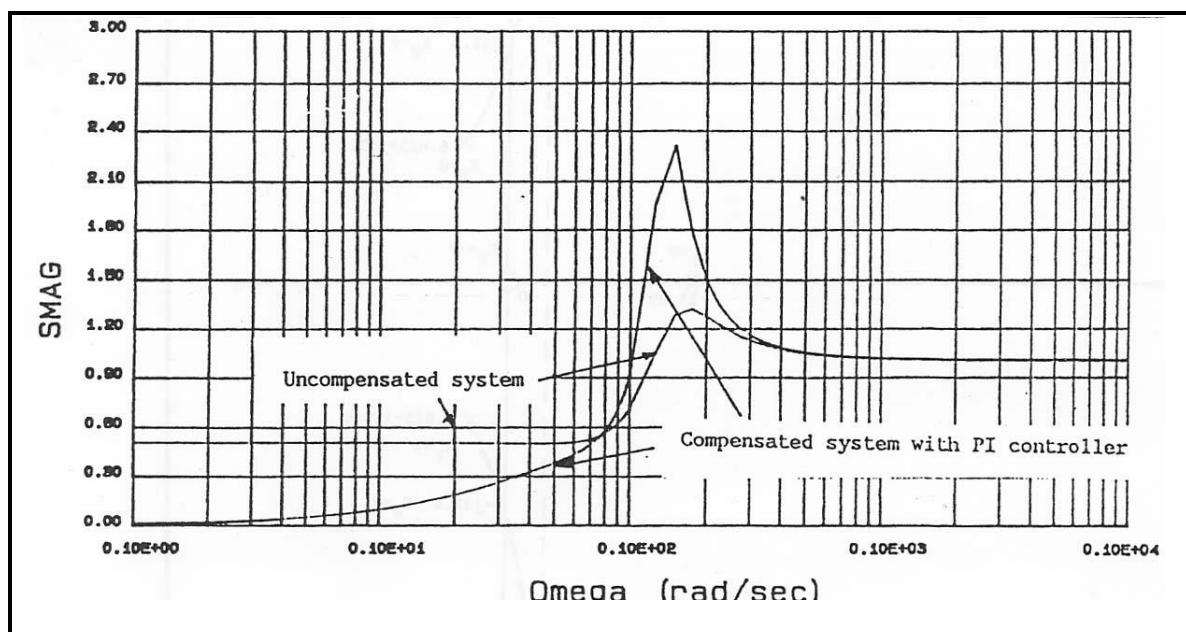
When $K_p = 1.1$ and $K_I = 10$, $K_v = 10$, the phase margin is 31.51 deg., and is maximum.

The corresponding roots of the characteristic equation roots are:

$$-5.4, \quad -2.3 + j13.41, \quad \text{and} \quad -2.3 - j13.41$$

Referring these roots to the root contours in Problem 10-8(c), the complex roots corresponds to a relative damping ratio that is near optimal.

(b) Sensitivity Function:



In the present case, the system with the PI controller has a higher maximum value for the sensitivity function.

11-20)**(a) Forward-path Transfer Function:**

$$G(s) = \frac{100(K_p s + K_I)}{s(s^2 + 10s + 100)}$$

For $K_v = 100$,

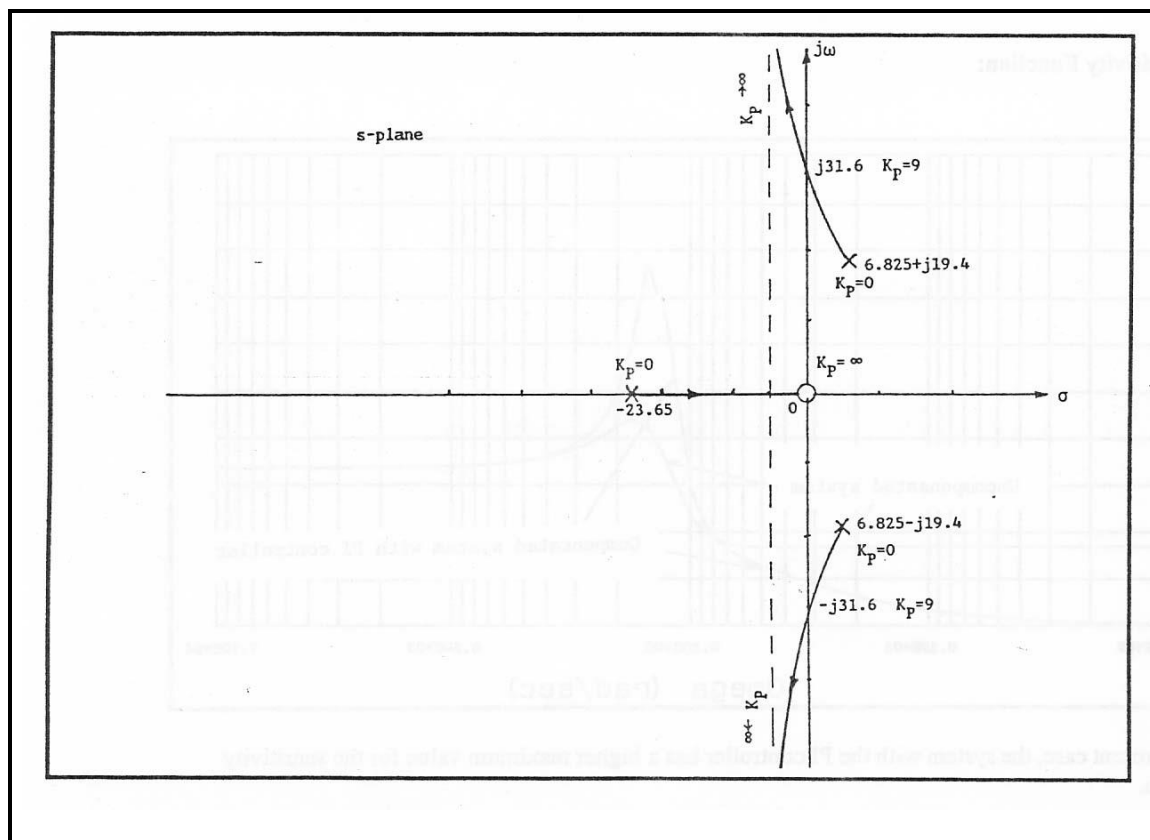
$$K_v = \lim_{s \rightarrow 0} sG(s) = \lim_{s \rightarrow 0} s \frac{100(K_p s + K_I)}{s(s^2 + 10s + 100)} = K_I = 100 \quad \text{Thus } K_I = 100.$$

(b) The characteristic equation is $s^3 + 10s^2 + (100 + 100K_p)s + 100K_I = 0$ **Routh Tabulation:**

s^3	1	$100 + 100K_p$	
s^2	10	10,000	
s^1	$100K_p - 900$	0	For stability, $100K_p - 900 > 0$
s^0	10,000		Thus $K_p > 9$

Root Contours:

$$G_{eq}(s) = \frac{100K_p s}{s^3 + 10s^2 + 100s + 10,000} = \frac{100K_p s}{(s + 23.65)(s - 6.825 + j19.4)(s - 6.825 - j19.4)}$$



(c) $K_I = 100$

$$G(s) = \frac{100(K_p s + 100)}{s(s^2 + 10s + 100)}$$

The following maximum overshoots of the system are computed for various values of K_p .

K_p	15	20	22	24	25	26	30	40	100	1000
y_{\max}	1.794	1.779	1.7788	1.7785	1.7756	1.779	1.782	1.795	1.844	1.859

When $K_p = 25$, minimum $y_{\max} = 1.7756$

11-21)

(a) Forward-path Transfer Function:

$$G(s) = \frac{100(K_p s + K_I)}{s(s^2 + 10s + 100)} \quad \text{For } K_v = \frac{100K_I}{100} = 10, \quad K_I = 10$$

(b) Characteristic Equation: $s^3 + 10s^2 + 100(K_p + 1)s + 1000 = 0$

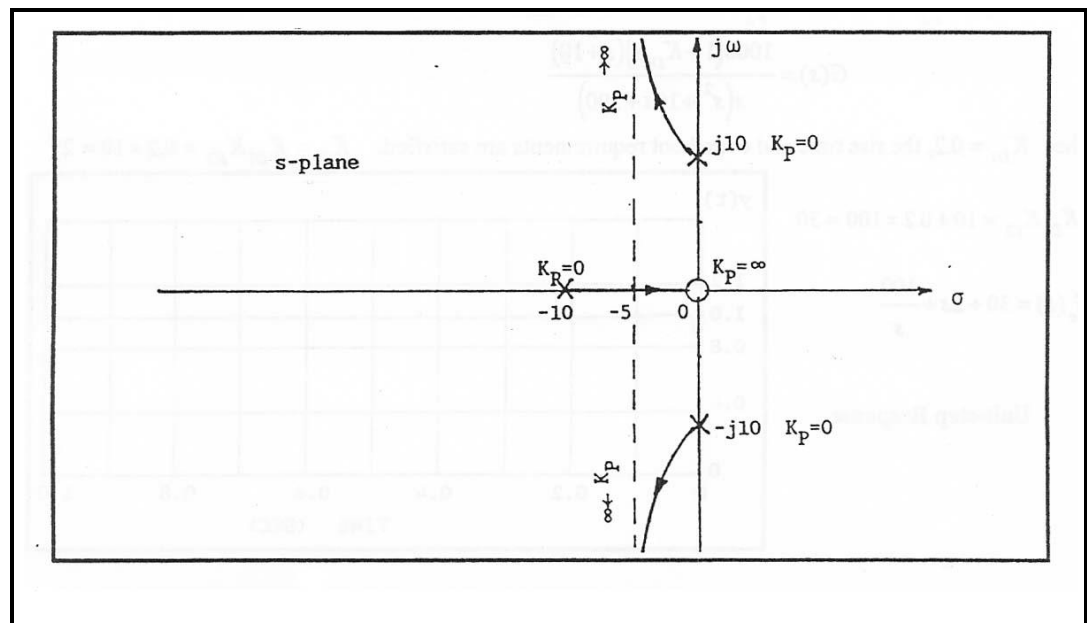
Routh Tabulation:

s^3	1	$100 + 100K_p$
s^2	10	1000
s^1	$100K_p$	0
s^0	1000	

For stability, $K_p > 0$

Root Contours:

$$G_{eq}(s) = \frac{100K_p s}{s^3 + 10s^2 + 100s + 1000}$$



- (c)** The maximum overshoots of the system for different values of K_p ranging from 0.5 to 20 are computed and tabulated below.

K_p	0.5	1.0	1.6	1.7	1.8	1.9	2.0	3.0	5.0	10	20
y_{\max}	1.393	1.275	1.2317	1.2416	1.2424	1.2441	1.246	1.28	1.372	1.514	1.642

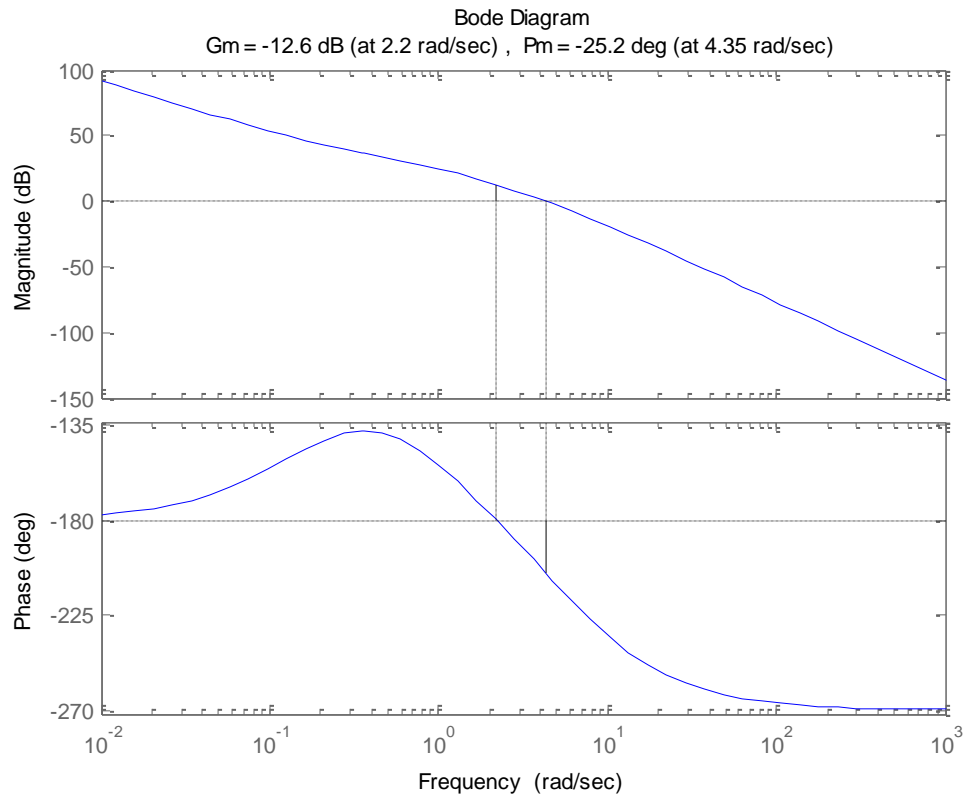
When $K_p = 1.7$, maximum $y_{\max} = 1.2416$

11-22) $K_v = \lim_{s \rightarrow 0} sG_c(s)G(s) > 20 \Rightarrow \frac{24K}{6} > 20 \Rightarrow K > 5$

let $K = 6$ and targeted $PM = 45^\circ$. To include some integral action, K_i is set to 1.

First, let's take a look at uncompensated system:

The open loop bode shows as PM of -25.2 @ 4.35 rad/sec:



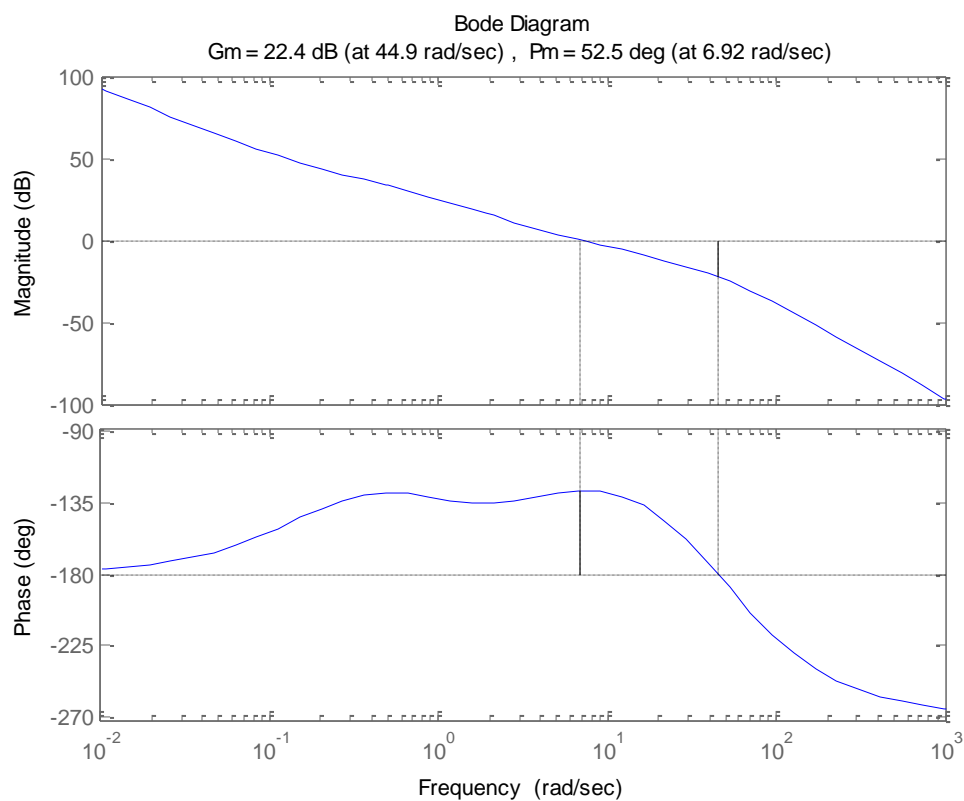
To achieve the PM of 45 deg, we need to add a phase lead of $(45 - (-25.2)) = 70.2$. By try and error, 2 compensators (a double lead compensator) each with phase lead of 55 deg was found suitable. Considering the change in cross over frequency after applying the lead filters, overall, a PM of 52 deg was obtained as seen in the bode diagram of compensated loop:

Double Lead filter design:

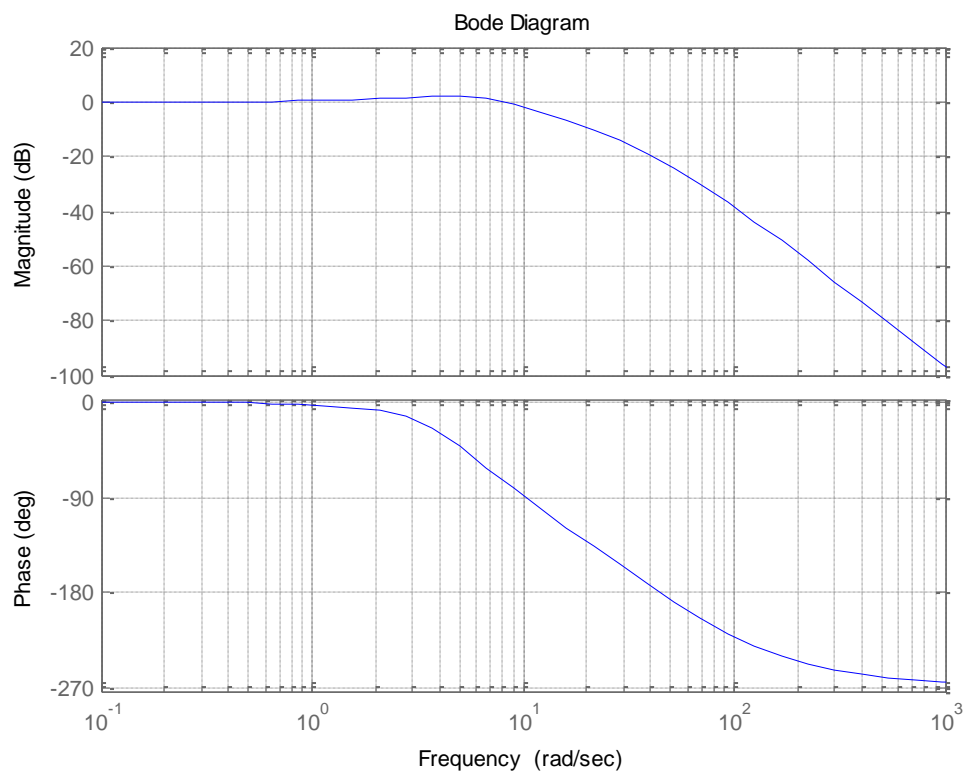
$$\alpha = \frac{1 + \sin \Phi_m}{1 - \sin \Phi_m} = \frac{1 + \sin 55}{1 - \sin 55} = 10.0590, \quad T = \frac{1}{\sqrt{10.059}\omega} = \frac{1}{15\sqrt{10.0590}} = 0.0210$$

The maximum phase lead of the compensators are placed at 15 rad/sec, which resulted in a larger PM (=52.5 deg) compared to applying this phase lead at original cross over frequency of 4.35 rad/sec. This was due to the shape of phase diagram affected by integral action (i.e. phase starts at -180 @ $\omega = 0$ rad/sec).

The gain crossover frequency is $\omega = 6.92$ rad/sec. Bode diagram of compensated loop transfer function can be observed in the following figure, showing a PM pf 52.5 deg:



Correspondingly, the Bode diagram of closed loop system can be shown as:



MATLAB code:

```
s = tf('s')
Kp = 6;
Ki = 1;
num_GH= 24*(Kp+Ki/s);
den_GH=s*(s+1)*(s+6);
GH=num_GH/den_GH;
%lead design
PL=55
CRover=15
alpha=(1+sin(PL/180*pi))/(1-sin(PL/180*pi))
T=1/alpha^0.5/CRover
lead=(1+T*alpha*s)/(1+T*s)
LT=GH*lead*lead %double lead compensation
CL = LT/(1+LT);
figure(1)
Margin(GH)
figure(2)
Margin(LT)
figure(3)
Bode(CL)
grid on;
```

11-23)

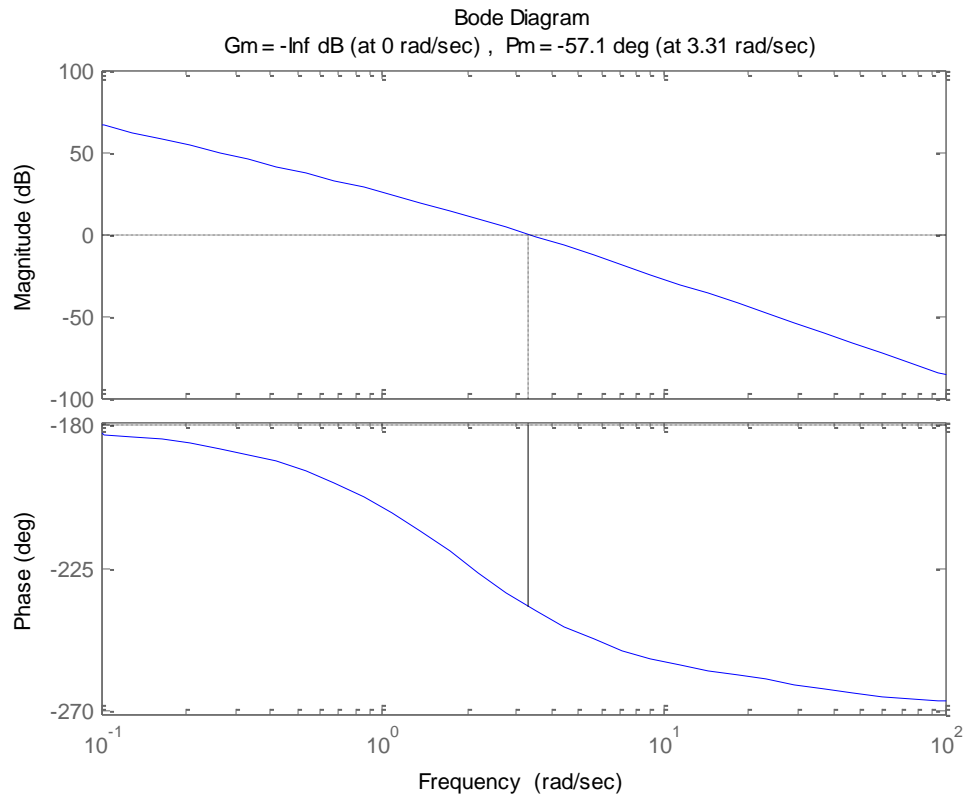
$$e_{ss} = \frac{1}{K_a} \leq 0.05$$

$$K_a = \lim_{s \rightarrow 0} s^2 G_c(s) G(s) = \lim_{s \rightarrow 0} \frac{40(K_p s + K_I)}{(s+2)(s+20)} = K_I > 20$$

Let's consider $K_I = 21$

As gain crossover frequency is $\omega = 1 \Rightarrow |G_c G(j\omega)|_{\omega=1} = 1 \Rightarrow K_p = 1.25$

Let's see if the PM is in the required range. The bode of the loop transfer function shows a PM of -57 deg at 3.31.



By try and error, a double lead compensator, each with phase lead of 53 deg was found suitable. Considering the change in cross over frequency after applying the lead filters, overall, a PM of 35.4 deg was obtained as seen in the bode diagram of compensated loop:

Double Lead filter design:

$$\alpha = \frac{1 + \sin \Phi_m}{1 - \sin \Phi_m} = \frac{1 + \sin 53}{1 - \sin 53} = 8.9322, \quad T = \frac{1}{\sqrt{8.9322}\omega} = \frac{1}{7.5\sqrt{8.9322}} = 0.0446$$

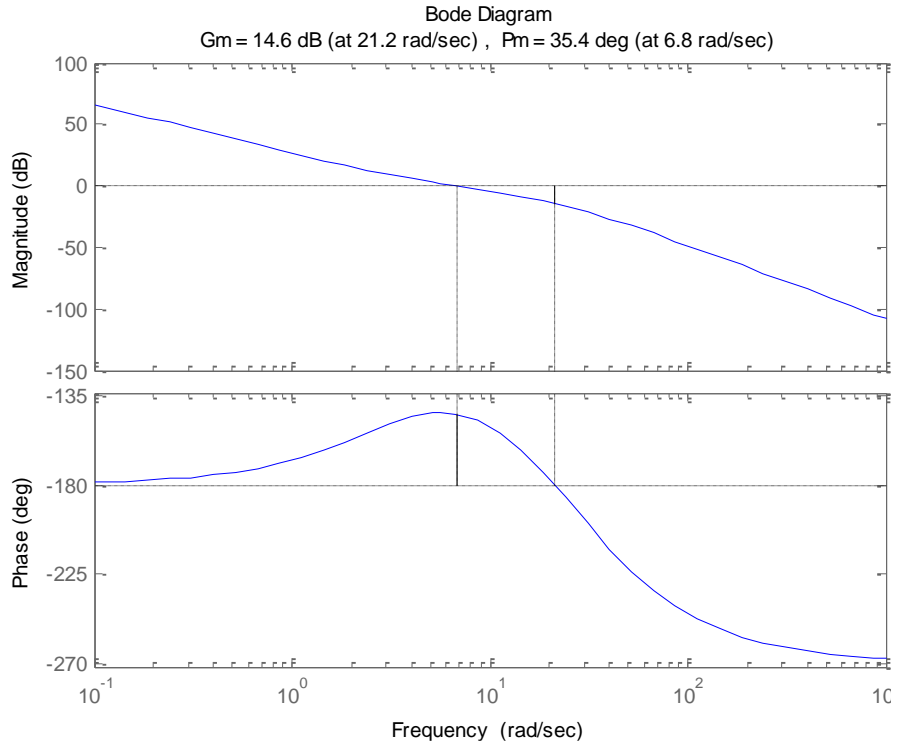
The maximum phase lead of the compensators are placed at 7.5 rad/sec, resulting in a larger PM (= 52.5 deg) compared to applying this phase lead at original cross over frequency of 3.31 rad/sec. This was due to the shape of phase diagram affected by integral action (i.e. phase starts at -180 @ $\omega = 0$ rad/sec).

Then the gain crossover frequency is $\omega = 6.8$ rad/sec. Bode diagram of compensated loop can be observed in the following figure, showing a PM of 35.4 deg:

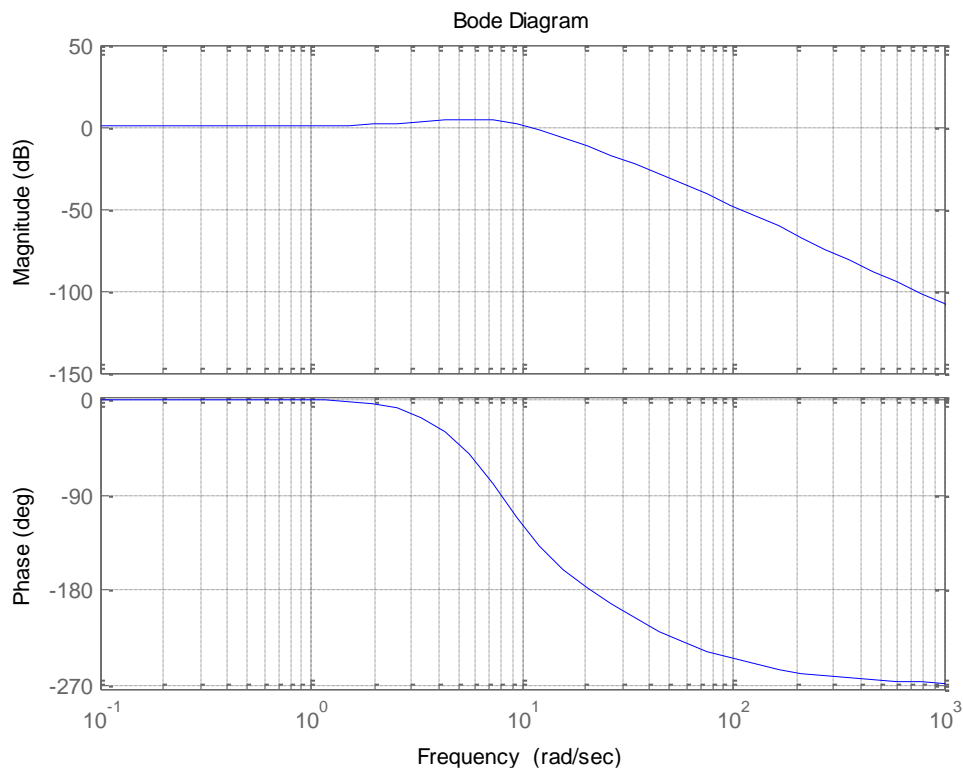
MATLAB code:

```
s = tf('s')
Kp = 1.25;
Ki = 21;
num_GH= 40*(Kp+Ki/s);
den_GH=s*(s+2)*(s+20);
GH=num_GH/den_GH;
CL = GH/(1+GH);
%lead design
PL=53
CRover=7.5
alpha=(1+sin(PL/180*pi))/(1-
sin(PL/180*pi))
T=1/alpha^0.5/CRover
lead=(1+T*alpha*s)/(1+T*s)
LT=GH*lead*lead
CL = LT/(1+LT);
```

```
figure(1)
Margin(GH)
figure(2)
Margin(LT)
figure(3)
Bode(CL)
grid on;
```



Correspondingly, the Bode diagram of closed loop system can be shown as:

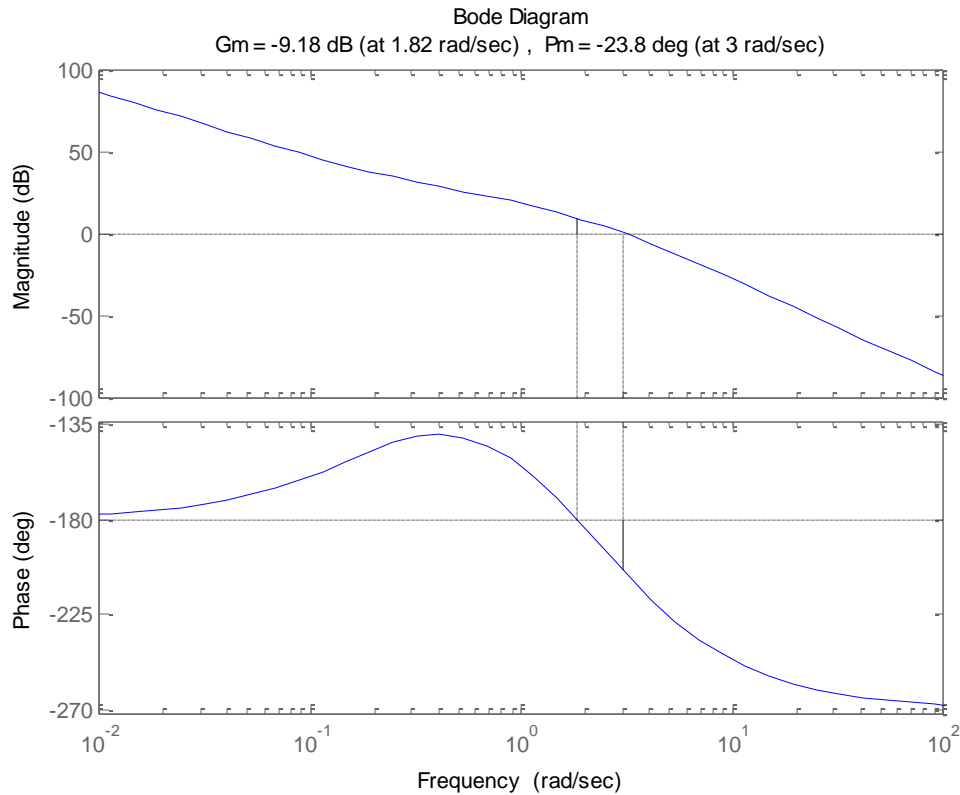


11-24)

To satisfy the unity DC gain ($G(0)/(1+K_p \cdot G(0)) = 1$), K_p should be equal to 1: $K_p=1$

In order to add some integral action, $K_I = 0.2$ was chosen as the integral gain.

First, the bode plot of the Loop transfer function is obtained demonstrating a PM of -23.8 deg at 3 rad/sec cross over frequency:



First, the bode plot of the Loop transfer function is obtained demonstrating a PM of -23.8 deg at 3 rad/sec cross over frequency:

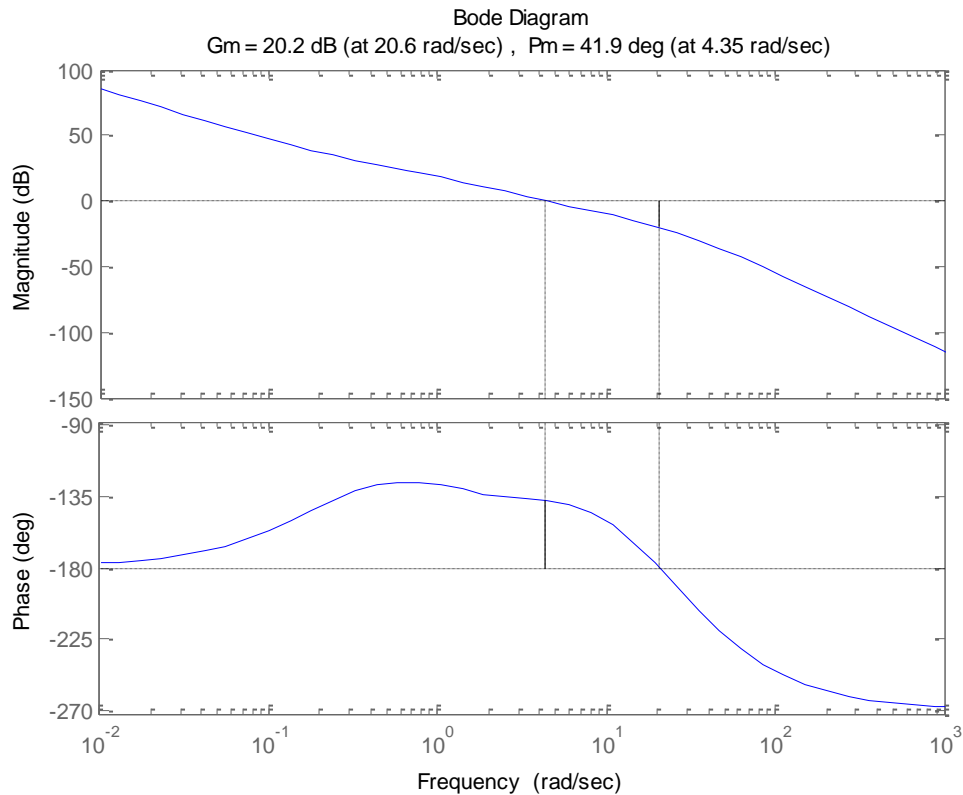
By try and error, a double lead compensator, each with phase lead of 48 deg was found suitable. Considering the change in cross over frequency after applying the lead filters, overall, a PM of 41.9 deg was obtained as seen in the bode diagram of compensated loop:

Double Lead filter design:

$$\alpha = \frac{1 + \sin \Phi_m}{1 - \sin \Phi_m} = \frac{1 + \sin 48}{1 - \sin 48} = 6.7865, \quad T = \frac{1}{\sqrt{6.7865}\omega} = \frac{1}{9\sqrt{8.9322}} = 0.0427$$

The maximum phase lead of the compensators are placed at 9 rad/sec, resulting in a larger PM (= 41.9 deg) compared to applying this phase lead at original cross over frequency of 3 rad/sec. This was due to the shape of phase diagram affected by integral action (i.e. phase starts at -180 @ $\omega = 0$ rad/sec).

Then the gain crossover frequency is $\omega = 4.35$ rad/sec. Bode diagram of compensated loop can be observed in the following figure, showing a PM pf 35.4 deg:



MATLAB code:

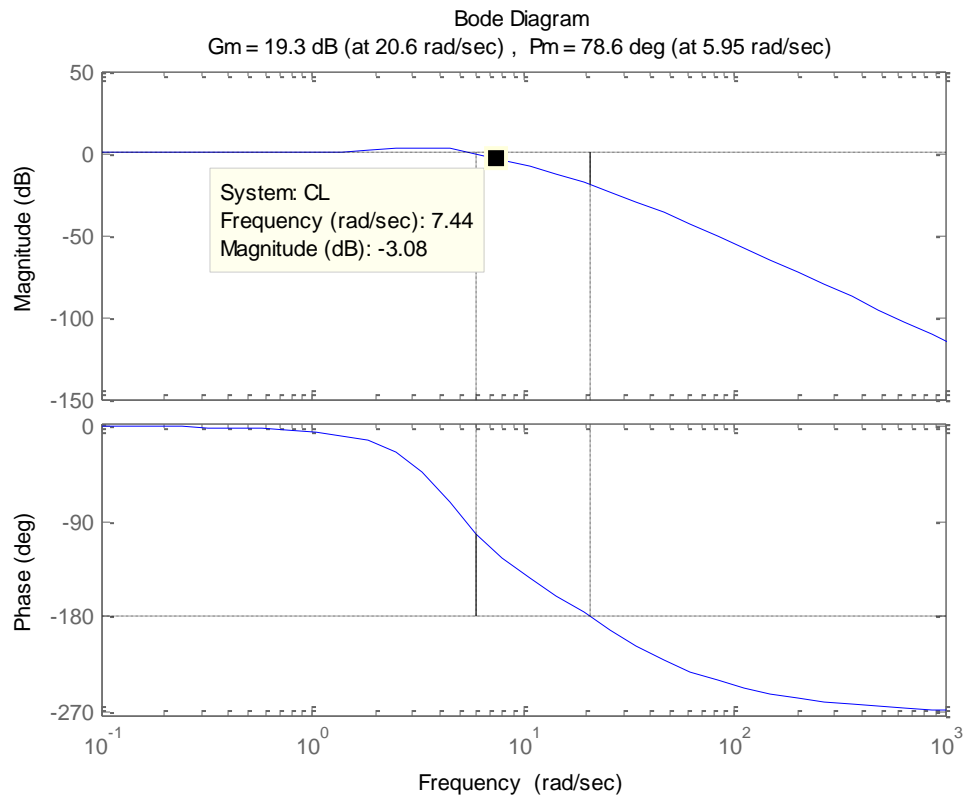
```
s = tf('s')
Kp=1;
Ki=0.2;
num_GH= 210*(Kp+Ki/s);
den_GH=s*(5*s+7)*(s+3);
GH=num_GH/den_GH;

%lead design
PL=48
CRover=9
alpha=(1+sin(PL/180*pi))/(1-sin(PL/180*pi))
T=1/alpha^0.5/CRover
lead=(1+T*alpha*s)/(1+T*s)

LT=GH*lead*lead
CL = LT/(1+LT);
```

figure(1)
 Margin(GH)
 figure(2)
 Margin(LT)
 figure(3)
 Margin(CL)

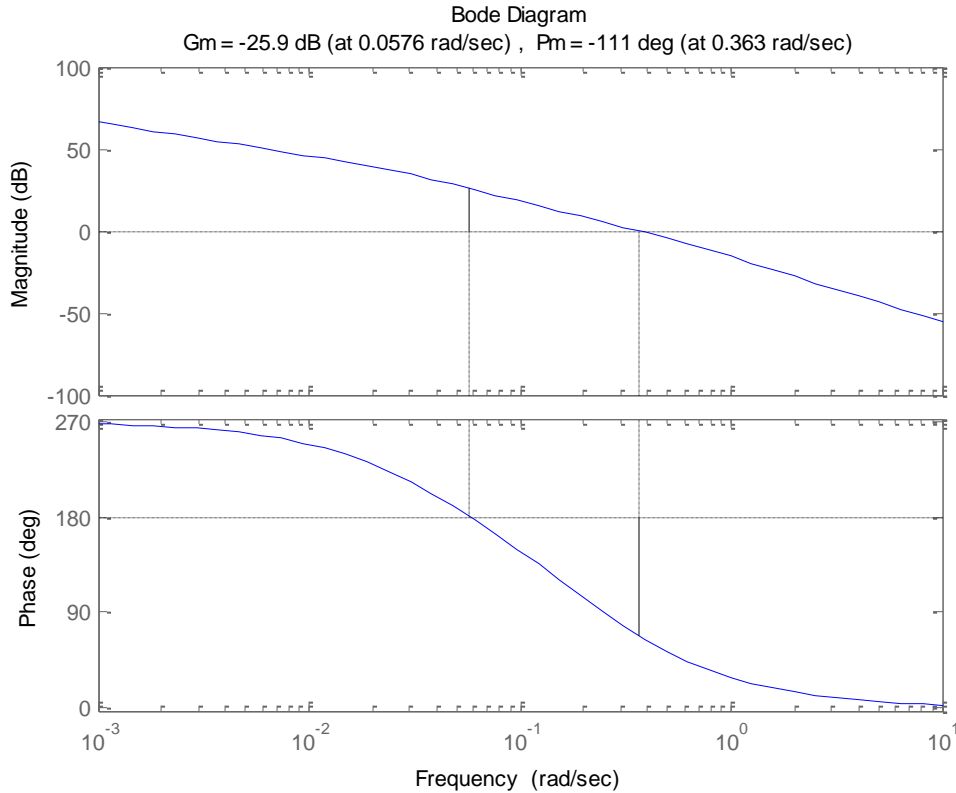
Correspondingly, the Bode diagram of closed loop system can be shown as:



The Bandwidth can be obtained from -3dB in magnitude diagram of the Bode plot. The above data point in the figure shows $BW = 7.44 \text{ rad/sec}$

11-25) $K_v = \lim_{s \rightarrow 0} sG_c(s)G(s) = \frac{2353K(71)}{(71)(13)(181)} = 2$, therefore, $K > 2$

From bode plot of uncompensated loop, we have PM = -111 at $\omega = 0.363$ rad/s:

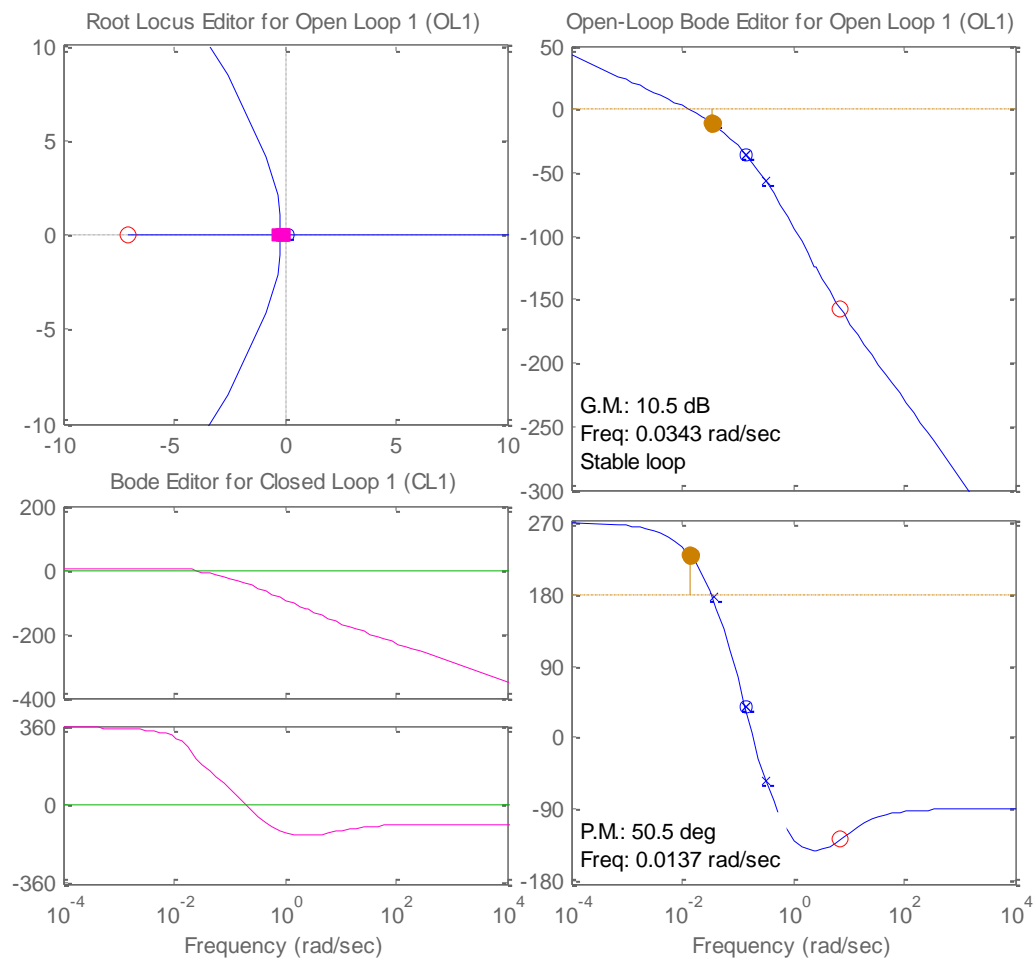


A PI controller can be expressed as $PI = (K_p + sK_i)$. The effect is similar to adding a Zero at $\frac{-K_p}{K_i}$. Let's place this zero at 71/500 to cancel the Phase lag originating from the unstable zero of G at +71/500:

$$(G(s) = \frac{2353K(71-500s)}{71s(40s+13)(5000s+181)}).$$

The compensator can be expressed as:

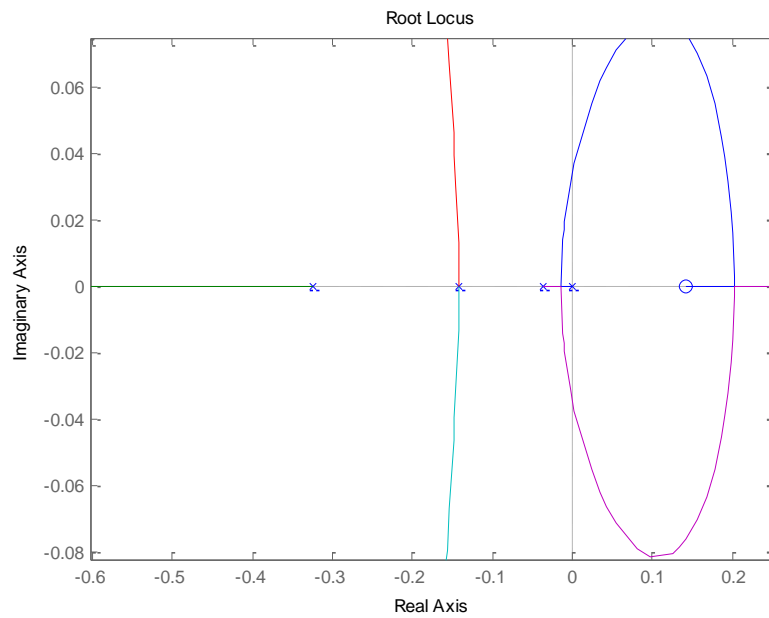
$PI = K_p \left(s + \frac{K_p}{K_i} \right) = K_p(s + 71/500)$, where K_p can be adjusted in sisotool as the overall gain of the loop, until the required PM is achieved. At $K_p = 37$, PM=50 deg as seen in the following sisotool results:



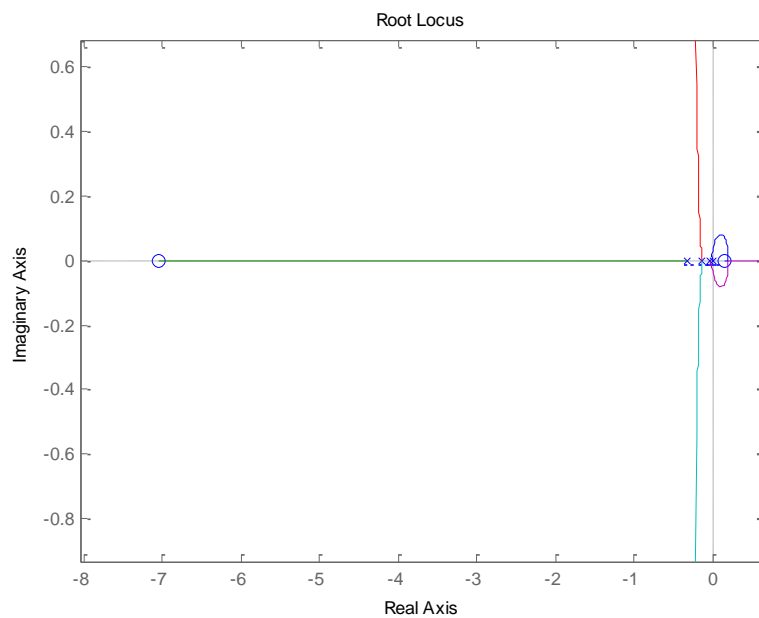
Considering the slow dynamic of the ship, and the RHS zero, the crossover frequency is relatively low.

The root locus diagram can be seen as:

Zoom in



Zoom out



MATLAB Code:

```
s = tf('s')
Kp=1;
Ki=1;

num_G= 2353*2*(71-500*s)
den_G=71*s*(40*s+13)*(5000*s+181)*(71+500*s)^2;
G=num_G/den_G;
```

%PI design

```

Kp=1
Ki=71/500
PI=Kp+Ki*s

figure(100)
Margin(G)

figure(101)
rlocus(G*PI)

sisotool

```

11-26) a) Transfer functions G and H are generated in MATLAB and imported into sisotool:

MATLAB Code:

```

s = tf('s')

num_G= 2*10^5;
den_G=s*(s+20)*(s^2+50*s+10000);
G=num_G/den_G;

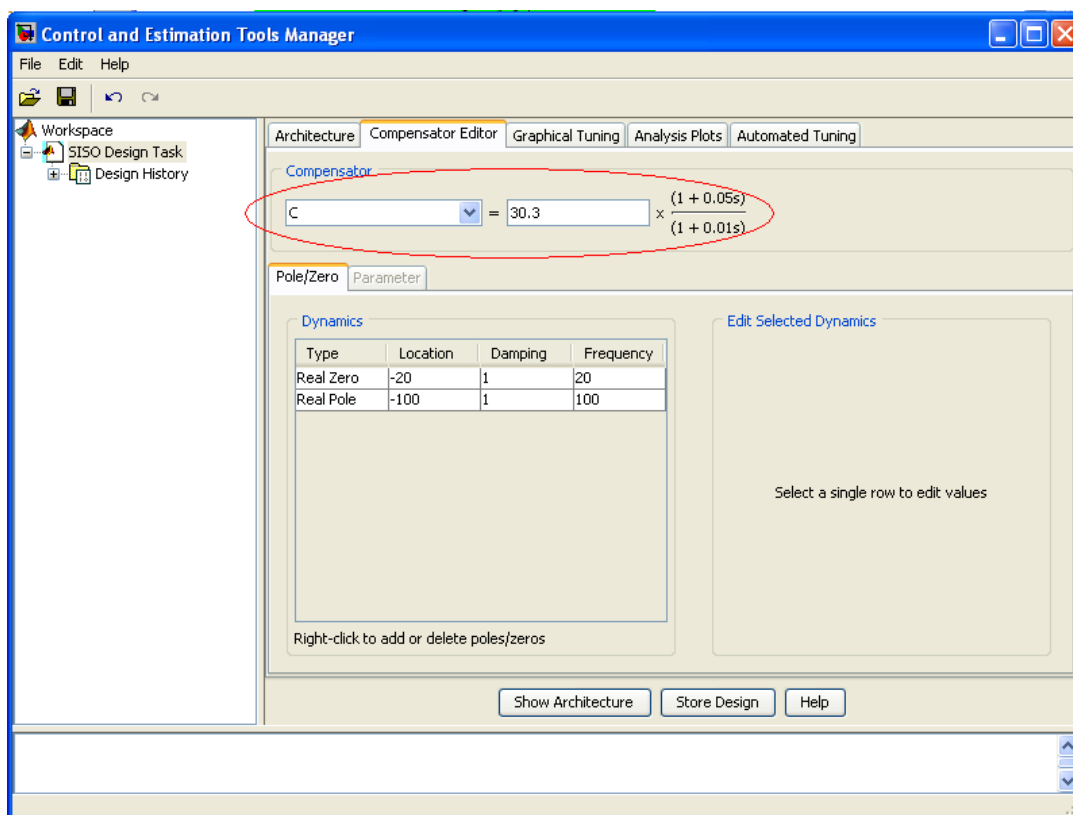
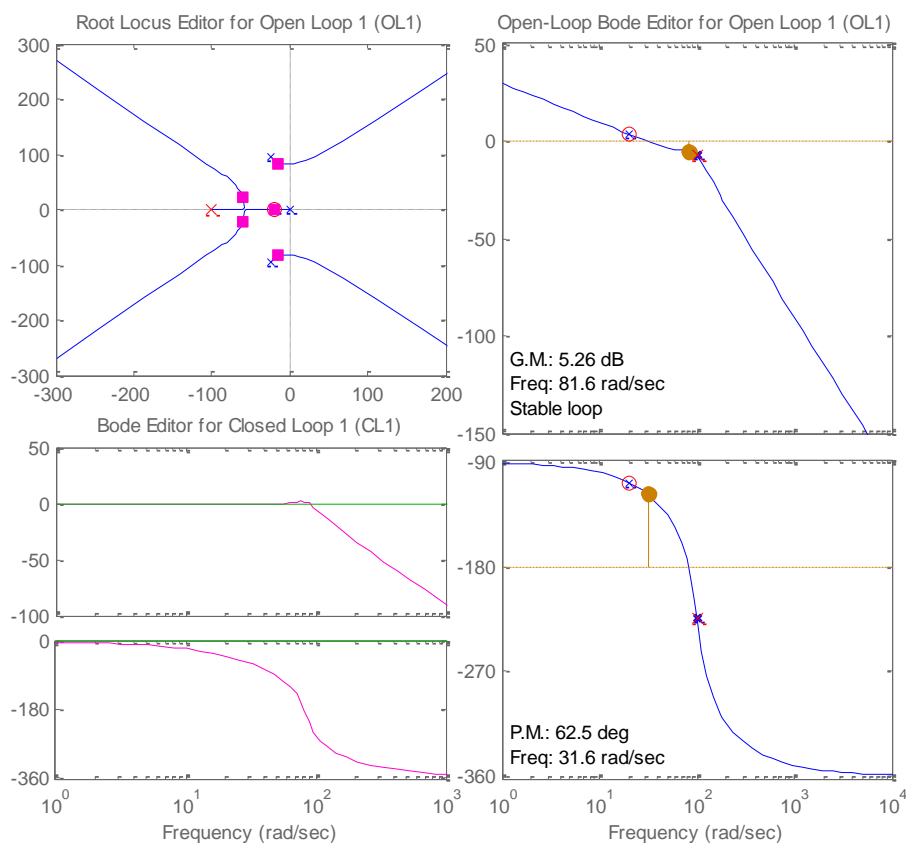
num_c1= 0.05*s+1;
den_c1=0.01*s+1;
c1=num_c1/den_c1;

num_c2= s/0.316+1;
den_c2=s/3.16+1;
c2=num_c2/den_c2;

sisotool

```

(a) The gain was changed until the cross over frequency matches 31.6 rad/sec as a requirement. At $K=30.3$, the desired cross over frequency of 31.6 rad/sec happens as can be seen in the following sisotool results:



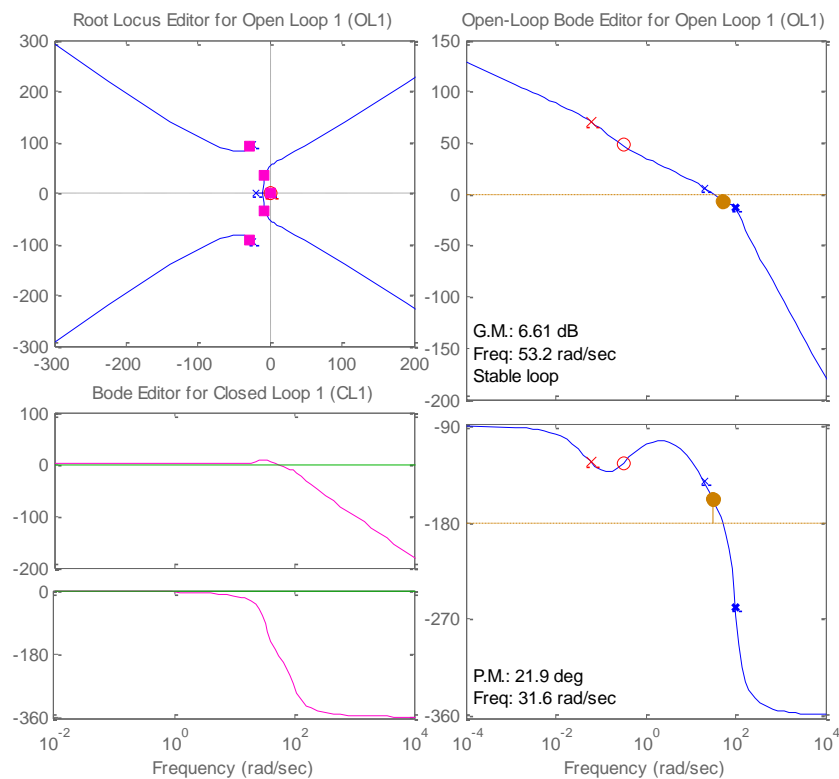
b) $K_v = \lim_{s \rightarrow 0} s G_c(s) G(s) \rightarrow K_v = \lim_{s \rightarrow 0} s \left(\frac{0.05s+1}{0.01s+1} \right) \left(\frac{2 \times 10^5 \times K}{s(s+20)(s^2+50s+10000)} \right) = K = 30.3$, then $K_v = 30.3$

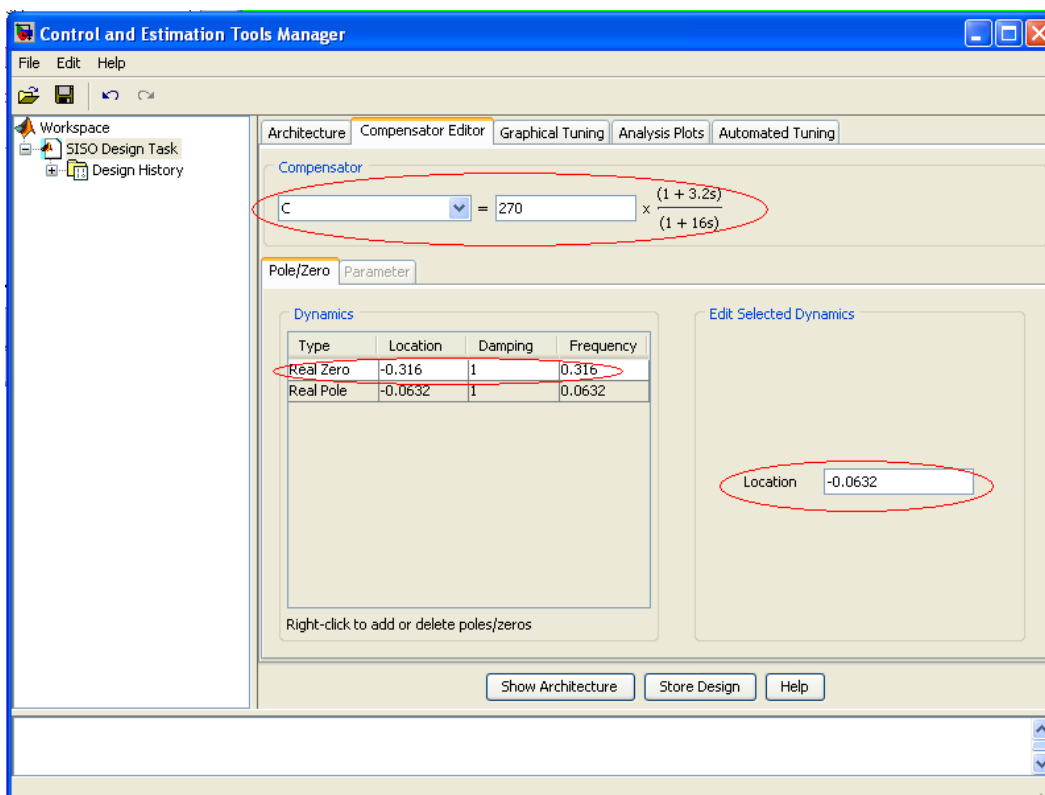
c) Again, $K_v = \lim_{s \rightarrow 0} s G_c(s) G(s) \rightarrow K_v = \lim_{s \rightarrow 0} s \left(\frac{r(\tau s+1)}{r\tau s+1} \right) \left(\frac{2 \times 10^5 \times r}{s(s+20)(s^2+50s+10000)} \right) = r$. To have $K_v = 100$, the overall gain of the PI controller should be equal to 100 ($r = 100$).

d & e) In this part, the PI pole is asked to be placed at -3.16 rad/sec and the crossover frequency needs to be at 31.6 rad/sec. The zero and the gain of the PI controller needs to be designed.

Considering the structure of the PI controller given in the question, $H(s) = \frac{r(\tau s+1)}{(r\tau s+1)}$, the corresponding pole is set to -3.16 in sisotool. The place of the zero and the overall gain is iteratively changed in the MATLAB sisotool to achieve the crossover frequency of 31.6 rad/sec.

With a zero at -0.06321 rad/sec and overall gain of $K=270$, required crossover frequency (31.6 rad/sec) and PM of 21.9 deg is obtained as shown in the following sisotool results:





e) the presented sisotool figure shows the compensated bode diagram and 21.9 deg of PM.

11-27)

$$G_c(s) = K_p + K_D s + \frac{K_I}{s} = \frac{K_D s^2 + K_p s + K_I}{s} = (1 + K_{D1}s) \left(K_{P2} + \frac{K_{I2}}{s} \right)$$

where

$$K_P = K_{P2} + K_{D1}K_{I2} \quad K_D = K_{D1}K_{P2} \quad K_I = K_{I2}$$

Forward-path Transfer Function:

$$G(s) = G_c(s)G_p(s) = \frac{100(1 + K_{D1}s)(K_{P2}s + K_{I2})}{s(s^2 + 10s + 100)} \quad K_v = \lim_{s \rightarrow 0} sG(s) = K_{I2} = 100$$

Thus

$$K_I = K_{I2} = 100$$

Consider only the PI controller, (with $K_{D1} = 0$)**Forward-path Transfer Function:****Characteristic Equation:**

$$G(s) = \frac{100(K_{P2}s + 100)}{s(s^2 + 10s + 100)} \quad s^3 + 10s^2 + (100 + 100K_{P2})s + 10,000 = 0$$

For stability, $K_{P2} > 9$. Select $K_{P2} = 10$ for fast rise time.

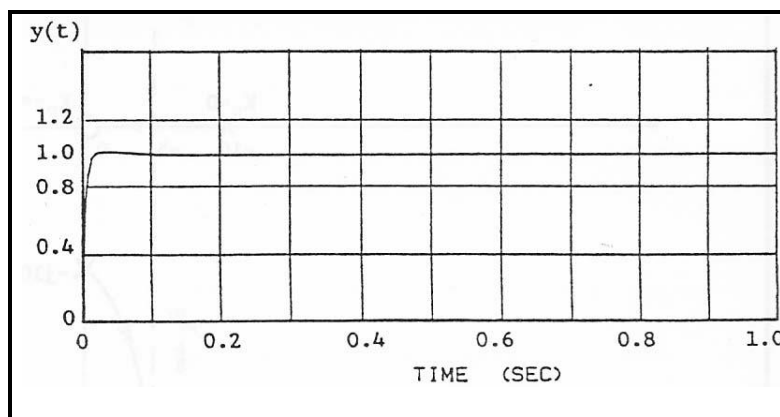
$$G(s) = \frac{1000(1 + K_{D1}s)(s + 10)}{s(s^2 + 10s + 100)}$$

When $K_{D1} = 0.2$, the rise time and overshoot requirements are satisfied.

$$K_D = K_{D1} K_{P2} = 0.2 \times 10 = 2$$

$$K_P = K_{P2} + K_{D1} K_{I2} = 10 + 0.2 \times 100 = 30$$

$$G_c(s) = 30 + 2s + \frac{100}{s}$$



Unit-step Response

11-28)**Process Transfer Function:**

$$G_p(s) = \frac{Y(s)}{U(s)} = \frac{e^{-0.2s}}{1+0.25s} \cong \frac{1}{(1+0.25s)(1+0.2s+0.02s^2)}$$

(a) PI Controller:

$$G(s) = G_c(s)G_p(s) \cong \frac{K_p + \frac{K_I}{s}}{(1+0.25s)(1+0.2s+0.02s^2)} = \frac{200(K_p s + K_I)}{s(s+4)(s^2+10s+50)}$$

$$\text{For } K_v = 2, \quad K_v = \lim_{s \rightarrow 0} sG(s) = \frac{200K_I}{4 \times 50} = K_I = 2 \quad \text{Thus } K_I = 2$$

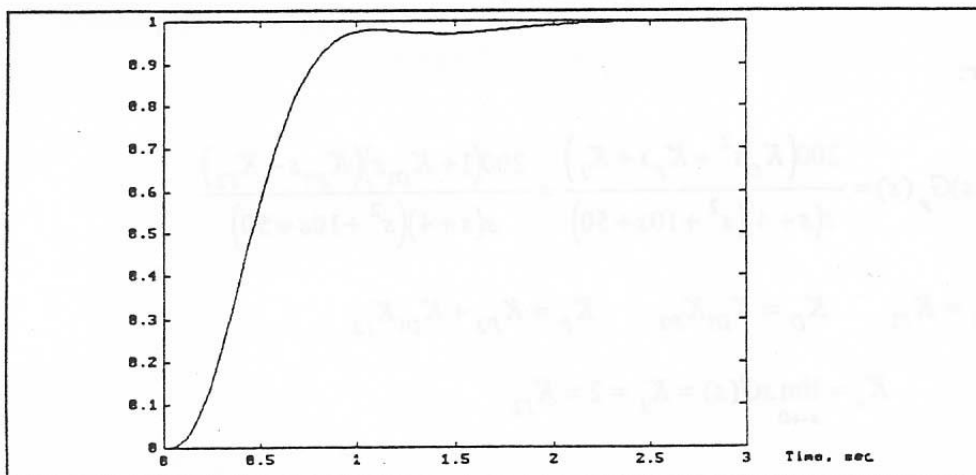
$$\text{Thus } G(s) = \frac{200(2 + K_p s)}{s(s+4)(s^2+10s+50)}$$

The following values of the attributes of the unit-step response are computed for the system with various values for K_p .

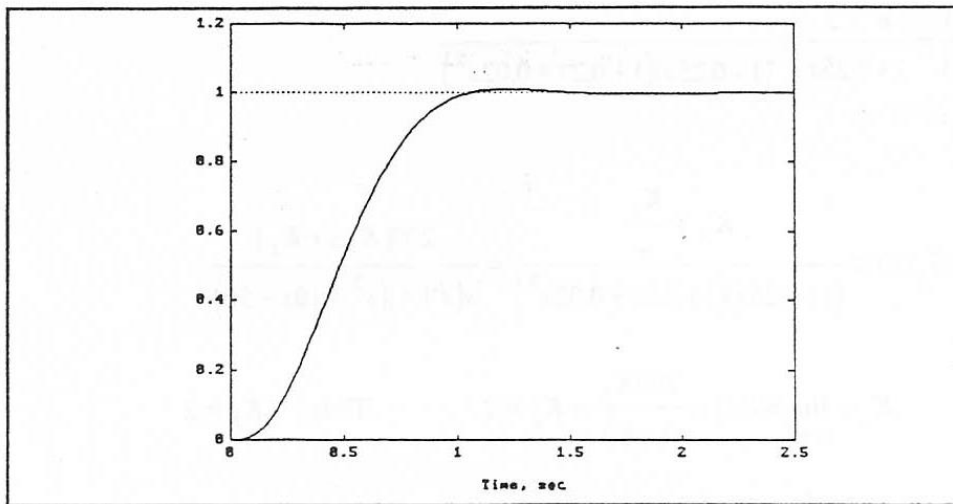
K_p	Max overshoot (%)	t_s (sec)	t_s (sec)
0.1	19.5	0.61	2.08
0.2	13.8	0.617	1.966
0.3	8.8	0.615	1.733
0.4	4.6	0.606	0.898
0.5	1.0	0.5905	0.878
0.6	0	0.568	0.851
0.7	0	0.541	1.464
0.8	0	0.5078	1.603
1.0	0	0.44	1.618

The settling time t_s is minimum (0.851 sec) when $K_p = 0.6$. Statistically, $K_p = 0.6$ is the best choice. The unit-step response is shown below. However, a better response is obtained when $K_p = 0.5$.

Unit-step Response: ($K_p = 0.6$, $K_I = 2$)



Unit-step Response: ($K_p = 0.5$, $K_I = 2$)



For stability check we perform the Routh tabulation. The characteristic equation with $K_I = 2$ is

$$s^4 + 14s^3 + 90s^2 + (200 + 200K_p)s + 400 = 0$$

Routh Tabulation:

s^4	1	90	400
s^3	14	$200 + 200K_p$	
s^2	$75.714 - 14.284K_p$	400	
s^1	$\frac{9542.8 + 12285.66K_p - 2857.14K_p^2}{75.714 - 14.284K_p}$		
s^0	400		

For the coefficients in the first row to be positive, from the s^2 row, $K_p < 5.3$. From the s^1 row,

$$9542.8 + 12285.66K_p - 2857.14K_p^2 > 0 \quad \text{or} \quad (K_p - 4.9718)(K_p + 0.6718) < 0$$

Thus $K_p < 4.9718$ which is the condition for stability.

(b) PID Controller:

$$G(s) = G_c(s)G_p(s) = \frac{200(K_D s^2 + K_p s + K_I)}{s(s+4)(s^2 + 10s + 50)} = \frac{200(1 + K_{D1}s)(K_{P2}s + K_{I2})}{s(s+4)(s^2 + 10s + 50)}$$

where

$$K_I = K_{I2} \quad K_D = K_{D1}K_{P2} \quad K_P = K_{P2} + K_{D1}K_{I2}$$

$$K_v = \lim_{s \rightarrow 0} sG(s) = K_I = 2 = K_{I2}$$

$$G(s) = \frac{200(1 + K_{D1}s)(K_{P2}s + 2)}{s(s+4)(s^2 + 10s + 50)} = \frac{200(K_D s^2 + K_P s + K_I)}{s(s+4)(s^2 + 10s + 50)}$$

From the results in part (a), we set $K_P = 0.6$. The following attributes of the unit-step response show that adding derivative control does not provide any further improvement to the system response.

K_D	Max Overshoot (%)	t_r (sec)	t_s (sec)
0.1	1.1	0.9568	1.247
0.05	0.1	0.792	1.14
0.01	0	0.608	0.9075
0.005	0	0.588	0.8828
0.001	0	0.572	0.8753
0.0005	0	0.570	0.8778

11-29)

Forward-path Transfer Function:

$$G(s) = G_c(s)G_p(s) = \frac{(K_p s + K_I)e^{-0.2s}}{s(1 + 0.25s)}$$

$$K_v = \lim_{s \rightarrow 0} sG(s) = K_I = 2 \quad \text{Thus } K_I = 2$$

The attributes of the frequency response for various values of K_p are computed and tabulated below.

K_p	PM (deg)	GM (deg)	M_r	BW (rad/sec)
0.1	49.72	10.91	1.196	3.55
0.2	54.5	12.58	1.092	3.54
0.3	59.0	13.15	1.027	3.56
0.5	67.07	11.88	1.000	3.81
0.6	70.50	10.92	1.000	4.20
0.7	73.41	9.98	1.000	5.09
0.8	75.65	9.10	1.000	6.62
0.9	76.93	8.27	1.000	7.99
1.0	77.04	7.50	1.000	9.05
1.1	75.81	6.78	1.033	9.90
2.0	31.08	2.03	4.029	13.64
2.4	8.51	0.52	12.55	14.52
2.5545	0	0	∞	

Maximum phase margin of 77.04 deg is obtained when $K_p = 1.0$. In Problem 10-13(a), $K_p = 0.6$ is chosen for 0 maximum overshoot and minimum settling time.

The critical value of K_p for stability is 2.5545. In Problem 10-13(a), the critical value of K_p is 4.9718.

11-30) If there is no disturbance then

$$G(s) = \frac{0.9}{s^2} \left(\frac{2}{s+2} \right)$$

Let's consider PID controller as:

$$G_c(s) = (1 + K_D(s)) (K_p + K_I(s)) = \frac{K}{s} [(\tau_D s + 1)(s + 1/\tau_I)]$$

If τ_D is sufficiently smaller than τ_I then the τ_I has minor effect in PID controller. Let's examine PID controller when τ_D is varied.

$$\text{If } \begin{cases} \tau_D \leq 0.5 \text{ then the } PM < 180^\circ \\ \tau_D \geq 100 \text{ then the } PM < -90^\circ \\ \tau_D = 10 \text{ then } PM = 65^\circ \text{ and } \omega = 0.5 \frac{\text{rad}}{\text{sec}} \\ \tau_D \leq 20 \text{ then } \max\{\omega_c\} = 1 \frac{\text{rad}}{\text{sec}} \end{cases}$$

so let's consider $\tau_D = 10$, then $\frac{1}{\tau_D} = 0.1$

As τ_I to be large enough with respect to τ_D , let $\tau_I = 20$ $\tau_D = 200 \Rightarrow \frac{1}{\tau_I} = 0.005$

Now we have to determine the value of K so that the gain at the crossover frequency remains at 1.

If K=1 then $|G_c(s)G(s)|_{\omega=0.5} = 20$. Therefore, $\frac{1}{K} = 20$, or K=0.05

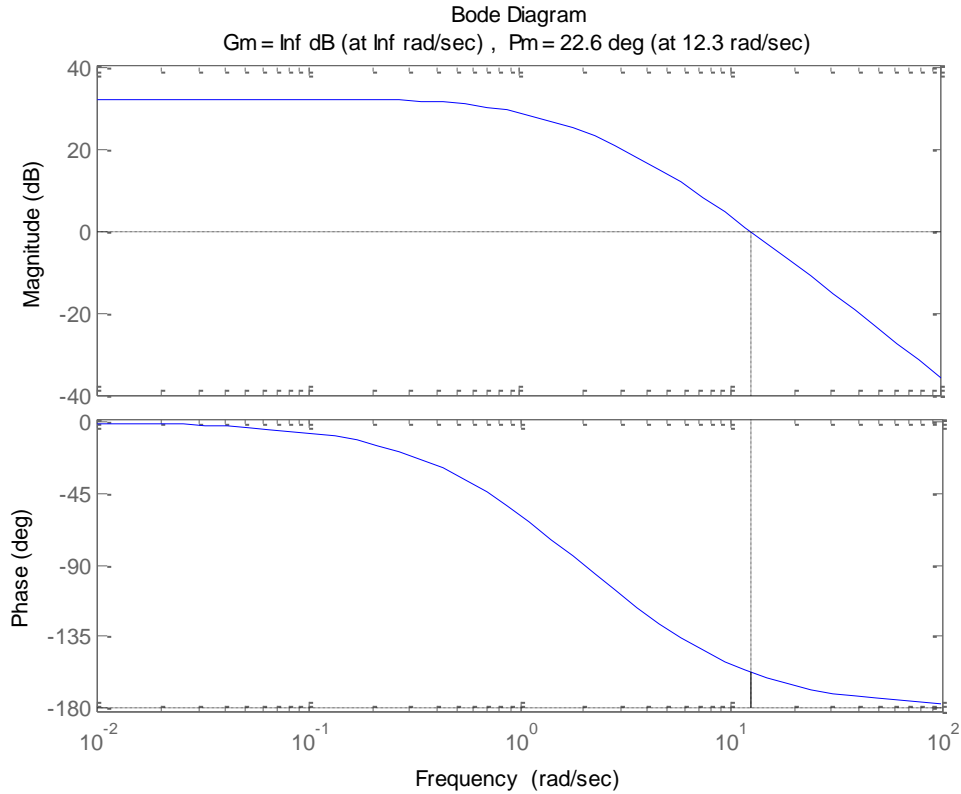
11-31) Let

$$G_c(s) = \frac{1 + \tau_2 s}{1 + \frac{\tau_2}{\alpha_2} s} \frac{K\alpha_1(1 + \tau_1 s)}{1 + \alpha_1 \tau_1 s}$$

where $e_{ss} \leq 0.01$. therefore, $\lim_{s \rightarrow 0} G_c(s)G(s) = 10K\alpha_1 + 1 > 100$

which gives $K\alpha_1 = 10$. Now $K\alpha_1 = 10$, then $PM = -40^\circ$. As a result, 91° phase lead is required to achieve $PM = 45^\circ$

The crossover frequency is 12.3 rad/sec as can be seen in the uncompensated bode diagram. The lag compensator must position $\omega_c = 5$ rad/sec, where its gain is 17.5 dB. Therefore the ratio of lag compensator can be chosen for this purpose as $3 < \frac{1}{\alpha_2} < 10$



Now the gain, which is obtained from combination of lead and lag compensator, is

$$Gain = K\alpha_1 \left| \frac{1 + j5\tau_1}{1 + j5\alpha_1\tau_1} \right| \cdot \left| \frac{1 + j5\tau_2}{1 + j5\frac{\tau_2}{\alpha_2}} \right|$$

or

$$gain_{dB} = 20 \log \left[K\alpha_1 \left| \frac{1 + j5\tau_1}{1 + j5\alpha_1\tau_1} \right| \right] + 20 \log \left| \frac{1 + j5\tau_2}{1 + j5\frac{\tau_2}{\alpha_2}} \right|$$

where $\left| \frac{1 + j5\tau_1}{1 + j5\alpha_1\tau_1} \right| > 1$ for $\alpha_1 < 1$. Since it is required that the final gain is increased by 17.5 dB,

let's choose $\alpha_2 = \frac{1}{15}$.

On the other hand, the corner frequency is $\frac{1}{\tau_2} = 0.5 \frac{rad}{sec} \Rightarrow \tau_2 = 2$. Therefore,

$$20 \log \left| \frac{1 + j5\tau_1}{1 + j5\alpha_1\tau_1} \right|_{\substack{\alpha_2 = \frac{1}{15} \\ \tau_2 = 2}} = 23.5$$

$$20 \log \left| \frac{1 + j5 \tau_2}{1 + j5 \frac{\tau_2}{\alpha_2}} \right| = 23.5 - 17.5 = 6 \text{ dB}$$

As a result, the actual phase reduction is $\Phi = \tan^{-1} \left(\frac{1}{\alpha_2} \tau_2 \omega \right) - \tan^{-1}(\tau_2 \omega) = 5.33$

The required phase lead is $\Phi = 45 + 5.33 + 3 = 53.33$, where $\Phi_m(\omega) = \tan^{-1}(\omega \tau_1) - \tan^{-1}(\omega \alpha_1 \tau_1) = 53.33$

By trial and error, we can find $\alpha_1 = 0.068$ and $\tau_1 = 0.35$. Therefore $K = 147$ where $K = \frac{10}{\alpha_1}$

11-32)**(a)**

$$G_p(s) = \frac{Z(s)}{F(s)} = \frac{1}{Ms^2 + K_s} = \frac{1}{150s^2 + 1} = \frac{0.00667}{s^2 + 0.00667}$$

The transfer function $G_p(s)$ has poles on the $j\omega$ axis. The natural undamped frequency is $\omega_n = 0.0816$ rad/sec.

(b) PID Controller:

$$G(s) = G_c(s)G_p(s) = \frac{0.00667(K_D s^2 + K_P s + K_I)}{s(s^2 + 0.00667)}$$

$$K_v = \lim_{s \rightarrow 0} sG(s) = K_I = 100 \quad \text{Thus} \quad K_I = 100$$

$$\text{Characteristic Equation: } s^3 + 0.00667K_D s^2 + 0.00667(1 + K_P)s + 0.00667K_I = 0$$

For $\zeta = 0.707$ and $\omega_n = 1$ rad/sec, the second-order term of the characteristic equation is $s^2 + 1.414s + 1 = 0$. Divide the characteristic equation by the second-order term.

$$\begin{aligned} & s + (0.00667K_D - 1.414) \\ s^2 + 1.414s + 1 & \left| s^3 + 0.00667K_D s^2 + (0.00667 + 0.00667K_P)s + 0.00667K_I \right. \\ & \underline{s^3 + 1.414s^2 + 1s} \\ & (0.00667K_D - 1.414)s^2 + (0.00667K_P - 0.99333)s + 0.00667K_I \\ & \underline{(0.00667K_D - 1.414)s^2 + (0.00943K_D - 2)s + 0.00667K_D - 1.414} \\ & (0.00667K_P - 0.00943K_D + 1.00667)s + 0.00667K_I - 0.00667K_D + 1.414 \end{aligned}$$

$$\text{For zero remainder, } 0.00667K_I - 0.00667K_D + 1.414 = 0 \quad (1)$$

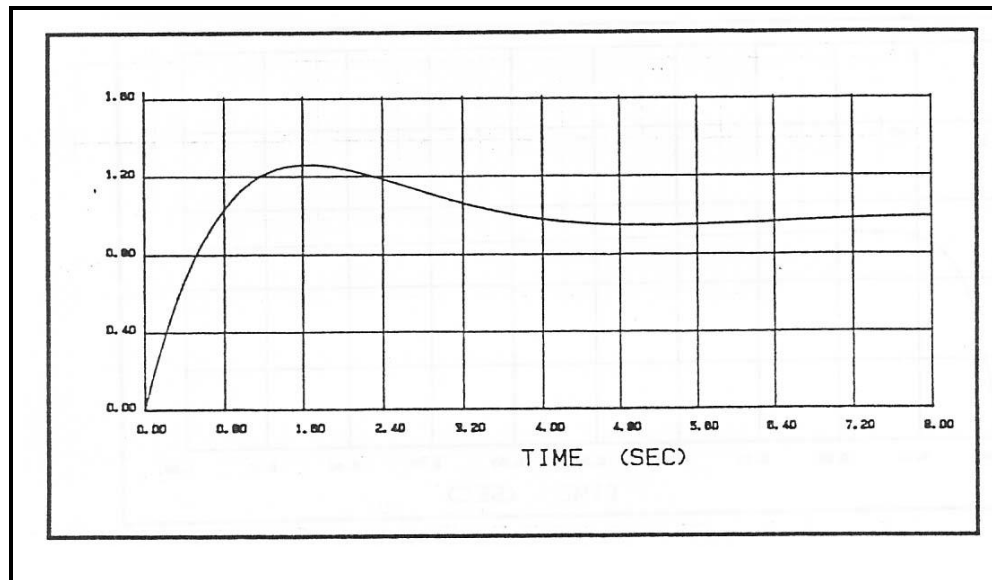
$$\text{and } 0.00667K_P - 0.00943K_D + 1.00667 = 0 \quad (2)$$

$$\text{From Eq. (1), } K_D = \frac{2.081}{0.00667} = 312$$

$$\text{From Eq. (2), } K_P = \frac{0.00943K_D - 1.00667}{0.00667} = 290.18$$

The forward-path transfer function becomes,

$$G(s) = \frac{2.081s^2 + 1.9355s + 0.667}{s(s^2 + 0.00667)}$$

Unit-step Response.

The unit-step response shows a maximum overshoot of 26%. Although the relative damping ratio of the complex roots is 0.707, the real pole of the third-order system transfer function is at -0.667 which adds to the overshoot.

(c)

$$G(s) = G_c(s)G_p(s) = \frac{0.00667(1 + K_{D1}s)(K_{P2}s + K_{I2})}{s(s^2 + 0.00667)}$$

For $K_v = 100$, $K_{I2} = K_I = 100$. Let us select $K_{P2} = 50$. Then

$$G(s) = \frac{0.00667(1 + K_{D1}s)(50s + 100)}{s(s^2 + 0.00667)}$$

For a small overshoot, K_{D1} must be relatively large. When $K_{D1} = 100$, the maximum overshoot is approximately 4.5%. Thus,

$$K_P = K_{P2} + K_{D1}K_{I2} = 50 + 100 \times 100 = 10050$$

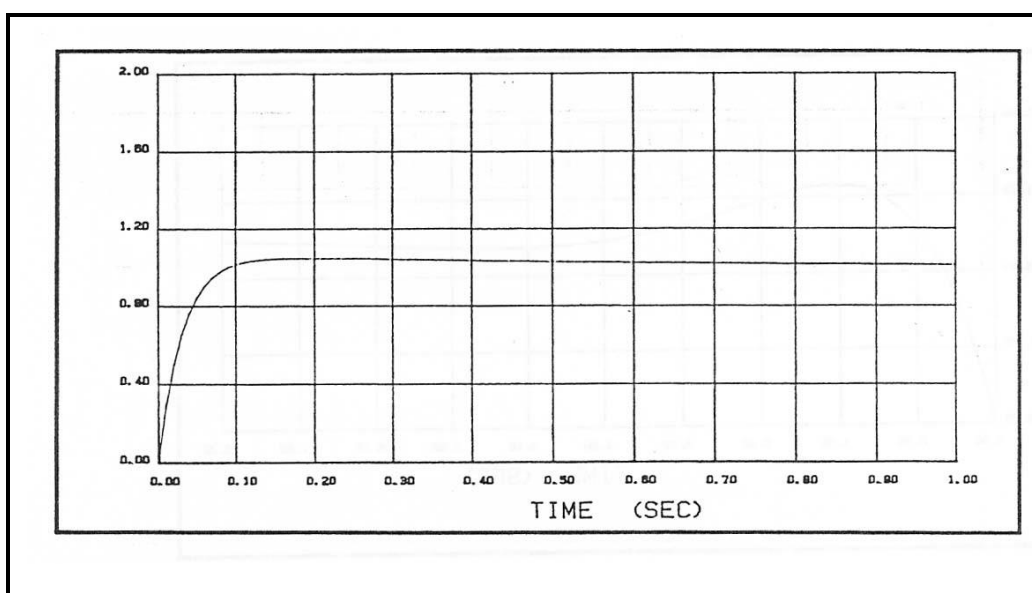
$$K_D = K_{D1}K_{P2} = 100 \times 50 = 5000$$

$$K_I = 100$$

System Characteristic Equation: $s^3 + 33.35s^2 + 67.04s + 0.667 = 0$

Roots: $-0.01, -2.138, -31.2$

Unit-step Response.



11-33)(a)

$$G_p(s) = \frac{Z(s)}{F(s)} = \frac{1}{Ms^2 + K_s} = \frac{1}{150s^2 + 1} = \frac{0.00667}{s^2 + 0.00667}$$

The transfer function $G_p(s)$ has poles on the $j\omega$ axis. The natural undamped frequency is

$$\omega_n = 0.0816 \text{ rad/sec.}$$

(b) PID Controller:

$$G(s) = G_c(s)G_p(s) = \frac{0.00667(K_D s^2 + K_P s + K_I)}{s(s^2 + 0.00667)}$$

$$K_v = \lim_{s \rightarrow 0} sG(s) = K_I = 100 \quad \text{Thus} \quad K_I = 100$$

$$\textbf{Characteristic Equation:} \quad s^3 + 0.00667K_D s^2 + 0.00667(1 + K_P)s + 0.00667K_I = 0$$

For $\zeta = 1$ and $\omega_n = 1 \text{ rad/sec}$, the second-order term of the characteristic equation is $s^2 + 2s + 1$.

Dividing the characteristic equation by the second-order term.

$$\begin{array}{r} s + (0.00667K_D - 2) \\ s^2 + 2s + 1 \overline{) s^3 + 0.00667K_D s^2 + (0.00667 + 0.00667K_P)s + 0.00667K_I} \\ \underline{s^3 + 2s^2 + s} \\ (0.00667K_D - 2)s^2 + (0.00667K_P - 0.99333)s + 0.00667K_I \\ \underline{(0.00667K_D - 2)s^2 + (0.01334K_D - 4)s + 0.00667K_D - 2} \\ (0.00667K_P - 0.01334K_D + 3.00667)s + 0.00667K_I - 0.00667K_D + 2 \end{array}$$

For zero remainder,

$$0.00667K_P - 0.01334K_D + 3.00667 = 0 \quad (1)$$

$$-0.00667K_D + 0.00667K_I + 2 = 0 \quad (2)$$

From Eq. (2),

$$0.00667K_D = 0.00667K_I + 2 = 2.667 \quad \text{Thus} \quad K_D = 399.85$$

From Eq. (1),

$$0.00667K_p = 0.01334K_D - 3.00667 = 2.3273 \quad \text{Thus} \quad K_p = 348.93$$

Forward-path Transfer Function:

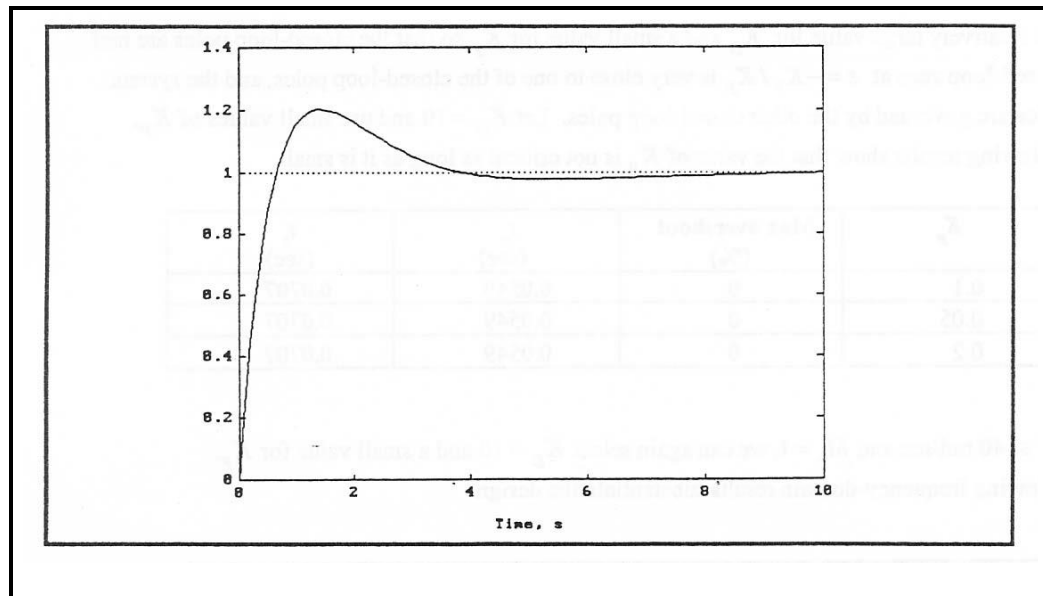
$$G(s) = \frac{0.00667(399.85s^2 + 348.93s + 100)}{s(s^2 + 0.00667)}$$

Characteristic Equation:

$$s^3 + 2.667s^2 + 2.334s + 0.667 = (s+1)^2(s+0.667) = 0$$

Roots: -1, -1, -0.667

Unit-step Response.



The maximum overshoot is 20%.

11-34) a) As $M\dot{v} + \mu v = u(t)$, therefore, $(Ms + \mu)V(s) = U(s)$ or $G(s) = \frac{V(s)}{U(s)} = \frac{1}{Ms + \mu}$

b) $\frac{V(s)}{U(s)} = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)}$

According to the second order system:

$$\begin{cases} \omega_n \geq \frac{1.8}{t_r} \rightarrow \omega_n \geq \frac{1.8}{5} \rightarrow \omega_n \geq 0.36 \\ \xi \geq \frac{\left(\frac{\ln M_p}{\pi}\right)^2}{\sqrt{1 + \left(\frac{\ln M_p}{K}\right)^2}} \rightarrow \xi \geq 0.6 \end{cases}$$

Let's first add a PD controller with $G_c(s) = 1 + K_D s$, and find K_D which satisfy the maximum overshoot requirement.

After writing the closed-loop transfer function including the PD controller, the characteristic equation (denominator of the closed loop T.F.) is: $s^2 + \frac{(\mu + K_d)}{M}s + \frac{K_p}{M}$

Therefore:

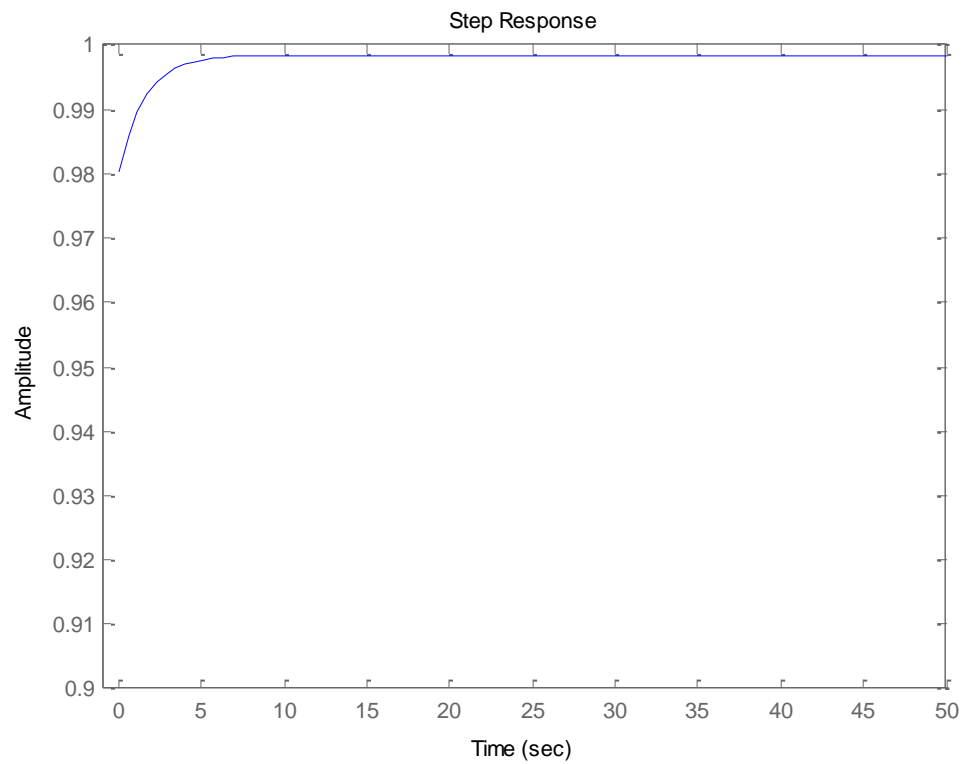
$$K_p = M\omega_n^2 = 1000(0.36^2) = 129.6$$

Also, $\frac{(\mu + K_d)}{M} = 2\xi\omega_n \rightarrow K_d = 2M\xi\omega_n - \mu = 2(1000)(0.6)(0.36) - 50 = 382 \text{ Nsec/m}$

Now let's add a PI controller with $G_c(s) = K_p + \frac{K_I}{s}$, and find K_p and K_I by using following table.

Now, for the PI part, K_I should be selected so that the additional pole at $-\frac{K_I}{K_p}$ does not interfere with the system dynamics. This pole is usually placed at least 1 decade lower (frequency wise) than the slowest existing poles of the system. In this case, since $\frac{\mu}{M} = \frac{50}{1000} = 0.05$, let's have $\frac{K_I}{K_p} = 0.005$, resulting in $K_I = K_p(0.005) = 129.6(0.005) = 0.648$

The step response is obtained through the following MATLAB code, showing the rise time of less than 5 sec, and almost no overshoot



MATLAB code:

```
s = tf('s')

Kp = 129.6
Kd = 382
Ki = 0.648

num_GH= (Kp*Kd*s)*(1+Ki/s);
den_GH=(1000*s+50);
GH=num_GH/den_GH;
CL = GH/(1+GH)

figure(1)
step(CL)
xlim([-1 50])
ylim([0.9 1])
```

11-35)(a) Process Transfer Function:**Forward-path Transfer Function**

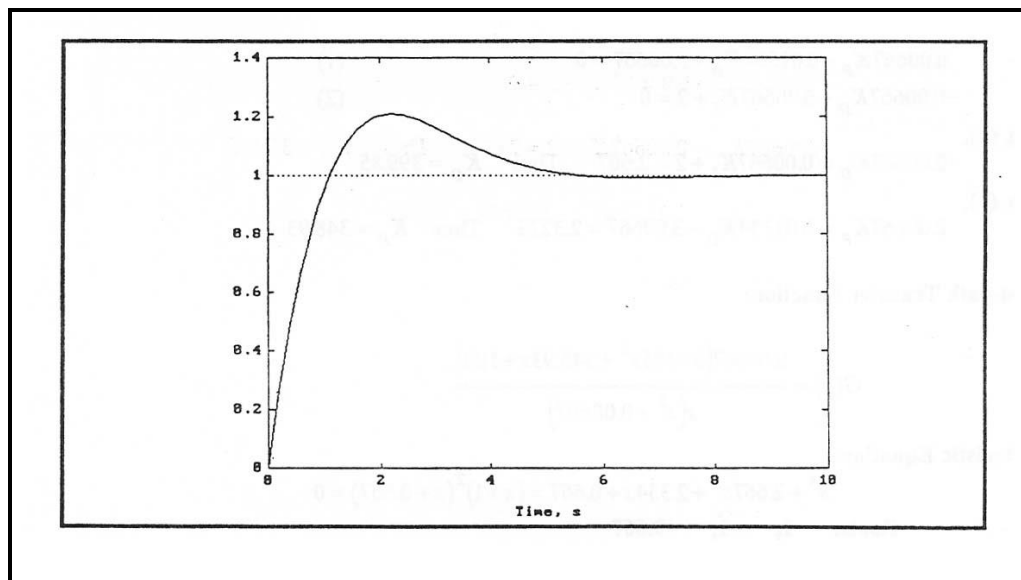
$$G_p(s) = \frac{4}{s^2}$$

$$G(s) = G_c(s)G_p(s) = \frac{4(K_p + K_D s)}{s^2}$$

Characteristic Equation: $s^2 + 4K_D s + 4K_P = s^2 + 1.414s + 1 = 0$ for $\zeta = 0.707$, $\omega_n = 1$ rad/sec

$$K_P = 0.25 \text{ and } K_D = 0.3535$$

Unit-step Response.



Maximum overshoot = 20.8%

(b) Select a relatively large value for K_D and a small value for K_P so that the closed-loop poles are real.

The closed-loop zero at $s = -K_P / K_D$ is very close to one of the closed-loop poles, and the system dynamics are governed by the other closed-loop poles. Let $K_D = 10$ and use small values of K_P .

The following results show that the value of K_P is not critical as long as it is small.

K_p	Max overshoot (%)	t_r (sec)	t_s (sec)
0.1	0	0.0549	0.0707
0.05	0	0.0549	0.0707
0.2	0	0.0549	0.0707

(c) For $BW \leq 40$ rad/sec and $M_r = 1$, we can again select $K_D = 10$ and a small value for K_p .

The following frequency-domain results substantiate the design.

K_p	PM (deg)	M_r	BW (rad/sec)
0.1	89.99	1	40
0.05	89.99	1	40
0.2	89.99	1	40

11-36) (a) Forward-path Transfer Function:

$$G(s) = G_c(s)G_p(s) = \frac{10,000(K_p + K_D s)}{s^2(s+10)}$$

Characteristic Equation:

$$s^3 + 10s^2 + 10,000K_D s + 10,000K_P = 0$$

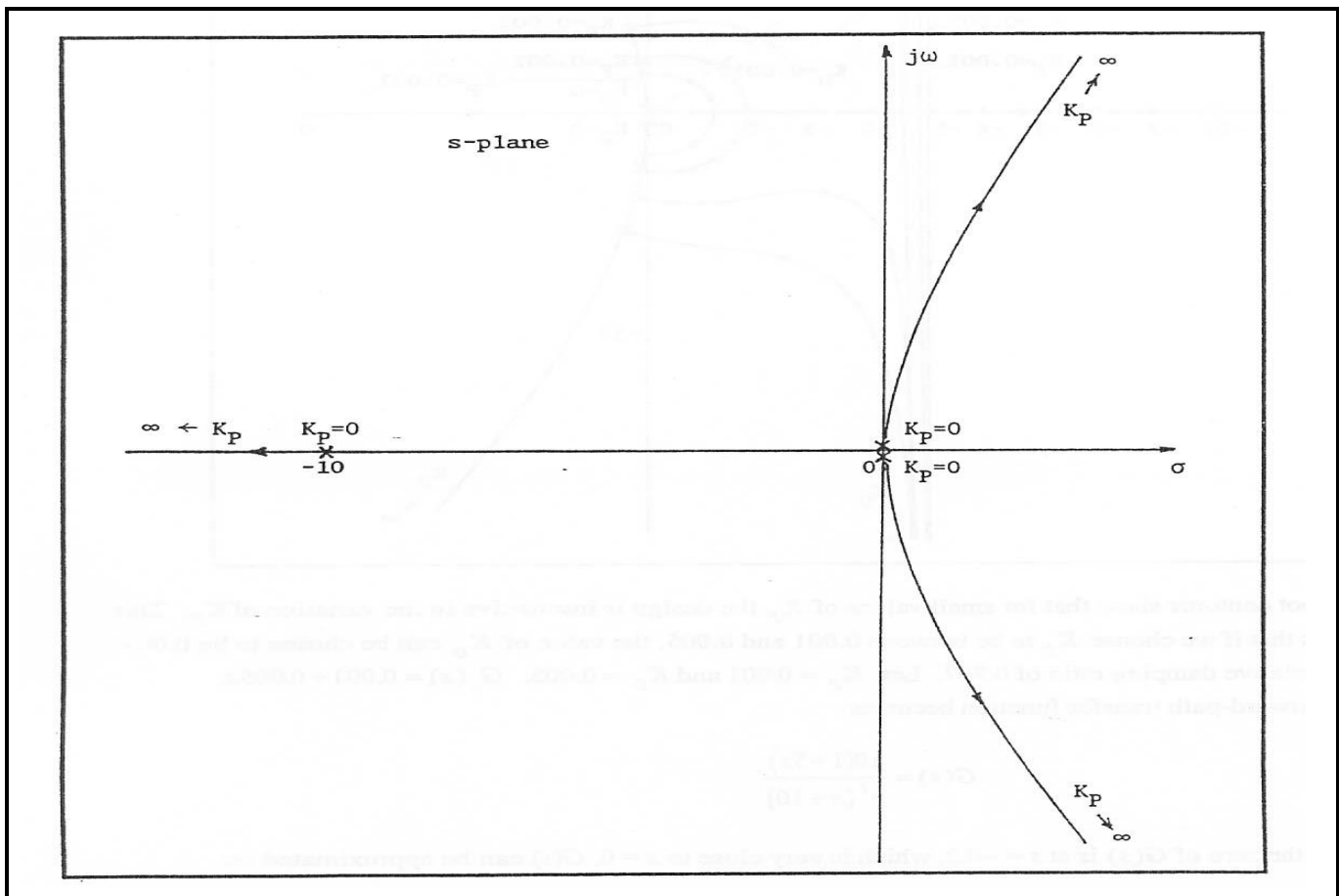
Routh Tabulation:

s^3	1	$10,000K_D$
s^2	10	$10,000K_P$
s^1	$10,000K_D - 1000K_P$	0
s^0	$10,000K_P$	

The system is stable for $K_P > 0$ and $K_D > 0.1K_P$

(b) Root Locus Diagram:

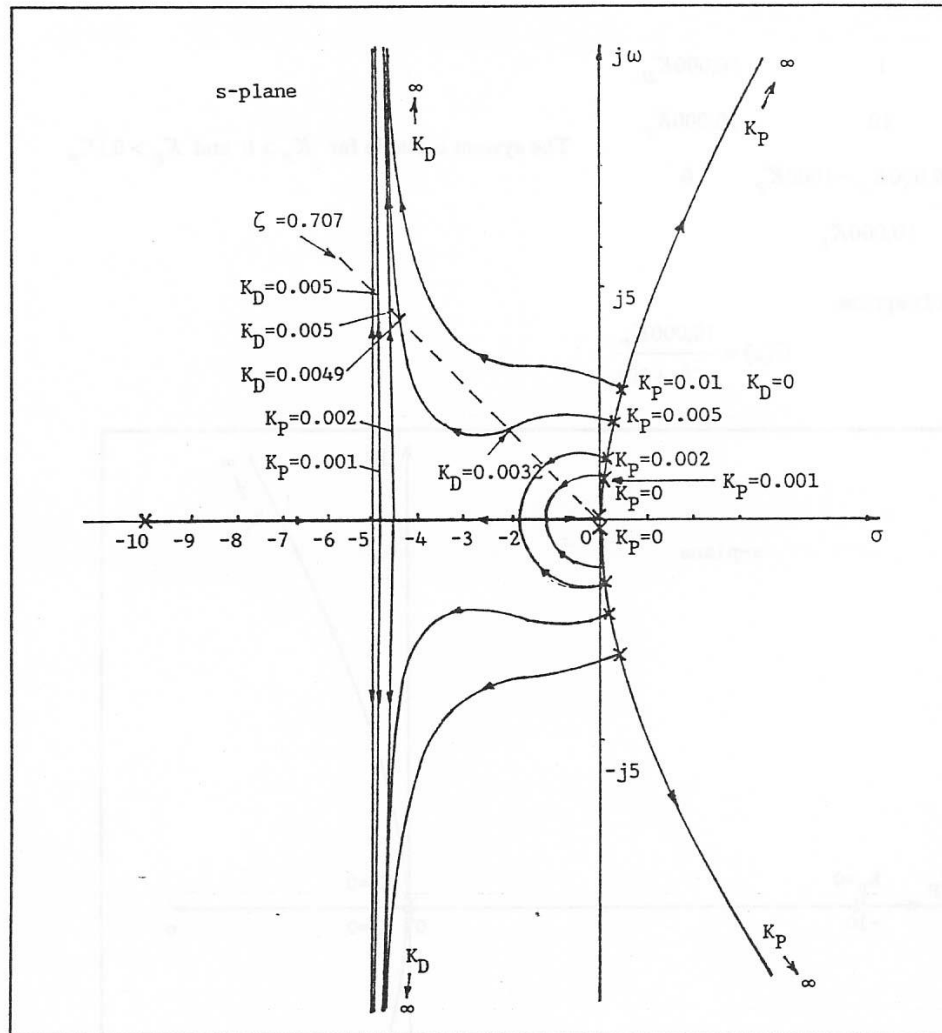
$$G(s) = \frac{10,000K_P}{s^2(s+10)}$$



Root Contours:

$$0 \leq K_D < \infty, \quad K_P = 0.001, 0.002, 0.005, 0.01.$$

$$G_{eq}(s) = \frac{10,000K_D s}{s^3 + 10s^2 + 10,000K_P}$$



- (c) The root contours show that for small values of K_P the design is insensitive to the variation of K_P . This means that if we choose K_P to be between 0.001 and 0.005, the value of K_D can be chosen to be 0.005 for a relative damping ratio of 0.707. Let $K_P = 0.001$ and $K_D = 0.005$. $G_c(s) = 0.001 + 0.005s$. The forward-path transfer function becomes

$$G(s) = \frac{10(1+5s)}{s^2(s+10)}$$

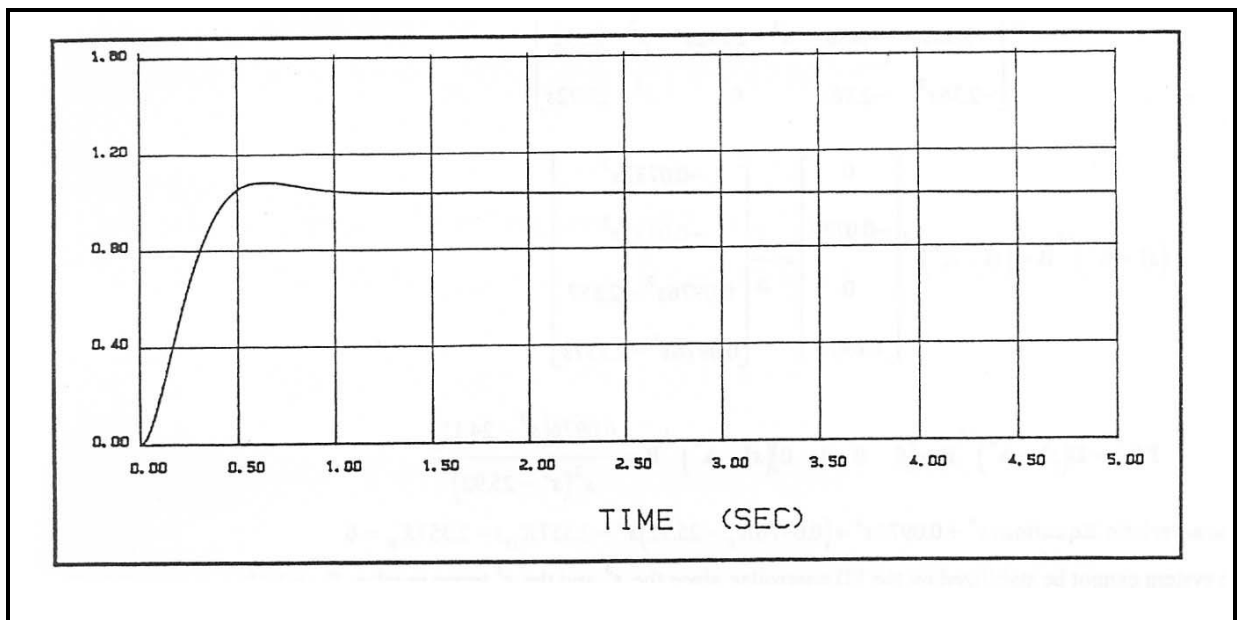
Since the zero of $G(s)$ is at $s = -0.2$, which is very close to $s = 0$, $G(s)$ can be approximated as:

$$G(s) \cong \frac{50}{s(s+10)}$$

For the second-order system, $\zeta = 0.707$. Using Eq. (7-104), the rise time is obtained as

$$t_r = \frac{1 - 0.4167\zeta + 2.917\zeta^2}{\omega_n} = 0.306 \text{ sec}$$

Unit-step Response:

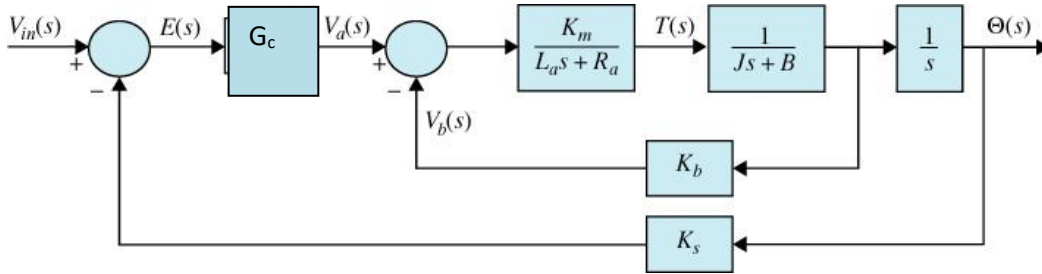


(d) Frequency-domain Characteristics:

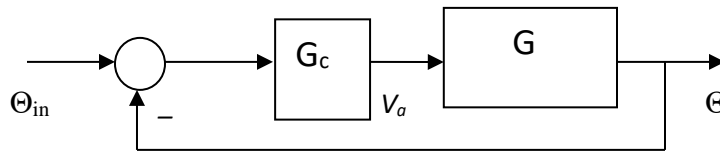
$$G(s) = \frac{10(1+5s)}{s^2(s+10)}$$

PM (deg)	GM (dB)	M_r	BW (rad/sec)
63	∞	1.041	7.156

11-37) This problem is extensively discussed in Chapters 5 and 6. Use the transfer function (5-123) for the open-loop system, and a series PID compensator in a unity feedback system.



Reduce to:



Where:

$$G(s) = \frac{\Theta(s)}{V_a(s)} = \frac{K_t}{s(L_a J s^2 + (L_a B + R_a J)s + R_a B + K_t K_b)}$$

$$G_c(s) = K_p + K_D s + \frac{K_I}{s} = \frac{K_D s^2 + K_p s + K_I}{s}$$

With

the rotor inertia (J) = 0.01 kg.m²/s²

damping ratio of the mechanical system (B) = 0.1 Nms

back-emf constant (K_b) = 0.01 Nm/Amp

torque constant (K_t) = 0.01 Nm/Amp

armature resistance (R_a) = 1 Ω

armature inductance (L_a) = 0.5 H

Starting systematically, set $K_I = K_D = 0$. Assume a small electric time constant (or small inductance) and simplify to Equation (5-126):

$$\frac{\Theta_m(s)}{\Theta_{in}(s)} = \frac{\frac{K K_m K_s}{R_a}}{(\tau_e s + 1) \left\{ J_m s^2 + \left(B + \frac{K_b K_m}{R_a} \right) s + \frac{K K_m K_s}{R_a} \right\}} \quad (5-125)$$

Where K_s is the sensor gain, and, as before, $\tau_e = (L_a/R_a)$ may be neglected for small L_a .

$$\begin{aligned}\frac{\Theta_m(s)}{\Theta_{in}(s)} &= \frac{\frac{KK_tK_s}{R_aJ}}{s^2 + \left(\frac{R_aB + K_tK_b}{R_aJ}\right)s + \frac{KK_tK_s}{R_aJ}} \\ \frac{\Theta_m(s)}{\Theta_{in}(s)} &= \frac{\frac{KK_tK_s}{R_aJ}}{s^2 + \left(\frac{R_aB + K_tK_b}{R_aJ}\right)s + \frac{KK_tK_s}{R_aJ}} \quad (5-126) \\ \frac{\Theta_m(s)}{\Theta_{in}(s)} &= \frac{\frac{K_p 0.1}{0.01}}{s^2 + \left(\frac{0.1 + (0.01)(0.01)}{0.01}\right)s + \frac{K_p 0.1}{0.01}} \\ \frac{\Theta_m(s)}{\Theta_{in}(s)} &= \frac{10K_p}{s^2 + 12s + 10K_p}\end{aligned}$$

Where $K_s=0$.

Using $t_s \cong \frac{3.2}{\zeta\omega_n}$; for a less than 2 sec settling time $\zeta\omega_n \leq 1.6$

For a PO of 4.3, $\zeta=0.707$, resulting in $\omega_n=2.26$.

Then a standard 2nd order prototype system that will have the desired response, with zero steady state error, takes the following form

$$\frac{\Theta_m(s)}{\Theta_{in}(s)} = \frac{5.12}{s^2 + 3.2s + 5.12}$$

For obvious reasons

$$\frac{\Theta_m(s)}{\Theta_{in}(s)} = \frac{10K_p}{s^2 + 12s + 10K_p} \neq \frac{5.12}{s^2 + 3.2s + 5.12}$$

Let's add a PD controller

$$G_c(s) = K_p + K_D s$$

$$\frac{\Theta_m(s)}{\Theta_{in}(s)} = \frac{10(K_D s + K_p)}{s^2 + (12 + 10K_D)s + 10K_p} \neq \frac{5.12}{s^2 + 3.2s + 5.12}$$

$$(12 + 10K_D) = 5.12$$

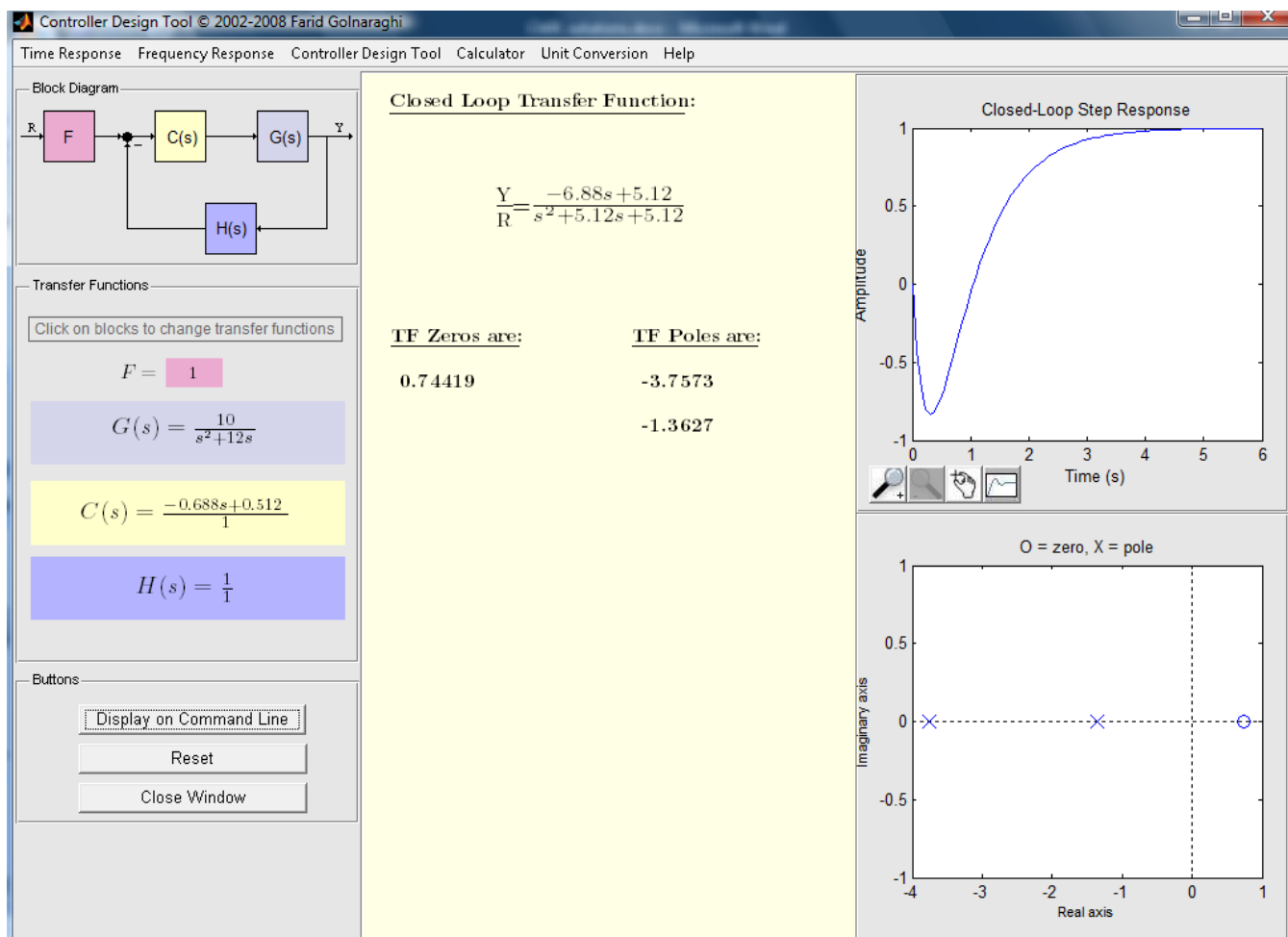
$$10K_p = 5.12$$

$$K_D = -0.688$$

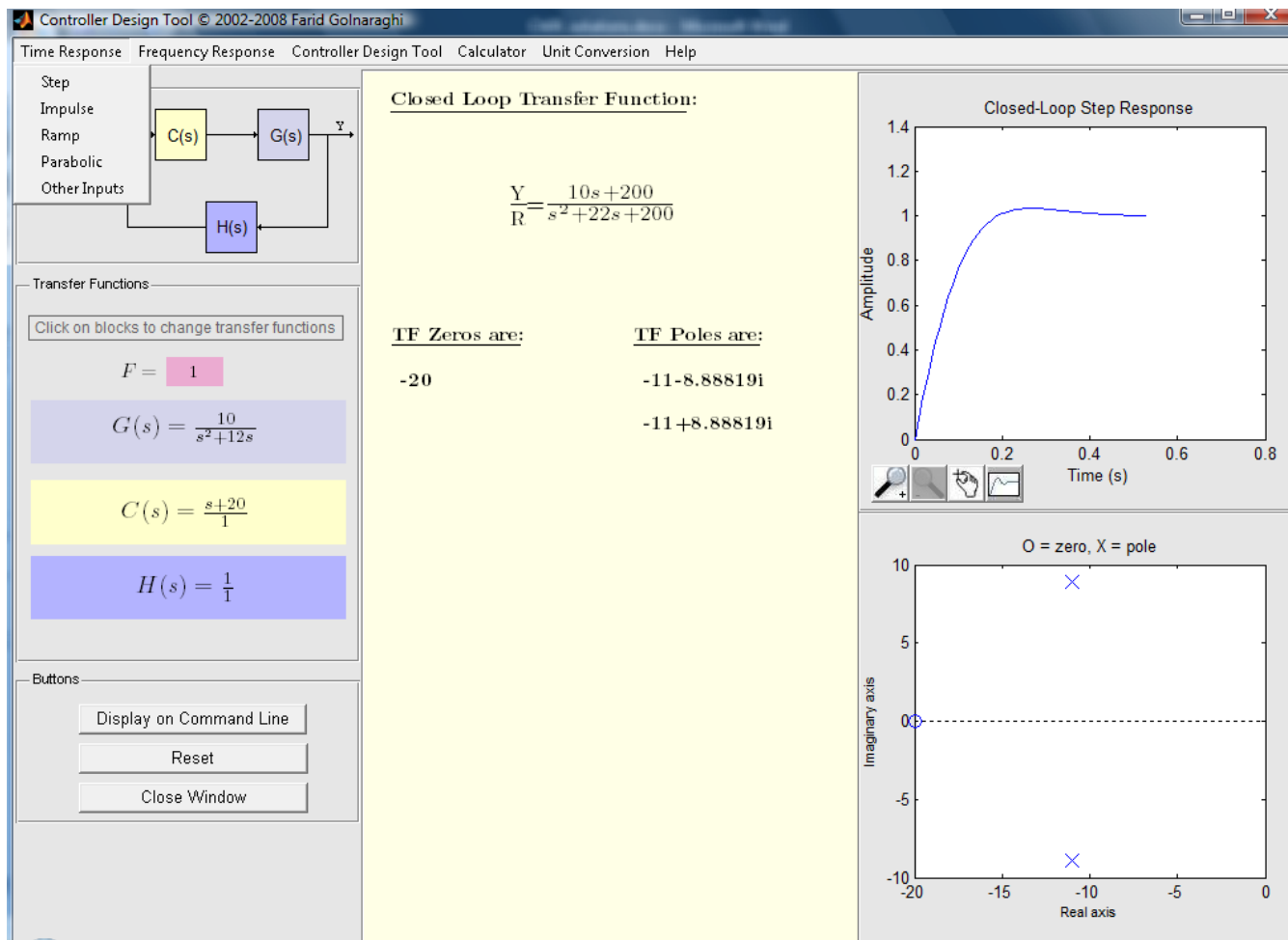
$$K_p = 0.512$$

Although the two systems are not the same because one has a zero, we chose the controller gain values by matching the two characteristic equations – as an initial approximation. The resulting zero in the right hand plane is troubling.

Lets find the response of the system through ACSYS:



Looking at the TF poles, it seems prudent to design the controller by placing its zero farther to LHS of the s-plane. Set $z = -K_p/K_D = -20$ and vary K_D to find the root locus or the response.



Done!

11-38) The same as 11-37

11-39)

$$\mathbf{A}^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 25.92 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -2.36 & 0 & 0 & 0 \end{bmatrix} \quad s\mathbf{I} - \mathbf{A}^* = \begin{bmatrix} s & -1 & 0 & 0 \\ -25.92 & s & 0 & 0 \\ 0 & 0 & s & -1 \\ 2.36 & 0 & 0 & s \end{bmatrix}$$

$$\Delta = |s\mathbf{I} - \mathbf{A}^*| = s \begin{vmatrix} s & 0 & 0 \\ 0 & s & -1 \\ 0 & 0 & s \end{vmatrix} + \begin{vmatrix} -25.92 & 0 & 0 \\ 0 & s & -1 \\ 2.36 & 0 & s \end{vmatrix} = s^2(s^2 - 25.92)$$

$$(s\mathbf{I} - \mathbf{A}^*)^{-1} = \frac{1}{\Delta} \begin{bmatrix} s^3 & s^2 & 0 & 0 \\ 25.92s^2 & s^3 & 0 & 0 \\ -2.36s & -2.36 & s^3 - 25.92s & s^2 - 25.92 \\ -2.36s^2 & -2.36s & 0 & s^3 - 25.92s \end{bmatrix}$$

$$(s\mathbf{I} - \mathbf{A}^*)^{-1} \mathbf{B} = (s\mathbf{I} - \mathbf{A}^*)^{-1} \begin{bmatrix} 0 \\ -0.0732 \\ 0 \\ 0.0976 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} -0.0732s^2 \\ -0.0732s^3 \\ 0.0976s^2 - 2.357 \\ 0.0976s^3 - 2.357s \end{bmatrix}$$

$$Y(s) = \mathbf{D}(s\mathbf{I} - \mathbf{A}^*)^{-1} \mathbf{B} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} (s\mathbf{I} - \mathbf{A}^*)^{-1} \mathbf{B} = \frac{0.0976(s^2 - 24.15)}{s^2(s^2 - 25.92)}$$

Characteristic Equation: $s^4 + 0.0976s^3 + (0.0976K_p - 25.92)s^2 - 2.357K_Ds - 2.357K_p = 0$

The system cannot be stabilized by the PD controller, since the s^3 and the s^1 terms involve K_D which require opposite signs for K_D .

11-40)

Let us first attempt to compensate the system with a PI controller.

$$G_c(s) = K_p + \frac{K_I}{s} \quad \text{Then} \quad G(s) = G_c(s)G_p(s) = \frac{100(K_p s + K_I)}{s(s^2 + 10s + 100)}$$

Since the system with the PI controller is now a type 1 system, the steady-state error of the system due to a step input will be zero as long as the values of K_p and K_I are chosen so that the system is stable.

Let us choose the ramp-error constant $K_v = 100$. Then, $K_I = 100$. The following frequency-domain performance characteristics are obtained with $K_I = 100$ and various value of K_p ranging from 10 to 100.

K_p	PM (deg)	GM (dB)	M_r	BW (rad/sec)
10	1.60	∞	29.70	50.13
20	6.76	∞	7.62	69.90
30	7.15	∞	7.41	85.40
40	6.90	∞	8.28	98.50
50	6.56	∞	8.45	106.56
100	5.18	∞	11.04	160.00

The maximum phase margin that can be achieved with the PI controller is only 7.15 deg when $K_p = 30$.

Thus, the overshoot requirement cannot be satisfied with the PI controller alone.

Next, we try a PID controller.

$$G_c(s) = K_p + K_D s + \frac{K_I}{s} = \frac{(1 + K_{D1}s)(K_{P2}s + K_{I2})}{s} = \frac{(1 + K_{D1}s)(K_{P2}s + 100)}{s}$$

Based on the PI-controller design, let us select $K_{P2} = 30$. Then the forward-path transfer function

becomes

$$G(s) = \frac{100(30s + 100)(1 + K_{D1}s)}{s(s^2 + 10s + 100)}$$

The following attributes of the frequency-domain performance of the system with the PID controller are obtained for various values of K_{D1} ranging from 0.05 to 0.4.

K_{D1}	PM (deg)	GM (dB)	M_r	BW (rad/sec)
0.05	85.0	∞	1.04	164.3
0.10	89.4	∞	1.00	303.8
0.20	90.2	∞	1.00	598.6
0.30	90.2	∞	1.00	897.0
0.40	90.2	∞	1.00	1201.0

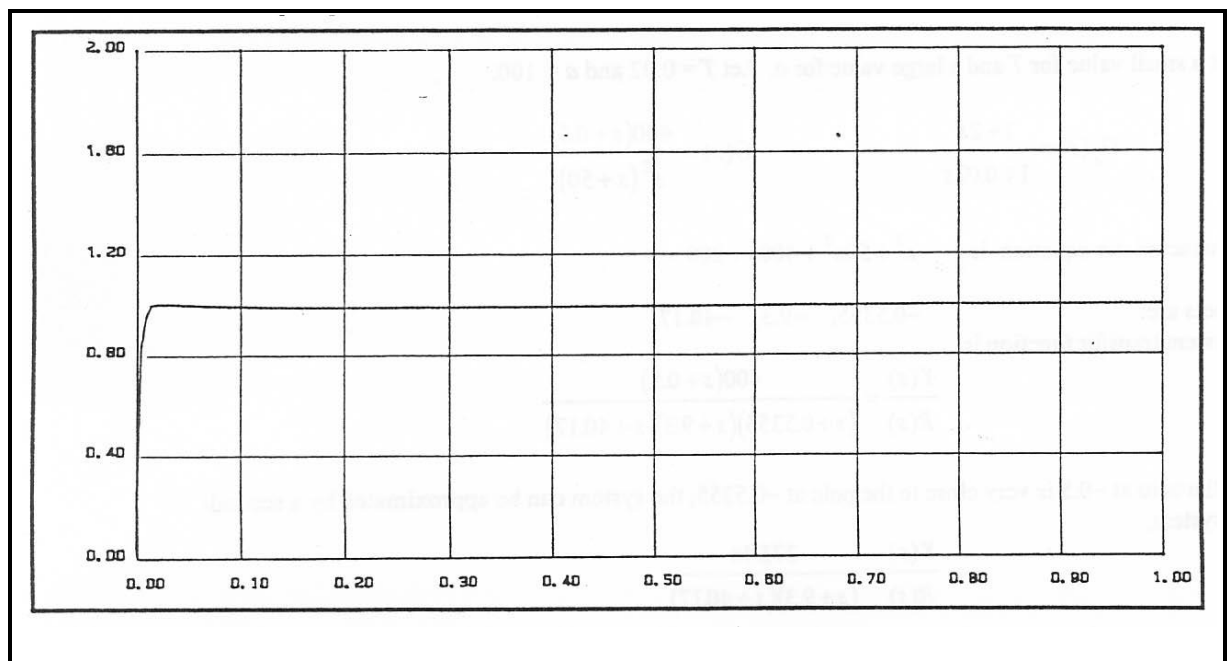
We see that for values of K_{D1} greater than 0.2, the phase margin no longer increases, but the bandwidth increases with the increase in K_{D1} . Thus we choose

$$K_{D1} = 0.2, \quad K_I = K_{I2} = 100, \quad K_D = K_{D1}K_{P2} = 0.2 \times 30 = 6,$$

$$K_P = K_{P2} + K_{D1}K_{I2} = 30 + 0.2 \times 100 = 50$$

The transfer function of the PID controller is $G_c(s) = 50 + 6s + \frac{100}{s}$

The unit-step response is show below. The maximum overshoot is zero, and the rise time is 0.0172 sec.



11-41)

$$\begin{aligned}\frac{Y(s)}{D(s)} &= \frac{s}{s(s^2 + 3.6s + 9) + K(\tau_1 s + 1)(\tau_2 s + 1)} \\ &= \frac{s}{s^3 + (3.6 + K\tau_1\tau_2)s^2 + (9 + K\tau_1 + K\tau_2)s + K}\end{aligned}$$

Let's consider the characteristic equation like:

$$s^3 + (3.6 + K\tau_1\tau_2)s^2 + (9 + K\tau_1 + K\tau_2)s + K = (s + p)(s^2 + 2\xi\omega_n s + \omega_n^2)$$

$$t_s = \frac{4}{\xi\omega_n} \text{ for 2\% settling time.}$$

Therefore we can choose $\xi = 0.5$ and $\omega_n = 4 \frac{\text{rad}}{\text{sec}}$ where $2 < t_s < 3$. Now, we can choose pole p far enough from pole dominant of second order. Let $p = 10$, then the characteristic equation would be:

$$s^3 + (3.6 + K\tau_1\tau_2)s^2 + (9 + K\tau_1 + K\tau_2)s + K = s^3 + 14s^2 + 56s + 160$$

where $K = 160$, $\tau_1 + \tau_2 = 0.29$, and $\tau_1\tau_2 = 0.065$

$$G_c(s) = \frac{160(0.065s^2 + 0.29s + 1)}{s}$$

Verify using MATLAB

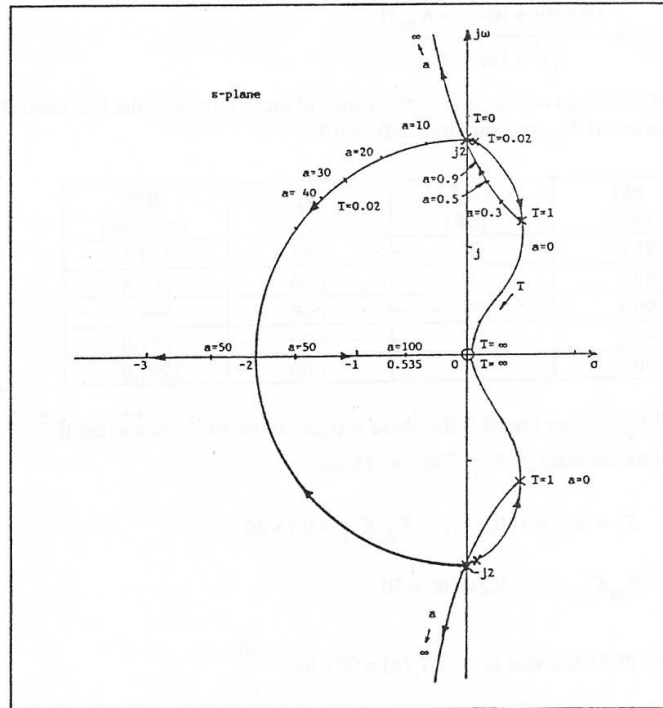
11-42)

(a)

$$G_p(s) = \frac{4}{s^2}$$

$$G(s) = G_c(s)G_p(s) = \frac{4(1+aTs)}{s^2(1+Ts)}$$

$$G_{eq}(s) = \frac{4aTs}{Ts^3 + s^2 + 4}$$

Root Contours: (T is fixed and a varies)Select a small value for T and a large value for a . Let $T = 0.02$ and $a = 100$.

$$G_c(s) = \frac{1+2s}{1+0.02s}$$

$$G(s) = \frac{400(s+0.5)}{s^2(s+50)}$$

The characteristic equation is $s^3 + 50s^2 + 400s + 200 = 0$ The roots are: $-0.5355, -9.3, -40.17$

The system transfer function is

$$\frac{Y(s)}{R(s)} = \frac{400(s+0.5)}{(s+0.5355)(s+9.3)(s+40.17)}$$

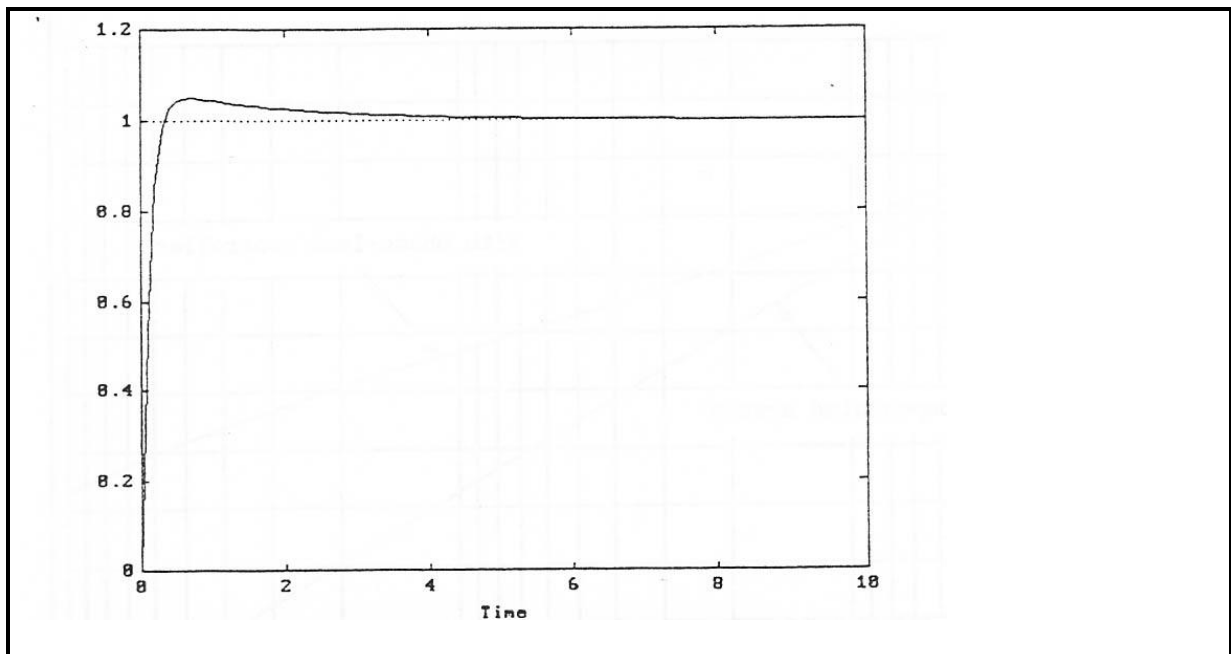
Since the zero at -0.5 is very close to the pole at -0.5355 , the system can be approximated by a second-order system,

$$\frac{Y(s)}{R(s)} = \frac{373.48}{(s + 9.3)(s + 40.17)}$$

The unit-step response is shown below. The attributes of the response are:

$$\text{Maximum overshoot} = 5\% \quad t_s = 0.6225 \text{ sec} \quad t_r = 0.2173 \text{ sec}$$

Unit-step Response.



The following attributes of the frequency-domain performance are obtained for the system with the phase-lead controller.

$$\text{PM} = 77.4 \text{ deg} \quad \text{GM} = \text{infinite} \quad M_r = 1.05 \quad \text{BW} = 9.976 \text{ rad/sec}$$

- (b)** The Bode plot of the uncompensated forward-path transfer function is shown below. The diagram shows that the uncompensated system is marginally stable. The phase of $G(j\omega)$ is -180 deg at all

frequencies. For the phase-lead controller we need to place ω_m at the new gain crossover frequency to realize the desired phase margin which has a theoretical maximum of 90 deg.

For a desired phase margin of 80 deg,

$$a = \frac{1 + \sin 80^\circ}{1 - \sin 80^\circ} = 130$$

The gain of the controller is $20\log_{10} a = 42$ dB. The new gain crossover frequency is at

$$|G(j\omega)| = -\frac{42}{2} = -21 \text{ dB}$$

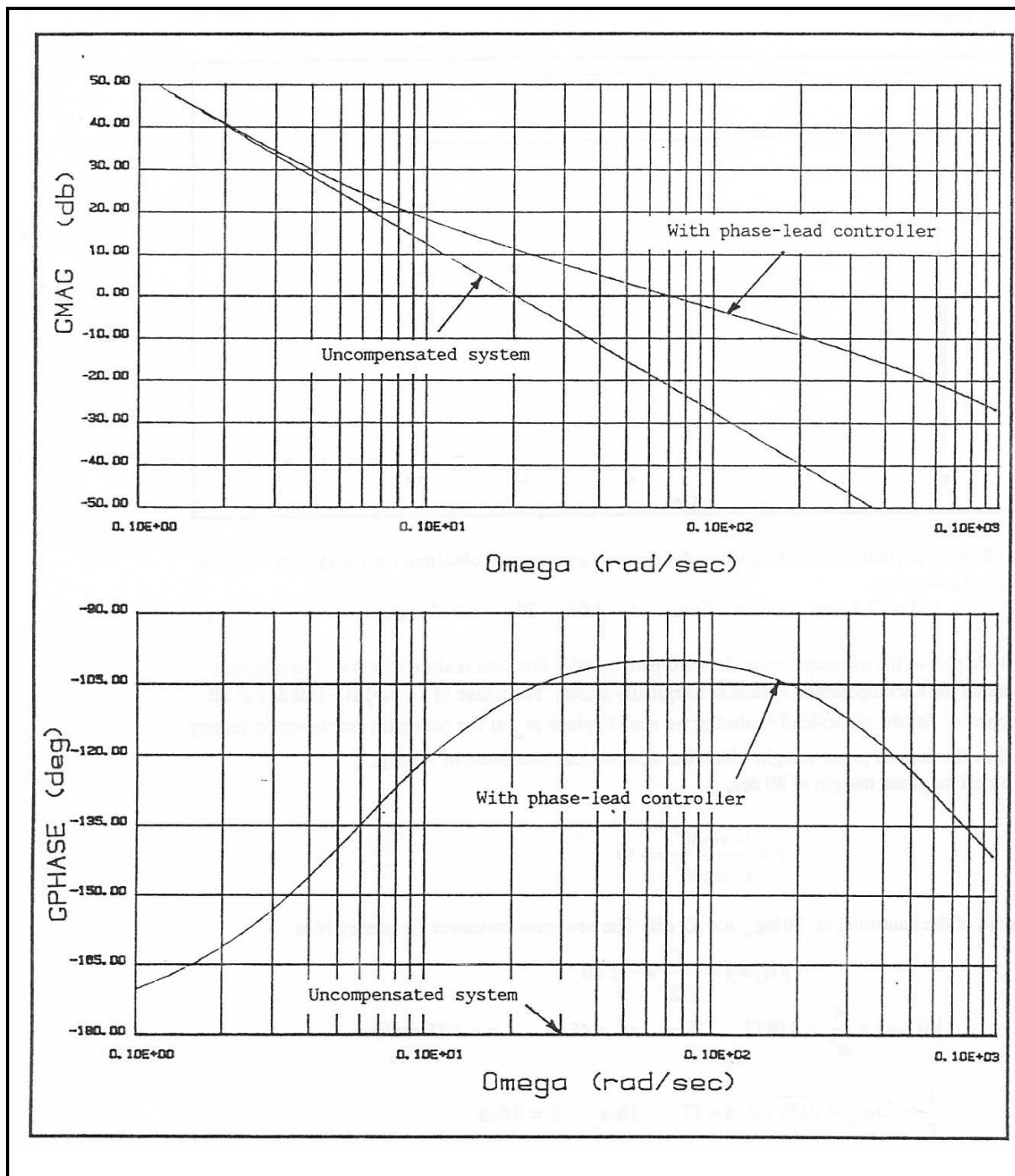
Or $|G(j\omega)| = \frac{4}{\omega^2} = 0.0877$ Thus $\omega^2 = 45.61$ $\omega = 6.75 \text{ rad/sec}$

$$\frac{1}{T} = \sqrt{a}\omega_m = \sqrt{130} \times 6.75 = 77 \quad \text{Thus} \quad T = 0.013$$

$$\frac{1}{aT} = 0.592 \quad \text{Thus} \quad aT = 1.69$$

$$G_c(s) = \frac{1 + aTs}{1 + Ts} = \frac{1 + 1.702s}{1 + 0.0131s} \quad G(s) = \frac{4(1 + 1.702s)}{s^2(1 + 0.0131s)}$$

Bode Plot.



11-43) (a) Forward-path Transfer Function:

$$G(s) = G_c(s)G_p(s) = \frac{1000(1+aTs)}{s(s+10)(1+Ts)} = \frac{1000a\left(s + \frac{1}{aT}\right)}{s(s+10)\left(s + \frac{1}{T}\right)}$$

Set $1/aT = 10$ so that the pole of $G(s)$ at $s = -10$ is cancelled. The characteristic equation of the system becomes

$$s^2 + \frac{1}{T}s + 1000a = 0$$

$$\omega_n = \sqrt{1000a} \quad 2\zeta\omega_n = \frac{1}{T} = 2\sqrt{1000a} \quad \text{Thus } a = 40 \quad \text{and } T = 0.0025$$

Controller Transfer Function:**Forward-path Transfer Function:**

$$G_c(s) = \frac{1+0.01s}{1+0.0025s}$$

$$G(s) = \frac{40,000}{s(s+400)}$$

The attributes of the unit-step response of the compensated system are:

$$\text{Maximum overshoot} = 0 \quad t_r = 0.0168 \text{ sec} \quad t_s = 0.02367 \text{ sec}$$

(b) Frequency-domain Design

The Bode plot of the uncompensated forward-path transfer function is made below.

$$G(s) = \frac{1000}{s(s+10)}$$

The attributes of the system are PM = 17.96 deg, GM = infinite.

$$M_r = 3.117, \text{ and BW} = 48.53 \text{ rad/sec.}$$

To realize a phase margin of 75 deg, we need more than 57 deg of additional phase. Let us add an additional 10 deg for safety. Thus, the value of ϕ_m for the phase-lead controller is chosen to be 67 deg. The value of a is calculated from

$$a = \frac{1 + \sin 67^\circ}{1 - \sin 67^\circ} = 24.16$$

The gain of the controller is $20 \log_{10} a = 20 \log_{10} 24.16 = 27.66$ dB. The new gain crossover frequency is at

$$\left| G(j\omega'_m) \right| = -\frac{27.66}{2} = -13.83 \text{ dB}$$

From the Bode plot ω'_m is found to be 70 rad/sec. Thus,

$$\frac{1}{T} = \sqrt{a}T = \sqrt{24.16} \times 70 = 344 \quad \text{or} \quad T = 0.0029 \quad aT = 0.0702$$

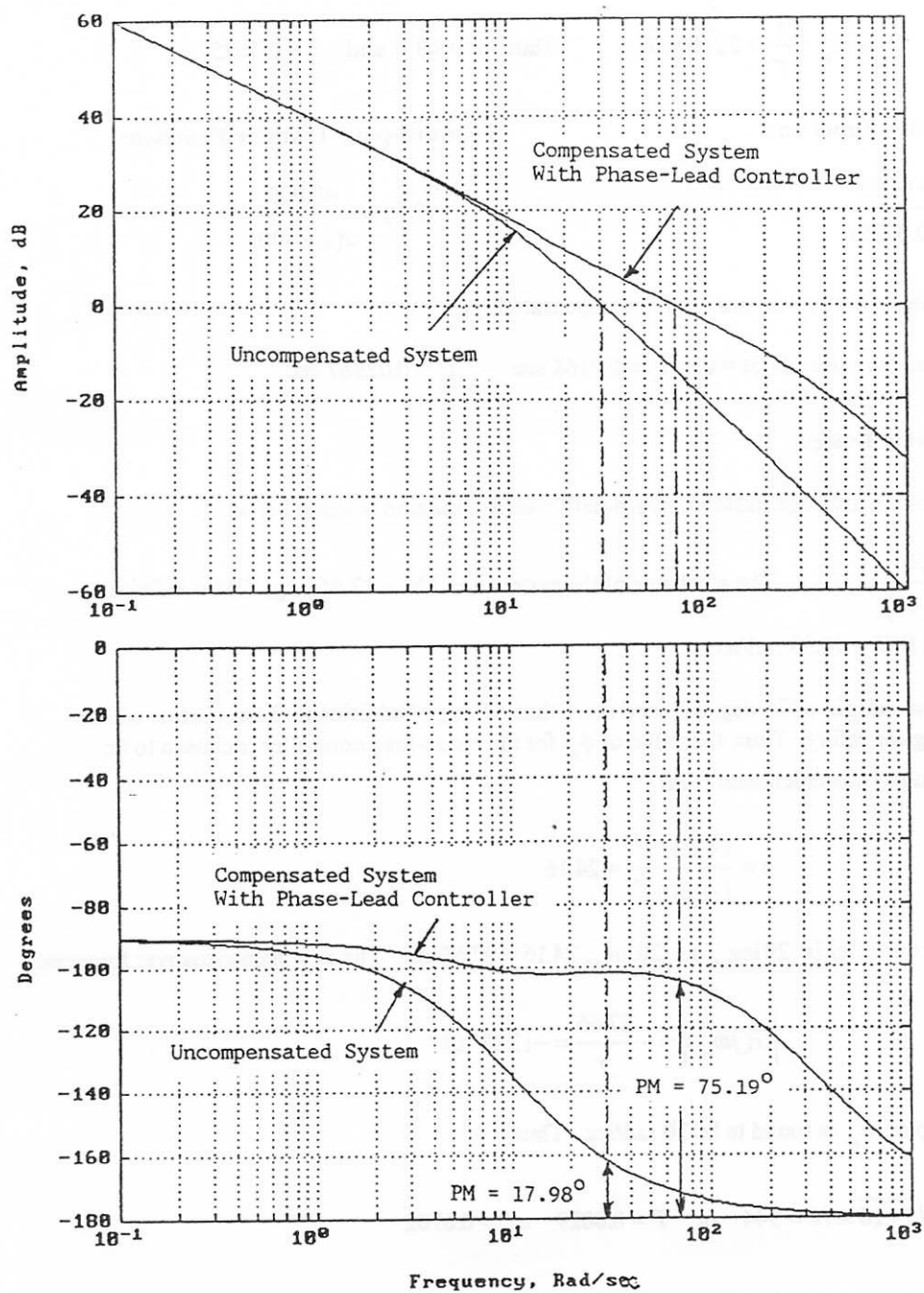
Thus
$$G_c(s) = \frac{1 + aTs}{1 + Ts} = \frac{1 + 0.0702s}{1 + 0.0029s}$$

The compensated system has the following frequency-domain attributes:

$$\text{PM} = 75.19 \text{ deg} \quad \text{GM} = \text{infinite} \quad M_r = 1.024 \quad \text{BW} = 91.85 \text{ rad/sec}$$

The attributes of the unit-step response are:

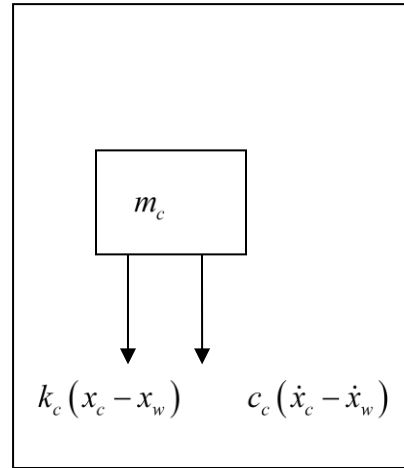
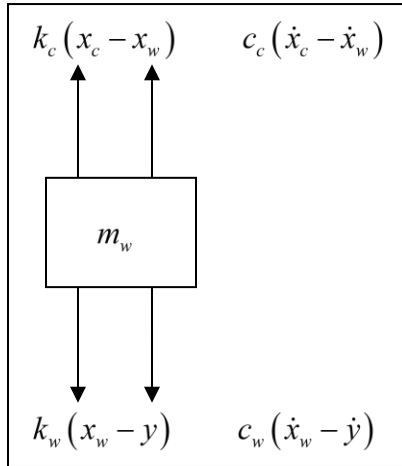
$$\text{Rise time } t_r = 0.02278 \text{ sec} \quad \text{Settling time } t_s = 0.02828 \text{ sec} \quad \text{Maximum overshoot} = 3.3\%$$



11-44) Also see Chapter 6 for solution to this problem.

Mathematical Model:

Draw free body diagrams (Assume both x_c and x_w are positive and are measured from equilibrium). Refer to Chapter 4 problems for derivation details.



$$\begin{bmatrix} m_w & 0 \\ 0 & m_c \end{bmatrix} \begin{bmatrix} \ddot{x}_w \\ \ddot{x}_c \end{bmatrix} + \begin{bmatrix} c_w + c_c & -c_c \\ c_c & c_c \end{bmatrix} \begin{bmatrix} \dot{x}_w \\ \dot{x}_c \end{bmatrix} + \begin{bmatrix} k_w + k_c & -k_c \\ k_c & k_c \end{bmatrix} \begin{bmatrix} x_w \\ x_c \end{bmatrix} = \begin{bmatrix} c_w & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{y} \\ 0 \end{bmatrix} + \begin{bmatrix} k_w & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ 0 \end{bmatrix}$$

To Solve we need to simplify, since the problem is very difficult.

Assume the wheel is very stiff; hence $k_w = \infty$, which implies $x_w = y$. Then

$$m_c \ddot{x}_c + c_c \dot{x}_c + k_c x_c = c_c \dot{y} + k_c y$$

or

$$\ddot{x}_c + 2\zeta\omega_n \dot{x}_c + \omega_n^2 x_c = 2\zeta\omega_n \dot{y} + \omega_n^2 y$$

Placing an actuator between the two masses (ignore actuator dynamics for simplicity), and use a PD control: the control force is (m_c is added to make the final equation look simpler):

$$F = m_c K_D (\dot{x}_c - \dot{x}_w) + m_c K_P (x_c - x_w)$$

where

$$x_w = y, \dot{x}_w = \dot{y}$$

The transfer function of the system is:

$$\frac{X_c}{Y} = \frac{(2\zeta\omega_n + K_D)s + (\omega_n^2 + K_P)}{s^2 + (2\zeta\omega_n + K_D)s + (2\zeta\omega_n + K_D)}$$

The rest is a standard PD controller design.

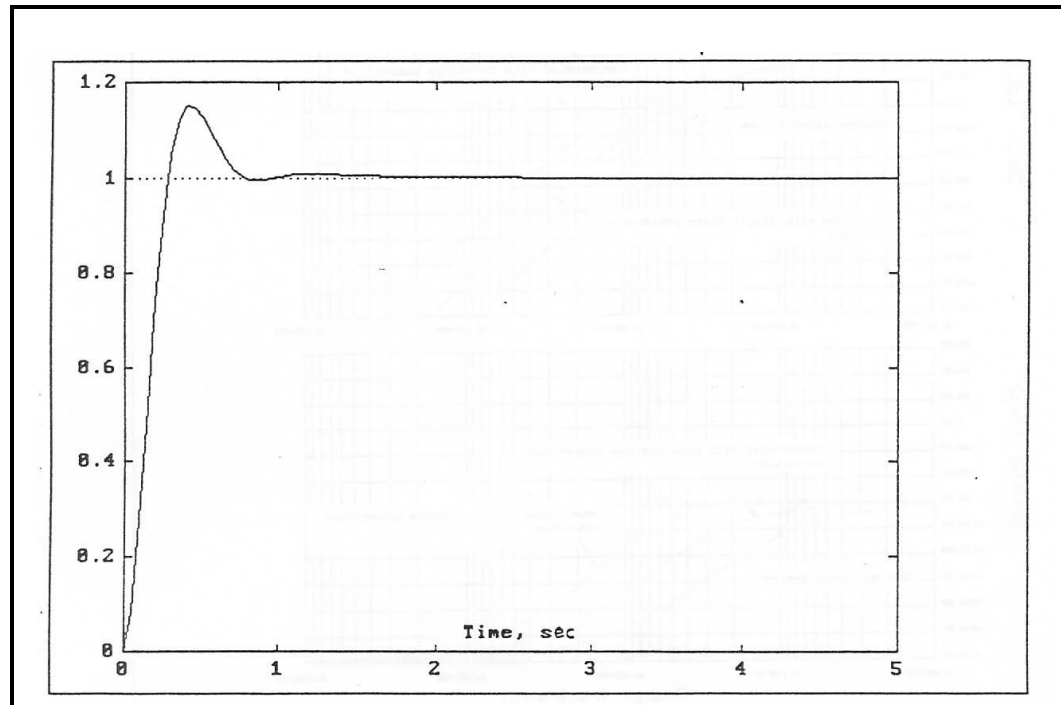
11-45) (a) Forward-path Transfer Function: ($N = 10$)

$$G(s) = G_c(s)G_p(s) = \frac{200(1 + aTs)}{s(s+1)(s+10)(1+Ts)}$$

Starting with $a = 1000$, we vary T first to stabilize the system. The following time-domain attributes are obtained by varying the value of T .

T	Max Overshoot (%)	t_r	t_s
0.0001	59.4	0.370	5.205
0.0002	41.5	0.293	2.911
0.0003	29.9	0.315	1.83
0.0004	22.7	0.282	1.178
0.0005	18.5	0.254	1.013
0.0006	16.3	0.230	0.844
0.0007	15.4	0.210	0.699
0.0008	15.4	0.192	0.620
0.0009	15.5	0.182	0.533
0.0010	16.7	0.163	0.525

The maximum overshoot is at a minimum when $T = 0.0007$ or $T = 0.0008$. The maximum overshoot is 15.4%.

Unit-step Response. ($T = 0.0008$ sec $\alpha = 1000$)**(b) Frequency-domain Design.**

Similar to the design in part (a), we set $\alpha = 1000$, and vary the value of T between 0.0001 and 0.001. The attributes of the frequency-domain characteristics are given below.

T	PM (deg)	GM (dB)	M_r	BW (rad/sec)
0.0001	17.95	60.00	3.194	4.849
0.0002	31.99	63.53	1.854	5.285
0.0003	42.77	58.62	1.448	5.941
0.0004	49.78	54.53	1.272	6.821
0.0005	53.39	51.16	1.183	7.817
0.0006	54.69	48.32	1.138	8.869
0.0007	54.62	45.87	1.121	9.913
0.0008	53.83	43.72	1.125	10.92
0.0009	52.68	41.81	1.140	11.88
0.0010	51.38	40.09	1.162	12.79

The phase margin is at a maximum of 54.69 deg when $T = 0.0006$. The performance worsens if the value of α is less than 1000.

11-46 (a) Bode Plot.

The attributes of the frequency response are:

$$\text{PM} = 4.07 \text{ deg} \quad \text{GM} = 1.34 \text{ dB} \quad M_r = 23.24 \quad \text{BW} = 4.4 \text{ rad/sec}$$

(b) Single-stage Phase-lead Controller.

$$G(s) = \frac{6(1 + aTs)}{s(1 + 0.2s)(1 + 0.5s)(1 + Ts)}$$

We first set $a = 1000$, and vary T . The following attributes of the frequency-domain characteristics are obtained.

T	PM (deg)	M_r
0.0050	17.77	3.21
0.0010	43.70	1.34
0.0007	47.53	1.24
0.0006	48.27	1.22
0.0005	48.06	1.23
0.0004	46.01	1.29
0.0002	32.08	1.81
0.0001	19.57	2.97

The phase margin is maximum at 48.27 deg when $T = 0.0006$.

Next, we set $T = 0.0006$ and reduce a from 1000. We can show that the phase margin is not very sensitive to the variation of a when a is near 1000. The optimal value of a is around 980, and the corresponding phase margin is 48.34 deg.

With $a = 980$ and $T = 0.0006$, the attributes of the unit-step response are:

$$\text{Maximum overshoot} = 18.8\% \quad t_r = 0.262 \text{ sec} \quad t_s = 0.851 \text{ sec}$$

(c) Two-stage Phase-lead Controller. ($a = 980$, $T = 0.0006$)

$$G(s) = \frac{6(1 + 0.588s)(1 + bT_2s)}{s(1 + 0.2s)(1 + 0.5s)(1 + 0.0006s)(1 + T_2s)}$$

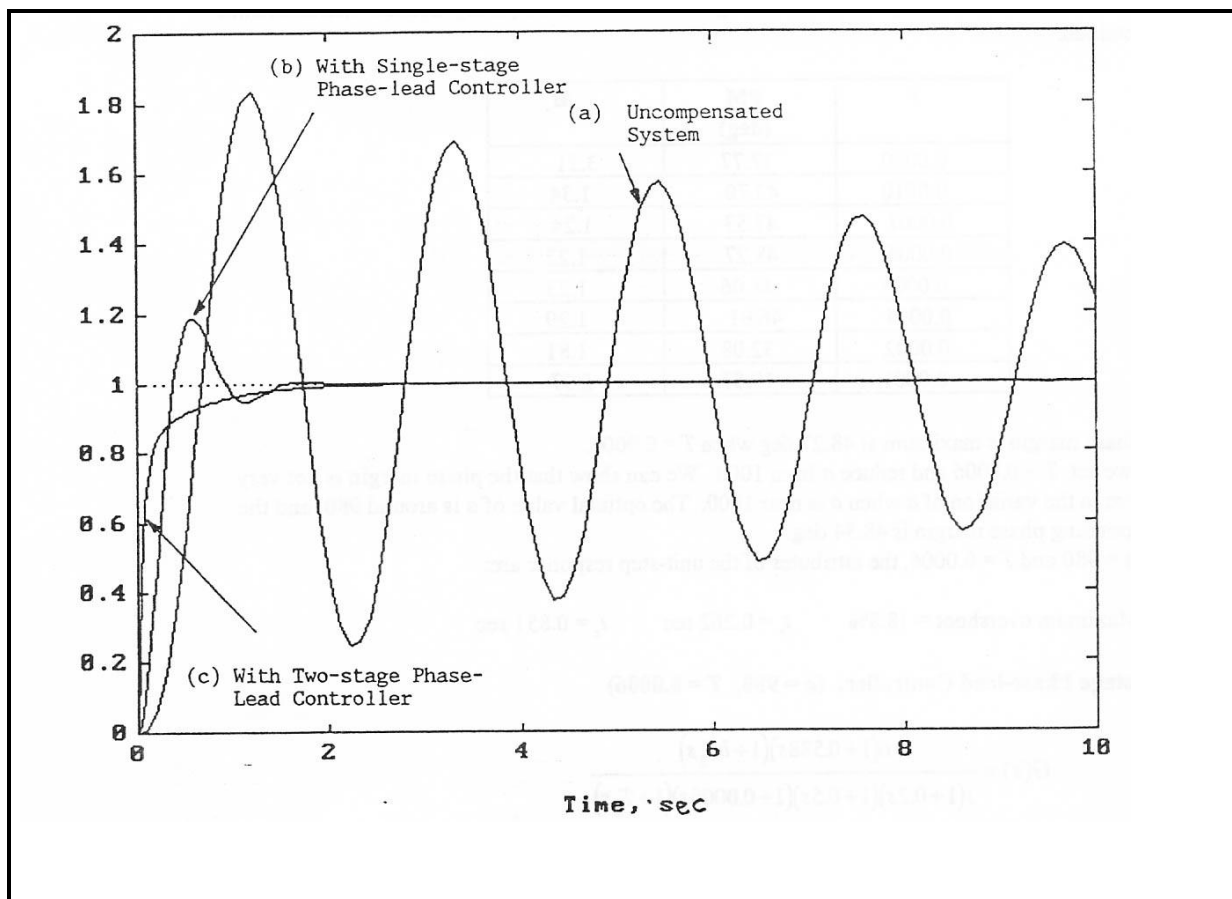
Again, let $b = 1000$, and vary T_2 . The following results are obtained in the frequency domain.

T_2	PM (deg)	M_r
0.0010	93.81	1.00
0.0009	94.89	1.00
0.0008	96.02	1.00
0.0007	97.21	1.00
0.0006	98.43	1.00
0.0005	99.61	1.00
0.0004	100.40	1.00
0.0003	99.34	1.00
0.0002	91.98	1.00
0.0001	73.86	1.00

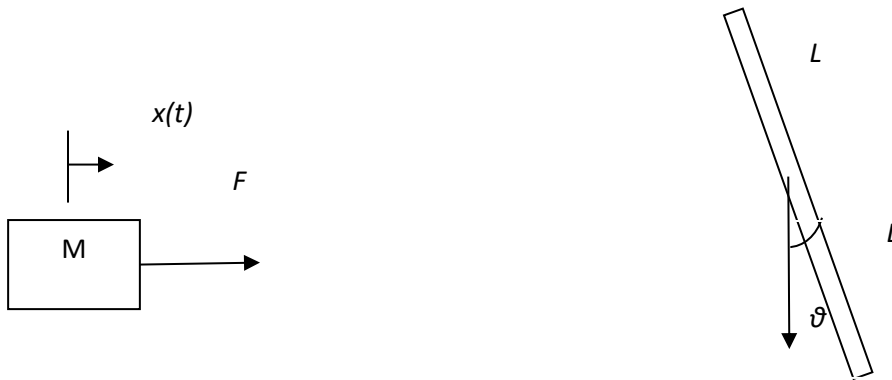
Reducing the value of b from 1000 reduces the phase margin. Thus, the maximum phase margin of 100.4 deg is obtained with $b = 1000$ and $T_2 = 0.0004$. The transfer function of the two-stage phase-lead controller is

$$G_c(s) = \frac{(1 + 0.588s)(1 + 0.4s)}{(1 + 0.0006s)(1 + 0.0004s)}$$

(c) Unit-step Responses.



11-47) Also see derivations in 4-9.



Here is an alternative representation including friction (damping) μ . In this case the angle θ is measured differently.

Let's find the dynamic model of the system:

$$1) (M + m)\ddot{x} + \mu\dot{x} - ml\ddot{\theta} \cos \theta - ml\dot{\theta}^2 \sin \theta = F$$

$$2) (I + ml^2)\ddot{\theta} + mgl \sin \theta = -ml\ddot{x} \cos \theta$$

Let $\theta = \pi + \Phi$. If Φ is small enough then $\cos \Phi \rightarrow 1$ and $\sin \Phi \rightarrow \Phi$, therefore

$$\begin{cases} (M + m)\ddot{x} + \mu\dot{x} - ml\ddot{\Phi} = F \\ (I + ml^2)\ddot{\Phi} - mgl\Phi = ml\ddot{x} \end{cases}$$

which gives:

$$\frac{\Phi(s)}{F(s)} = \frac{mls^2}{[(M + m)(I + ml^2) - (ml)^2]s^3 + \mu(I + ml^2)s^2 - (M + m)mgl - \mu mgl}$$

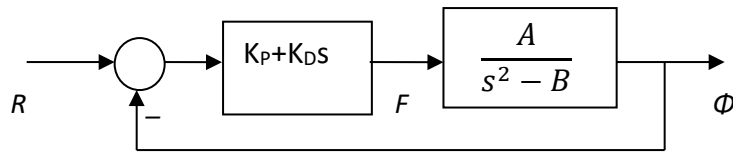
Ignoring friction $\mu = 0$.

$$\frac{\Phi(s)}{F(s)} = \frac{ml}{[(M + m)(I + ml^2) - (ml)^2]s^2 - (M + m)mgl} = \frac{A}{s^2 - B}$$

where

$$A = \frac{ml}{[(M + m)(I + ml^2) - (ml)^2]}; B = \frac{(M + m)mgl}{[(M + m)(I + ml^2) - (ml)^2]}$$

Ignoring actuator dynamics (DC motor equations), we can incorporate feedback control using a series PD compensator and unity feedback. Hence,



$$F(s) = K_p (R(s) - \Phi) - K_D s (R(s) - \Phi)$$

The system transfer function is:

$$\frac{\Phi}{R} = \frac{A(K_p + K_D s)}{(s^2 + K_D s + A(K_p - B))}$$

Control is achieved by ensuring stability ($K_p > B$)

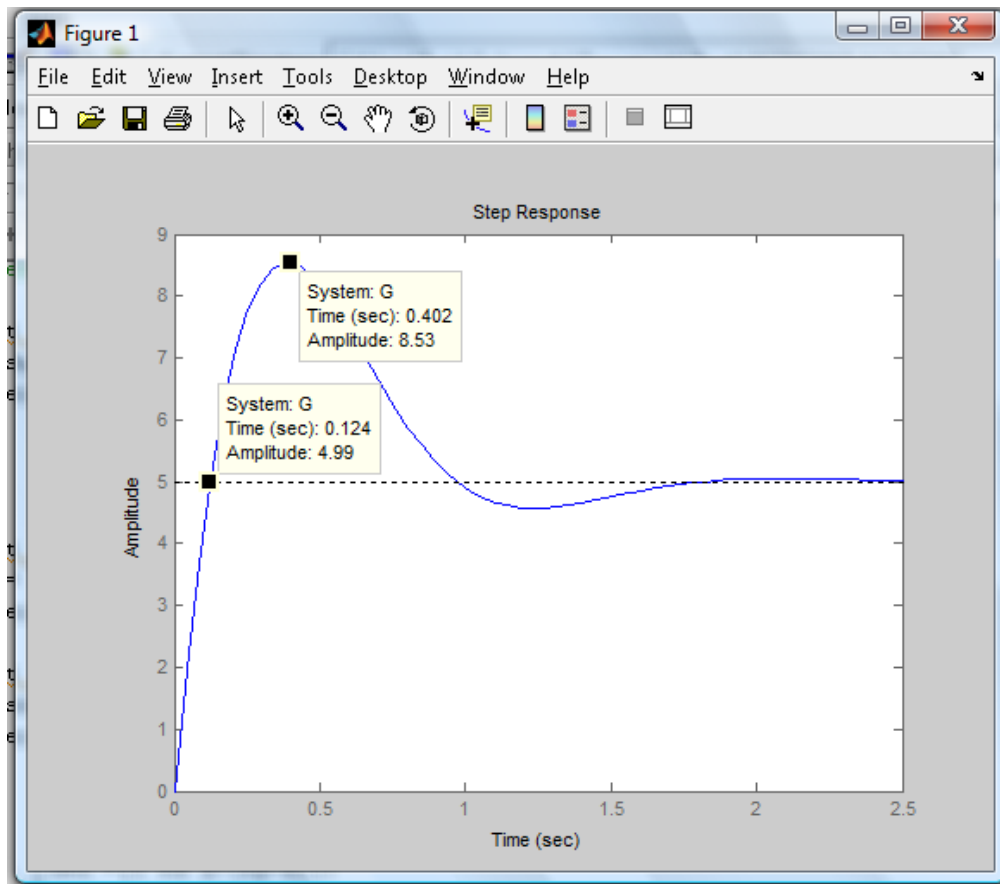
Use Routh Hurwitz to establish stability first. Use Acsys to do that as demonstrated in this chapter problems. Also Chapter 2 has many examples.

Use MATLAB to simulate response:

```
clear all
Kp=10;
Kd=5;
A=10;
B=8;
num = [A*Kd A*Kp];
den = [1 Kd A*(Kp-B)];
G=tf(num,den)
step(G)
```

Transfer function:

```
50 s + 100
-----
s^2 + 5 s + 20
```



Adjust parameters to achieve desired response. Use THE PROCEDURE in Example 5-11-1.

You may look at the root locus of the forward path transfer function to get a better perspective.

$$\frac{\Phi}{E} = \frac{A(K_p + K_D s)}{s^2 - AB} = \frac{AK_D(z + s)}{s^2 - AB}$$

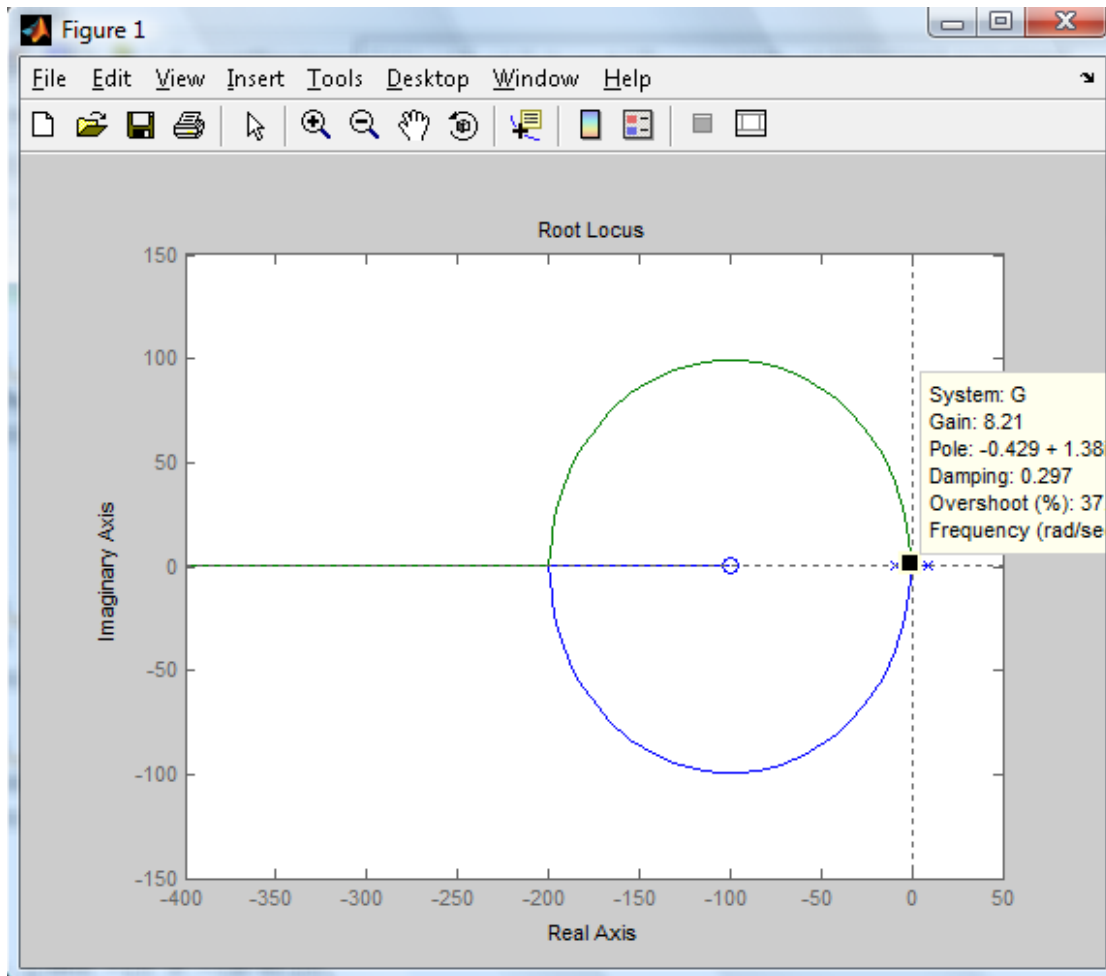
fix z and vary K_D .

```
clear all
z=100;
Kd=0.01;
A=10;
B=8;
num = [A*Kd A*Kd*z];
den = [1 0 -(A*B)];
G=tf(num,den)
rlocus(G)
```

Transfer function:

0.1 s + 10

s^2 - 80



For $z=10$, a large $K_D=0.805$ results in:

```
clear all
Kd=0.805;
Kp=10*Kd;
A=10;
B=8;
num = [A*Kd A*Kp];
den = [1 Kd A*(Kp-B)];
G=tf(num,den)
pole(G)
zero(G)
step(G)
```

Transfer function:

$$\frac{8.05 s + 80.5}{s^2 + 0.805 s + 0.5}$$

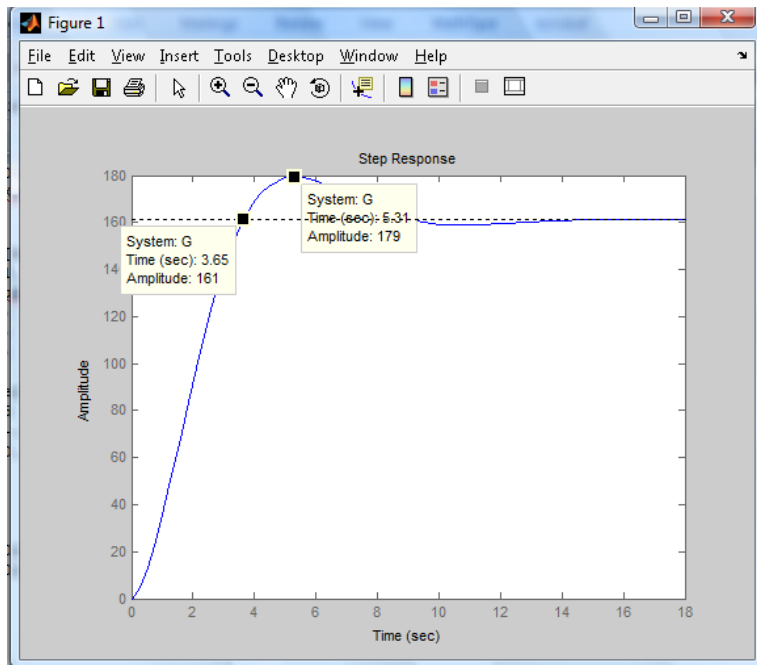
ans =

$$\begin{aligned} &-0.4025 + 0.5814i \\ &-0.4025 - 0.5814i \end{aligned}$$

ans =

-10

Looking at dominant poles we expect to see an oscillatory response with overshoot close to desired values.



For a better design, and to meet rise time criterion, use Example 5-11-1 and Chapter 9 PD design examples.

11-48) (a) The loop transfer function of the system is

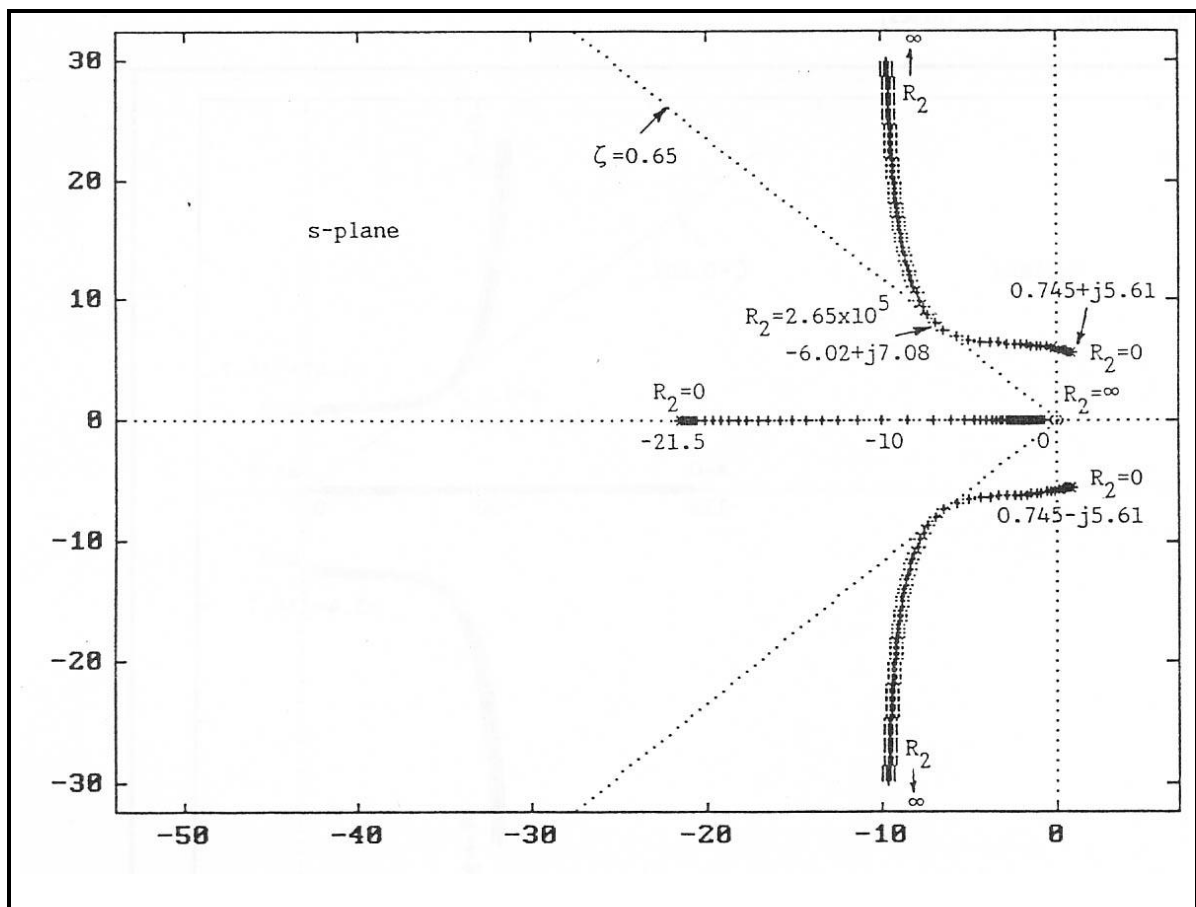
$$G(s)H(s) = \frac{10K_p K_a K_e}{Ns(1+0.05s)} \frac{1+R_2Cs}{R_1Cs} = \frac{68.76}{s(1+0.05s)} \frac{1+R_2 \times 10^{-6}s}{2s}$$

The characteristic equation is $s^3 + 20s^2 + 6.876 \times 10^{-4}R_2s + 687.6 = 0$

For root locus plot with R_2 as the variable parameter, we have

$$G_{eq}(s) = \frac{6.876 \times 10^{-4}R_2s}{s^3 + 20s^2 + 687.6} = \frac{6.876 \times 10^{-4}R_2s}{(s+21.5)(s-0.745+j5.61)(s-0.745-j5.61)}$$

Root Locus Plot.



When $R_2 = 2.65 \times 10^5$, the roots are at $-6.02 \pm j7.08$, and the relative damping ratio is 0.65 which is maximum. The unit-step response is plotted at the end together with those of parts (b) and (c).

(b) Phase-lead Controller.

$$G(s)H(s) = \frac{68.76(1 + aTs)}{s(1 + 0.05s)(1 + Ts)}$$

Characteristic Equation: $Ts^3 + (1 + 20T)s^2 + (20 + 1375.2aT)s + 1375.2 = 0$

With $T = 0.01$, the characteristic equation becomes

$$s^3 + 120s^2 + (2000 + 1375.2a)s + 137520 = 0$$

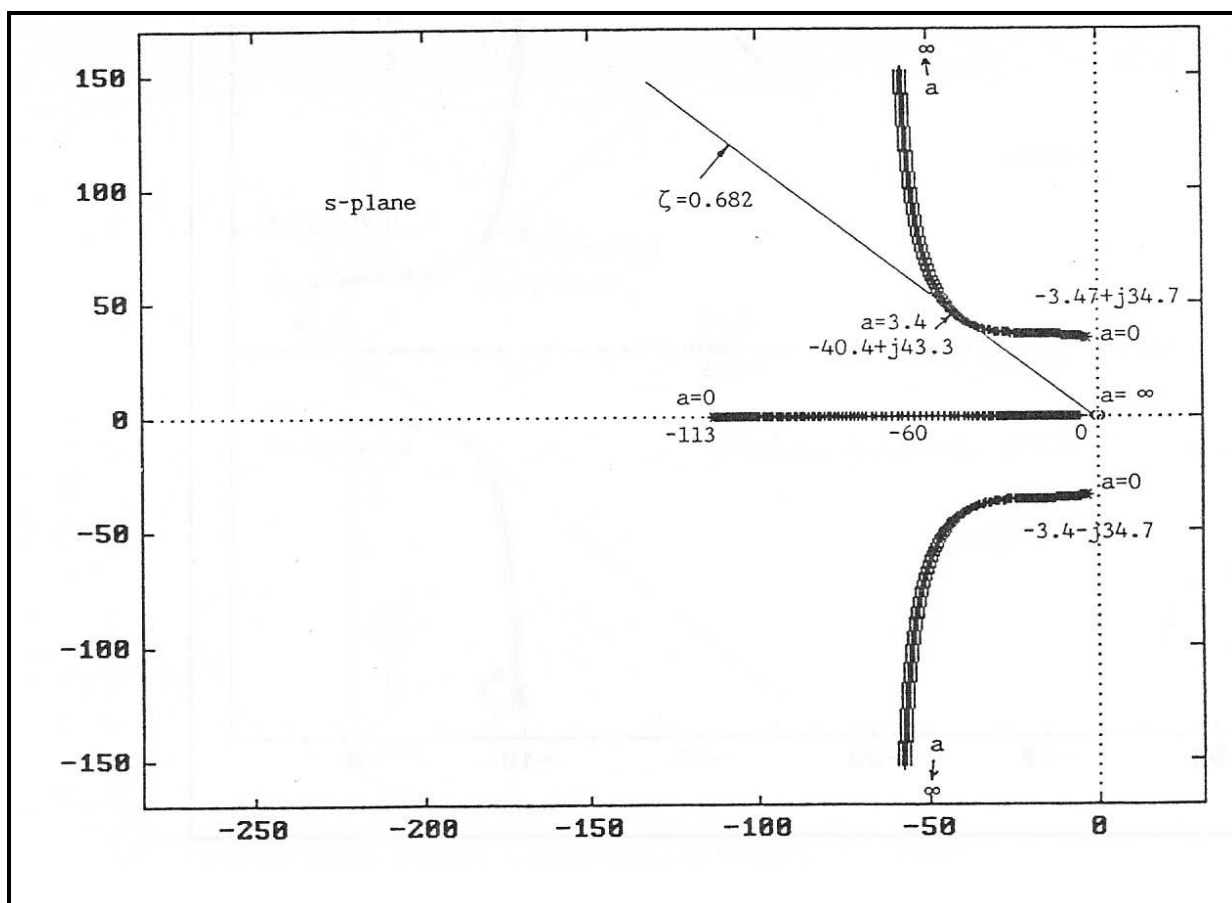
The last equation is conditioned for a root contour plot with a as the variable parameter.

Thus

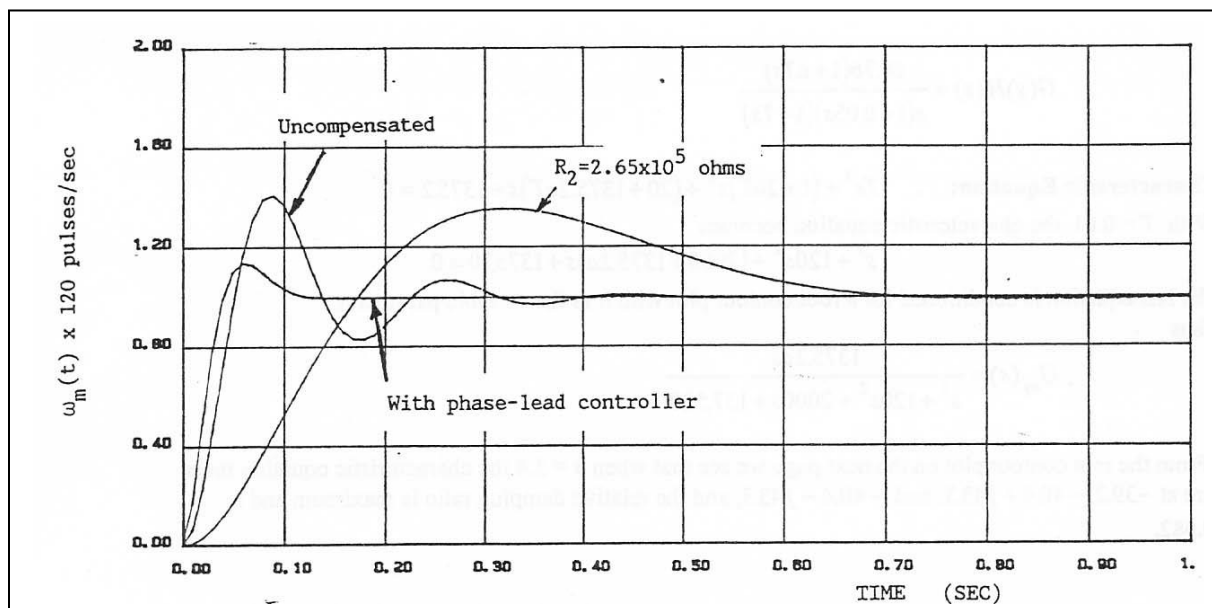
$$G_{eq}(s) = \frac{1375.2as}{s^3 + 120s^2 + 2000s + 137,520}$$

From the root contour plot on the next page we see that when $a = 3.4$ the characteristic equation roots are at -39.2 , $-40.4 + j43.3$, and $-40.4 - j43.3$, and the relative damping ratio is maximum and is 0.682.

Root Contour Plot (a varies).



Unit-step Responses.



(c) Frequency-domain Design of Phase-lead Controller.

For a phase margin of 60 deg, $a = 4.373$ and $T = 0.00923$. The transfer function of the controller is

$$G_c(s) = \frac{1 + aTs}{1 + Ts} = \frac{1 + 0.04036s}{1 + 0.00923s}$$

11-49 (a) Time-domain Design of Phase-lag Controller.

Process Transfer Function:

$$G_p(s) = \frac{200}{s(s+1)(s+10)}$$

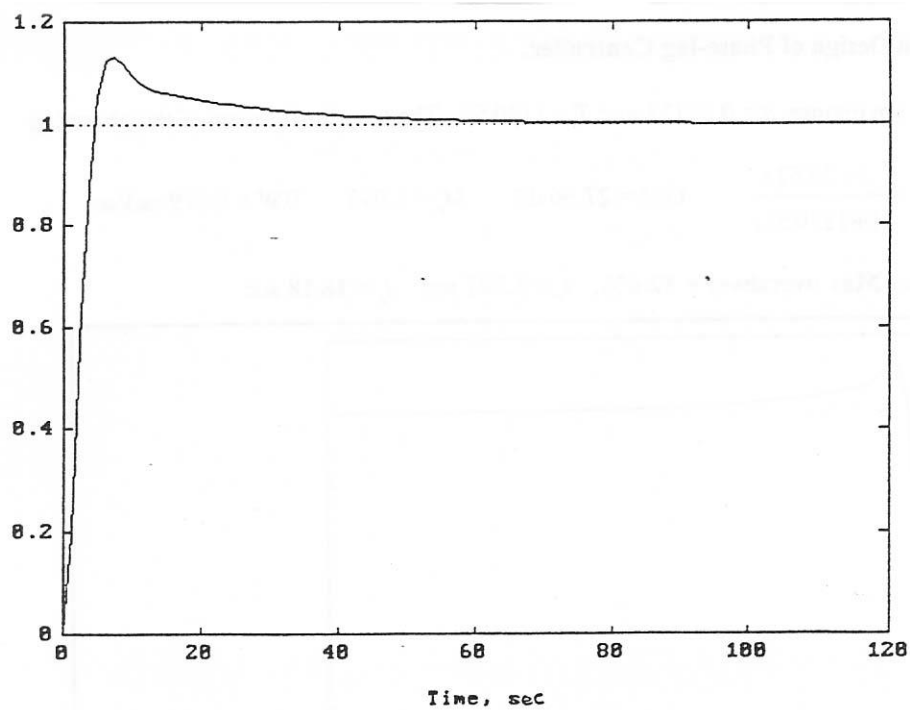
For the uncompensated system, the two complex characteristic equation roots are at $s = -0.475 + j0.471$ and $-0.475 - j0.471$ which correspond to a relative damping ratio of 0.707, when the forward path gain is 4.5 (as against 200). Thus, the value of a of the phase-lag controller is chosen to be

$$a = \frac{4.5}{200} = 0.0225 \quad \text{Select } T = 1000 \quad \text{which is a large number.}$$

Then

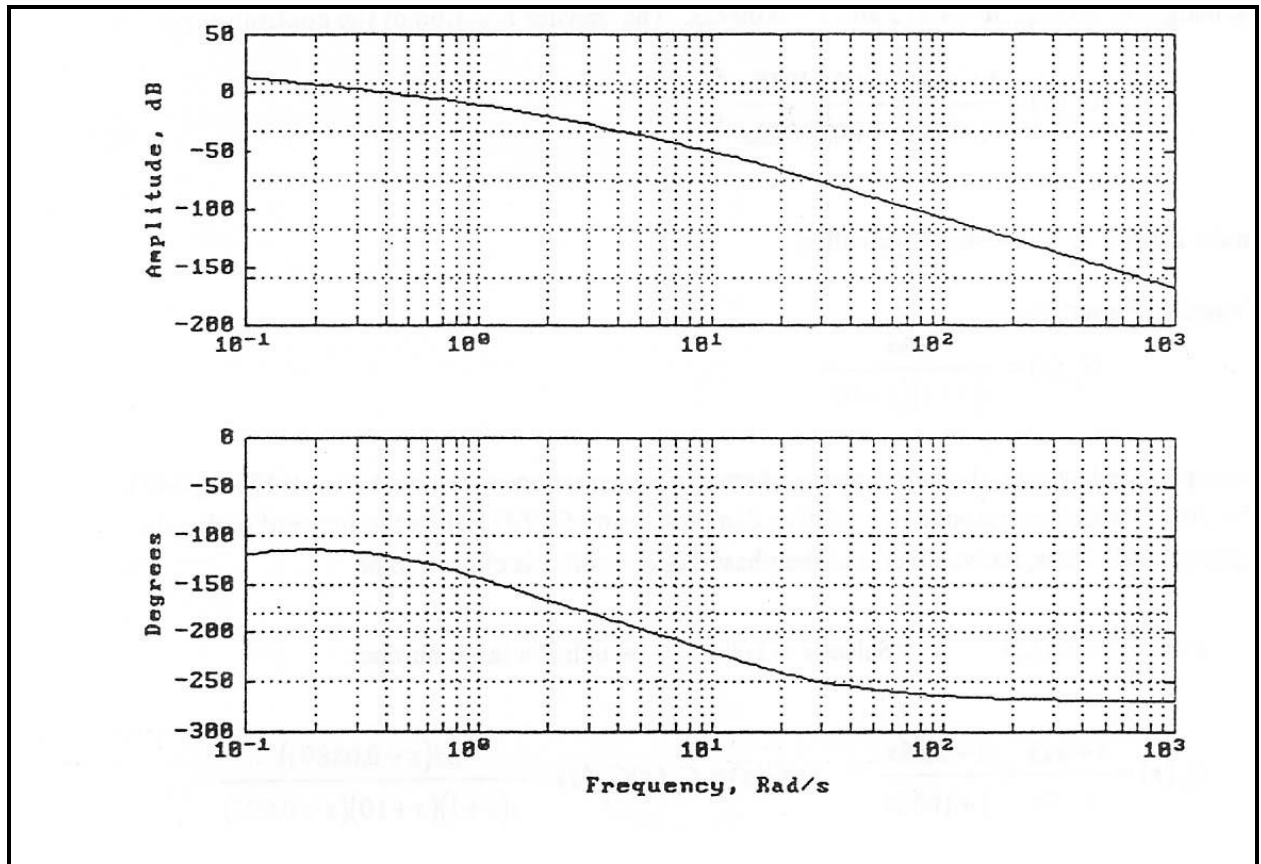
$$G_c(s) = \frac{1 + aTs}{1 + Ts} = \frac{1 + 22.5s}{1 + 1000s} \quad G(s) = G_c(s)G_p(s) = \frac{4.5(s + 0.0889)}{s(s+1)(s+10)(s+0.001)}$$

Unit-step Response.



Maximum overshoot = 13.6 $t_r = 3.238$ sec $t_s = 18.86$ sec

Bode Plot (with phase-lag controller, $\alpha = 0.0225$, $T = 1000$)



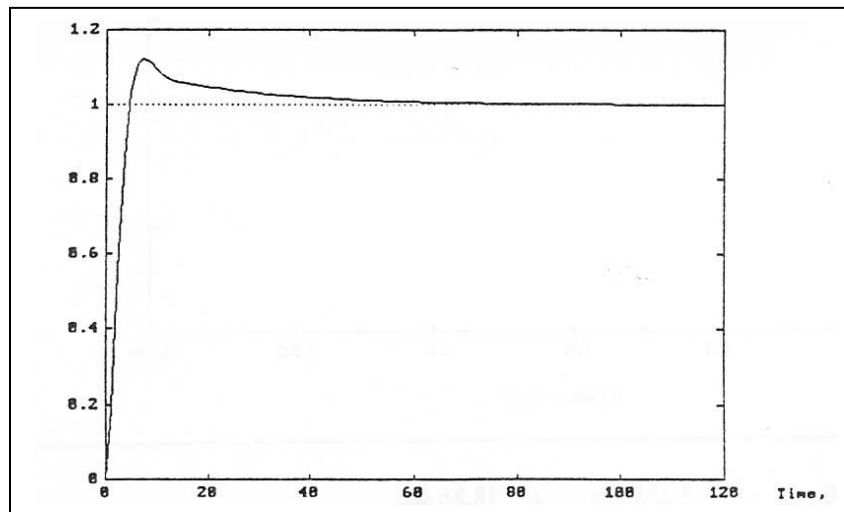
PM = 59 deg. GM = 27.34 dB $M_r = 1.1$ BW = 0.6414 rad/sec

(b) Frequency-domain Design of Phase-lag Controller.

For PM = 60 deg, we choose $\alpha = 0.02178$ and $T = 1130.55$. The transfer function of the phase-lag controller is

$$G_c(s) = \frac{1 + 24.62s}{1 + 1130.55s} \quad \text{GM} = 27.66 \text{ dB} \quad M_r = 1.093 \quad \text{BW} = 0.619 \text{ rad/sec}$$

Unit-step Response. Max overshoot = 12.6%, $t_r = 3.297$ sec $t_s = 18.18$ sec



11-50 (a) Time-domain Design of Phase-lead Controller

Forward-path Transfer Function.

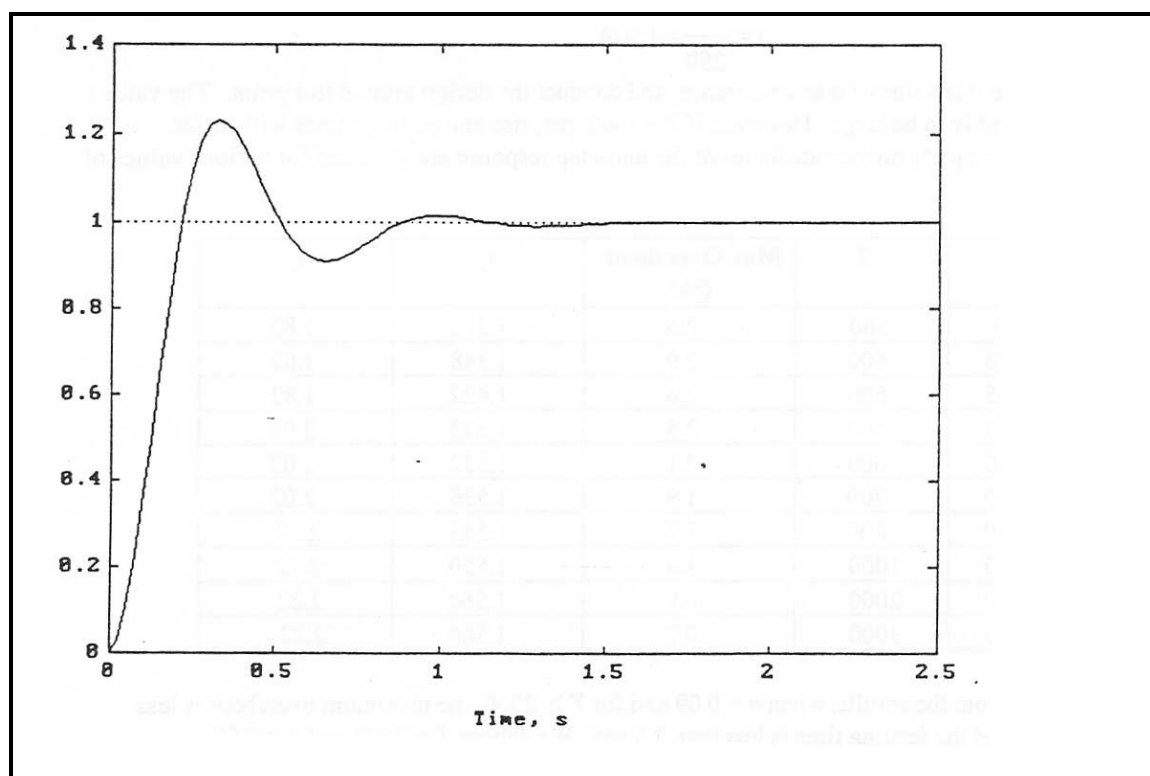
$$G(s) = G_c(s)G_p(s) = \frac{K(1+aTs)}{s(s+5)^2(1+Ts)} \quad K_v = \lim_{s \rightarrow 0} sG(s) = \frac{K}{25} = 10 \quad \text{Thus } K = 250$$

With $K = 250$, the system without compensation is marginally stable. For $a > 1$, select a small value for T and a large value for a . Let $a = 1000$. The following results are obtained for various values of T ranging from 0.0001 to 0.001. When $T = 0.0004$, the maximum overshoot is near minimum at 23%.

T	Max Overshoot (%)	t_r (sec)	t_s (sec)
0.0010	33.5	0.0905	0.808
0.0005	23.8	0.1295	0.6869
0.0004	23.0	0.1471	0.7711

0.0003	24.4	0.1689	0.8765
0.0002	30.6	0.1981	1.096
0.0001	47.8	0.2326	2.399

As it turns out $a = 1000$ is near optimal. A higher or lower value for a will give larger overshoot.



Unit-step Response.

(b) Frequency-domain Design of Phase-lead Controller

$$G(s) = \frac{250(1 + aTs)}{s^2(s + 5)^2(1 + Ts)}$$

Setting $a = 1000$, and varying T , the following attributes are obtained.

T	PM (deg)	M_r	BW (rad/sec)
0.00050	41.15	1.418	16.05
0.00040	42.85	1.369	14.15
0.00035	43.30	1.355	13.16
0.00030	43.10	1.361	12.12
0.00020	38.60	1.513	10.04

When $a = 1000$, the best value of T for a maximum phase margin is 0.00035, and PM = 43.3 deg.

As it turns out varying the value of a from 1000 does not improve the phase margin. Thus the transfer function of the controller is

$$G_c(s) = \frac{1 + aTs}{1 + Ts} = \frac{1 + 0.35s}{1 + 0.00035s} \quad \text{and} \quad G(s) = \frac{250(1 + 0.35s)}{s(s + 5)^2(1 + 0.00035s)}$$

(c) Time-domain Design of Phase-lag Controller

Without compensation, the relative damping is critical when $K = 18.5$. Then, the value of a is chosen to be

$$a = \frac{18.5}{250} = 0.074$$

We can use this value of a as a reference, and conduct the design around this point. The value of T is preferably to be large. However, if T is too large, rise and settling times will suffer.

The following performance attributes of the unit-step response are obtained for various values of a and T .

α	T	Max Overshoot (%)	t_r	t_s
0.105	500	2.6	1.272	1.82
0.100	500	2.9	1.348	1.82
0.095	500	2.6	1.422	1.82
0.090	500	2.5	1.522	2.02
0.090	600	2.1	1.532	2.02
0.090	700	1.9	1.538	2.02
0.090	800	1.7	1.543	2.02
0.090	1000	1.4	1.550	2.22
0.090	2000	0.9	1.560	2.22
0.090	3000	0.7	1.566	2.22

As seen from the results, when $\alpha = 0.09$ and for $T \geq 2000$, the maximum overshoot is less than 1% and the settling time is less than 2.5 sec. We choose $T = 2000$ and $\alpha = 0.09$.

The corresponding frequency-domain characteristics are:

$$PM = 69.84 \text{ deg}$$

$$GM = 20.9 \text{ dB}$$

$$M_r = 1.004$$

$$BW = 1.363 \text{ rad/sec}$$

(d) Frequency-domain Design of Phase-lag Controller

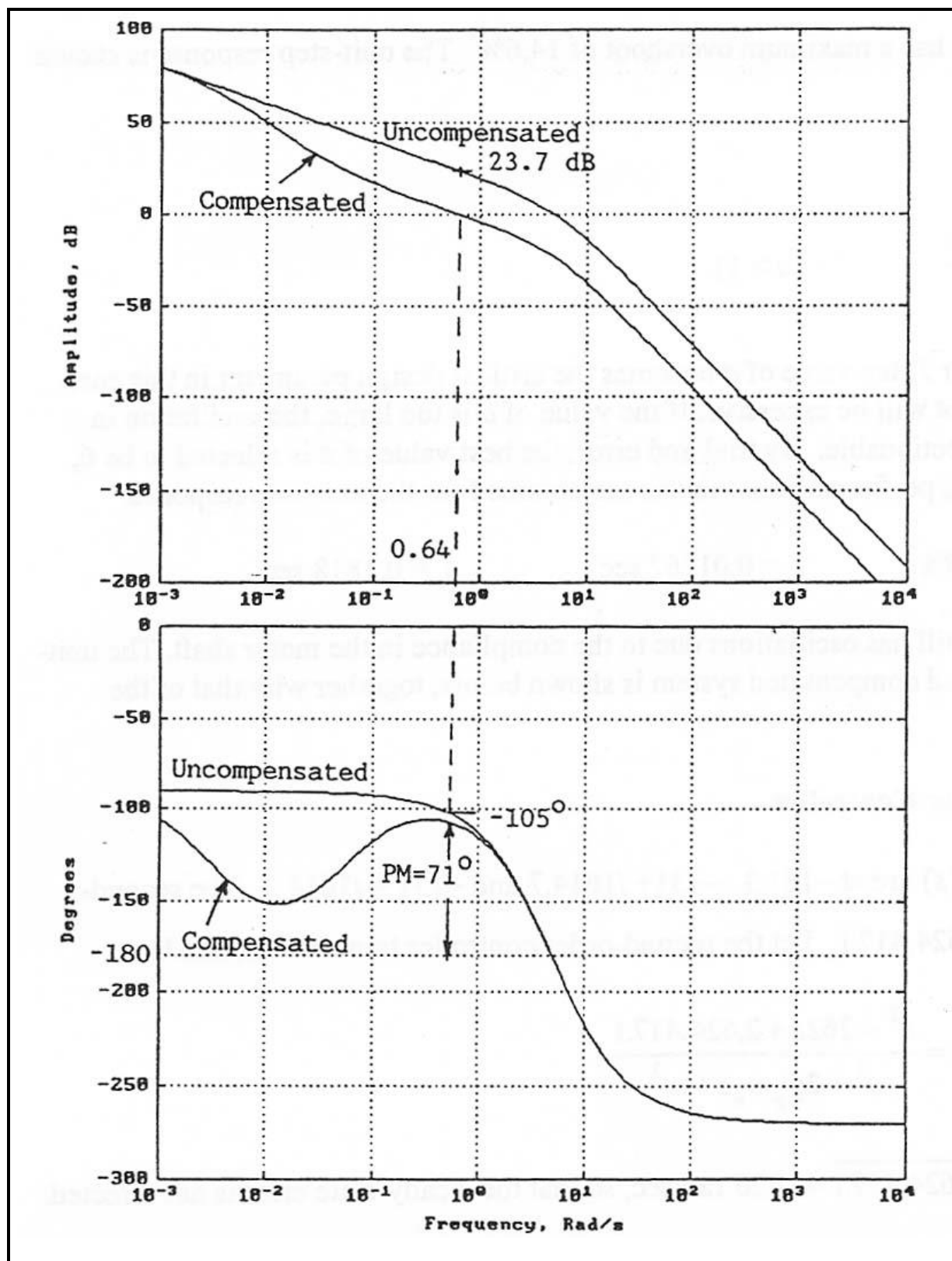
$$G(s) = \frac{250(1 + aTs)}{s(s+5)^2(1+Ts)} \quad a < 1$$

The Bode plot of the uncompensated system is shown below. Let us add a safety factor by requiring that the desired phase margin is 75 degrees. We see that a phase margin of 75 degrees can be realized if the gain crossover is moved to 0.64 rad/sec. The magnitude of $G(j\omega)$ at this frequency is 23.7 dB. Thus the phase-lag controller must provide an attenuation of -23.7 dB at the new gain crossover frequency. Setting

$$20\log_{10} a = -23.7 \text{ dB} \quad \text{we have} \quad a = 0.065$$

We can set the value of $1/aT$ to be at least one decade below 0.64 rad/sec, or 0.064 rad/sec. Thus, we get $T = 236$. Let us choose $T = 300$. The transfer function of the phase-lag controller becomes

$$G_c(s) = \frac{1 + aTs}{1 + Ts} = \frac{1 + 19.5s}{1 + 300s}$$



The attributes of the frequency response of the compensated system are:

$$\text{PM} = 71 \text{ deg} \quad \text{GM} = 23.6 \text{ dB} \quad M_r = 1.065 \quad \text{BW} = 0.937 \text{ rad/sec}$$

The attributes of the unit-step response are:

$$\text{Maximum overshoot} = 6\% \quad t_r = 2.437 \text{ sec} \quad t_s = 11.11 \text{ sec}$$

Comparing with the phase-lag controller designed in part (a) which has $\alpha = 0.09$ and $T = 2000$, the time response attributes are:

$$\text{Maximum overshoot} = 0.9\% \quad t_r = 1.56 \text{ sec} \quad t_s = 2.22 \text{ sec}$$

The main difference is in the large value of T used in part (c) which resulted in less overshoot, rise and settling times.

11-51)**11-52) Forward-path Transfer Function (No compensation)**

$$G(s) = G_p(s) = \frac{6.087 \times 10^7}{s(s^3 + 423.42s^2 + 2.6667 \times 10^6 s + 4.2342 \times 10^8)}$$

The uncompensated system has a maximum overshoot of 14.6%. The unit-step response is shown below.

(a) Phase-lead Controller

$$G_c(s) = \frac{1 + aTs}{1 + Ts} \quad (a > 1)$$

By selecting a small value for T , the value of a becomes the critical design parameter in this case.

If a is too small, the overshoot will be excessive. If the value of a is too large, the oscillation in the step response will be objectionable. By trial and error, the best value of a is selected to be 6, and $T = 0.001$. The following performance attributes are obtained for the unit-step response.

$$\text{Maximum overshoot} = 0\% \quad t_r = 0.01262 \text{ sec} \quad t_s = 0.1818 \text{ sec}$$

However, the step response still has oscillations due to the compliance in the motor shaft. The unit-step response of the phase-lead compensated system is shown below, together with that of the uncompensated system.

(b) Phase-lead and Second-order Controller

The poles of the process $G_p(s)$ are at -161.3 , $-131 + j1614.7$ and $-131 - j1614.7$. The second-order term is $s^2 + 262s + 2,624,417.1$. Let the second-order controller transfer function be

$$G_{c1}(s) = \frac{s^2 + 262s + 2,624,417.1}{s^2 + 2\zeta_p \omega_n s + \omega_n^2}$$

The value of ω_n is set to $\sqrt{2,624,417.1} = 1620$ rad/sec, so that the steady-state error is not affected.

Let the two poles of $G_{c1}(s)$ be at $s = -1620$ and -1620 . Then, $\zeta_p = 405$.

$$G_{c1}(s) = \frac{s^2 + 262s + 2,624,417.1}{s^2 + 3240s + 2,624,417.1}$$

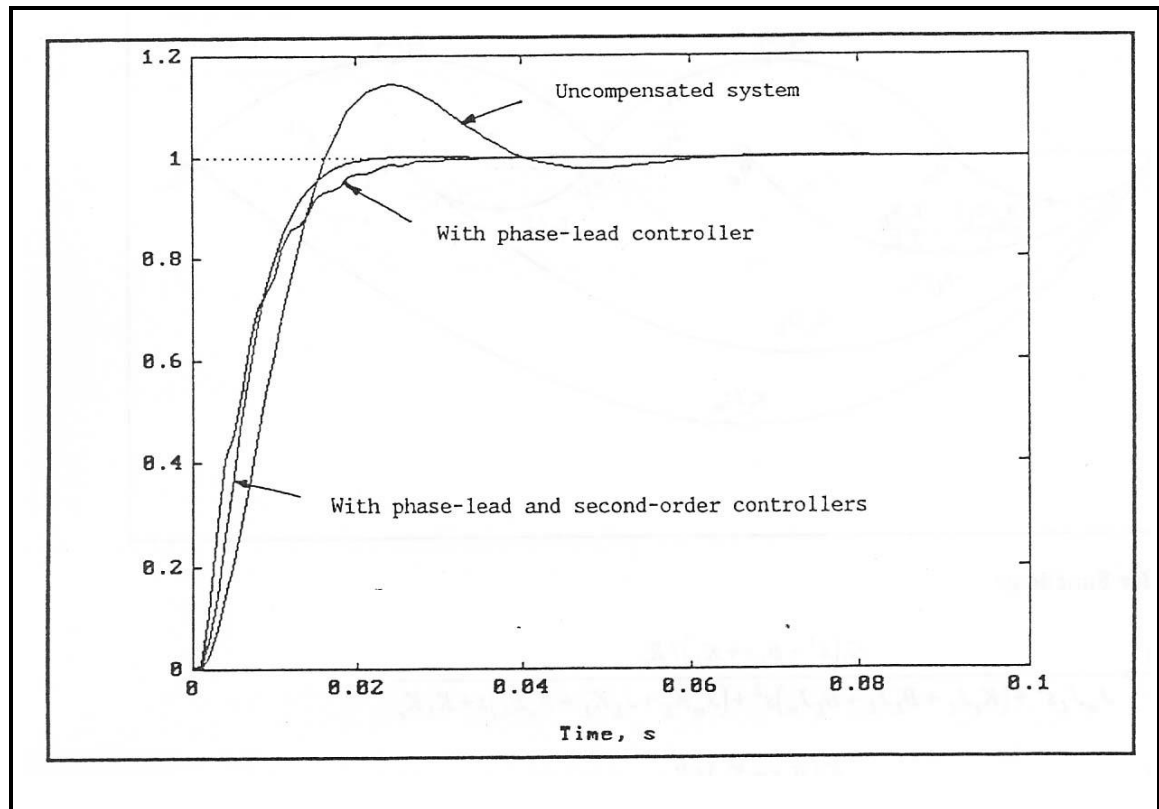
$$G(s) = G_c(s)G_{c1}(s)G_p(s) = \frac{6.087 \times 10^{10} (1 + 0.006s)}{s(s + 161.3)(s^2 + 3240s + 2,624,417.1)(1 + 0.001s)}$$

The unit-step response is shown below, and the attributes are:

$$\text{Maximum overshoot} = 0.2 \quad t_r = 0.01012 \text{ sec} \quad t_s = 0.01414 \text{ sec}$$

The step response does not have any ripples.

Unit-step Responses



11-53 (a) System Equations.

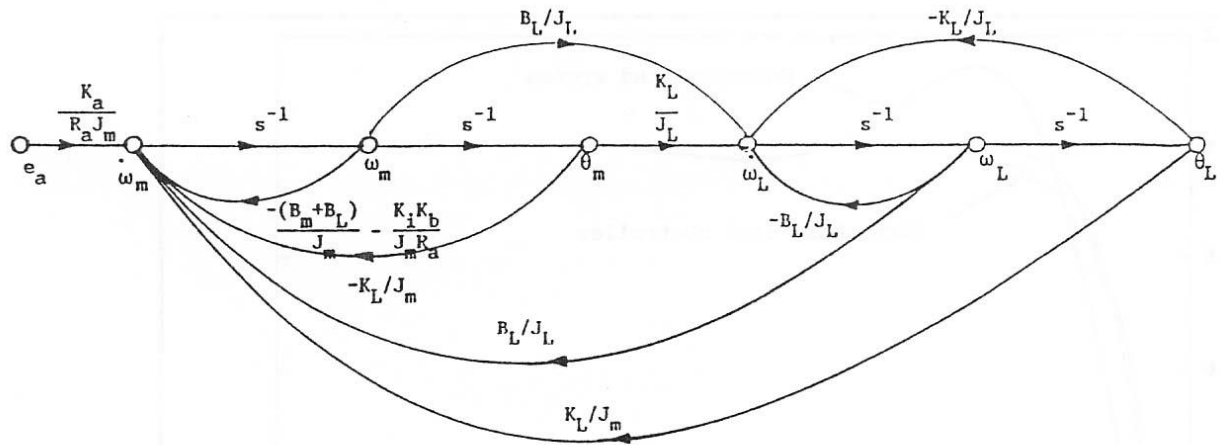
$$e_a = R_a i_a + K_b \omega_m \quad T_m = K_t i_a \quad T_m = J_m \frac{d\omega_m}{dt} + B_m \omega_m + K_L (\theta_m - \theta_L) + B_L (\omega_m - \omega_L)$$

$$K_L (\theta_m - \theta_L) + B_L (\omega_m - \omega_L) = J_L \frac{d\omega_L}{dt}$$

State Equations in Vector-matrix Form:

$$\begin{bmatrix} \frac{d\theta_L}{dt} \\ \frac{d\omega_L}{dt} \\ \frac{d\theta_m}{dt} \\ \frac{d\omega_m}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{K_L}{J_L} & -\frac{B_L}{J_L} & \frac{K_L}{J_L} & \frac{B_L}{J_L} \\ 0 & 0 & 0 & 1 \\ \frac{K_L}{J_m} & \frac{B_L}{J_m} & -\frac{K_L}{J_m} & -\frac{B_m + B_L}{J_m} - \frac{K_i K_b}{J_m R_a} \end{bmatrix} \begin{bmatrix} \theta_L \\ \omega_L \\ \theta_m \\ \omega_m \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{K_a}{R_a J_m} \end{bmatrix} e_a$$

State Diagram:



Transfer Functions:

$$\frac{\Omega_m(s)}{E_a(s)} = \frac{K_i (s^2 + B_L s + K_L) / R_a}{J_m J_L s^2 + (K_e J_L + B_L J_L + B_L J_m) s^2 + (J_m K_L + J_L K_L + K_e B_L) s + K_L K_e}$$

$$\frac{\Omega_L(s)}{E_a(s)} = \frac{K_i (B_L s + K_L) / R_a}{J_m J_L s^3 + (K_e J_L + B_L J_L + B_L J_m) s^2 + (J_m K_L + J_L K_L + K_e B_L) s + K_L K_e}$$

$$\frac{\Omega_m(s)}{E_a(s)} = \frac{133.33(s^2 + 10s + 3000)}{s^3 + 318.15s^2 + 60694.13s + 58240} = \frac{133.33(s^2 + 10s + 3000)}{(s + 0.9644)(s + 158.59 + j187.71)(s + 158.59 - j187.71)}$$

$$\frac{\Omega_L(s)}{E_a(s)} = \frac{1333.33(s + 300)}{(s + 0.9644)(s + 158.59 + j187.71)(s + 158.59 - j187.71)}$$

(b) Design of PI Controller.

$$G(s) = \frac{\Omega_L(s)}{E(s)} = \frac{1333.33K_p \left(s + \frac{K_I}{K_p} \right) (s + 300)}{s(s + 0.9644)(s^2 + 317.186s + 60388.23)}$$

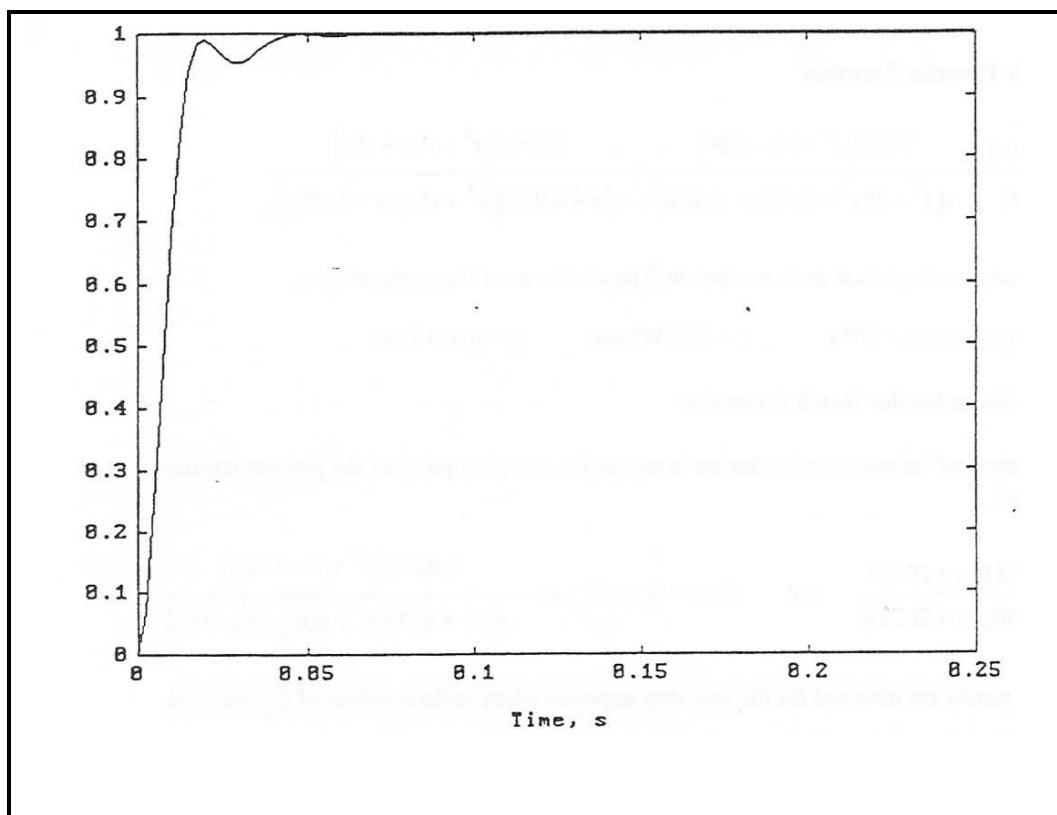
$$K_v = \lim_{s \rightarrow 0} sG(s) = \frac{1333.33 \times 300K_I}{0.9644 \times 60388.23} = 6.87K_I = 100 \quad \text{Thus } K_I = 14.56$$

With $K_I = 14.56$, we study the effects of varying K_p . The following results are obtained.

K_p	t_r (sec)	t_s (sec)	Max Overshoot (%)
20	0.00932	0.02778	4.2
18	0.01041	0.01263	0.7
17	0.01113	0.01515	0
16	0.01184	0.01515	0
15	0.01303	0.01768	0
10	0.02756	0.04040	0.6

With $K_I = 14.56$ and K_p ranging from 15 to 17, the design specifications are satisfied.

Unit-step Response:



(c) Frequency-domain Design of PI Controller ($K_I = 14.56$)

$$G(s) = \frac{1333.33(K_p s + 14.56)(s + 300)}{s(s^3 + 318.15s^2 + 60694.13s + 58240)}$$

The following results are obtained by setting $K_I = 14.56$ and varying the value of K_P .

K_p	PM (deg)	GM (dB)	M_r	BW (rad/sec)	Max Overshoot (%)	t_r (sec)	t_s (sec)
20	65.93	∞	1.000	266.1	4.2	0.00932	0.02778
18	69.76	∞	1.000	243	0.7	0.01041	0.01263
17	71.54	∞	1.000	229	0	0.01113	0.01515
16	73.26	∞	1.000	211.6	0	0.01184	0.01515
15	74.89	∞	1.000	190.3	0	0.01313	0.01768
10	81.11	∞	1.005	84.92	0.6	0.0294	0.0404
8	82.66	∞	1.012	63.33	1.3	0.04848	0.03492
7	83.14	∞	1.017	54.19	1.9	0.03952	0.05253
6	83.29	∞	1.025	45.81	2.7	0.04697	0.0606
5	82.88	∞	1.038	38.12	4.1	0.05457	0.0606

From these results we see that the phase margin is at a maximum of 83.29 degrees when $K_P = 6$.

However, the maximum overshoot of the unit-step response is 2.7%, and M_r is slightly greater than one. In part (b), the optimal value of K_P from the standpoint of minimum value of the maximum overshoot is between 15 and 17. Thus, the phase margin criterion is not a good indicator in the present case.

11-54 (a) Forward-path Transfer Function

$$G_p(s) = \frac{K\Theta_m(s)}{T_m(s)} = \frac{100K(s^2 + 10s + 100)}{s(s^3 + 20s^2 + 2100s + 10,000)} = \frac{10,000(s^2 + 10s + 100)}{s(s + 4.937)(s^2 + 15.06s + 2025.6)}$$

The unit-step response is plotted as shown below. The attributes of the response are:

$$\text{Maximum overshoot} = 57\% \quad t_r = 0.01345 \text{ sec} \quad t_s = 0.4949 \text{ sec}$$

(b) Design of the Second-order Notch Controller

The complex zeros of the notch controller are to cancel the complex poles of the process transfer function. Thus

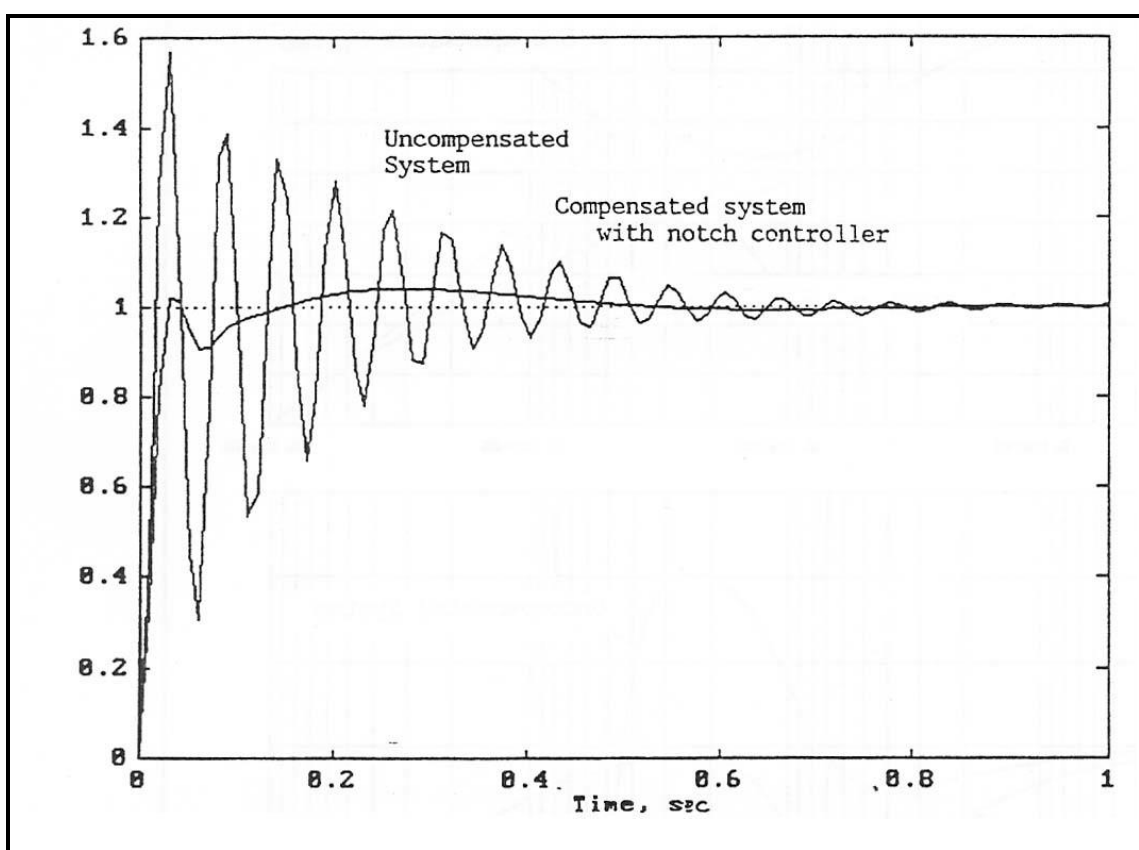
$$G_c(s) = \frac{s^2 + 15.06s + 2025.6}{s^2 + 90\zeta_p s + 2025.6} \quad \text{and} \quad G(s) = G_c(s)G_p(s) = \frac{10,000(s^2 + 10s + 100)}{s(s + 4.937)(s^2 + 90\zeta_p s + 2025.6)}$$

The following results are obtained for the unit-step response when various values of ζ_p are used.

The maximum overshoot is at a minimum of 4.1% when $\zeta_p = 1.222$. The unit-step response is plotted below, along with that of the uncompensated system.

ζ_p	$2\zeta\omega_n$	Max Overshoot (%)
2.444	200	7.3
2.333	210	6.9
2.222	200	6.5

1.667	150	4.9
1.333	120	4.3
1.222	110	4.1
1.111	100	5.8
1.000	90	9.8

Unit-step Response**(c) Frequency-domain Design of the Notch Controller**

The forward-path transfer function of the uncompensated system is

$$G(s) = \frac{10000(s^2 + 10s + 100)}{s(s + 4.937)(s^2 + 15.06s + 2025.6)}$$

The Bode plot of $G(j\omega)$ is constructed in the following. We see that the peak value of $|G(j\omega)|$ is approximately 22 dB. Thus, the notch controller should provide an attenuation of -22 dB or 0.0794 at the resonant frequency of 45 rad/sec. Using Eq. (10-155), we have

$$\left|G_c(j45)\right| = \frac{\zeta_z}{\zeta_p} = \frac{0.167}{\zeta_p} = 0.0794 \quad \text{Thus} \quad \zeta_p = 2.1024$$

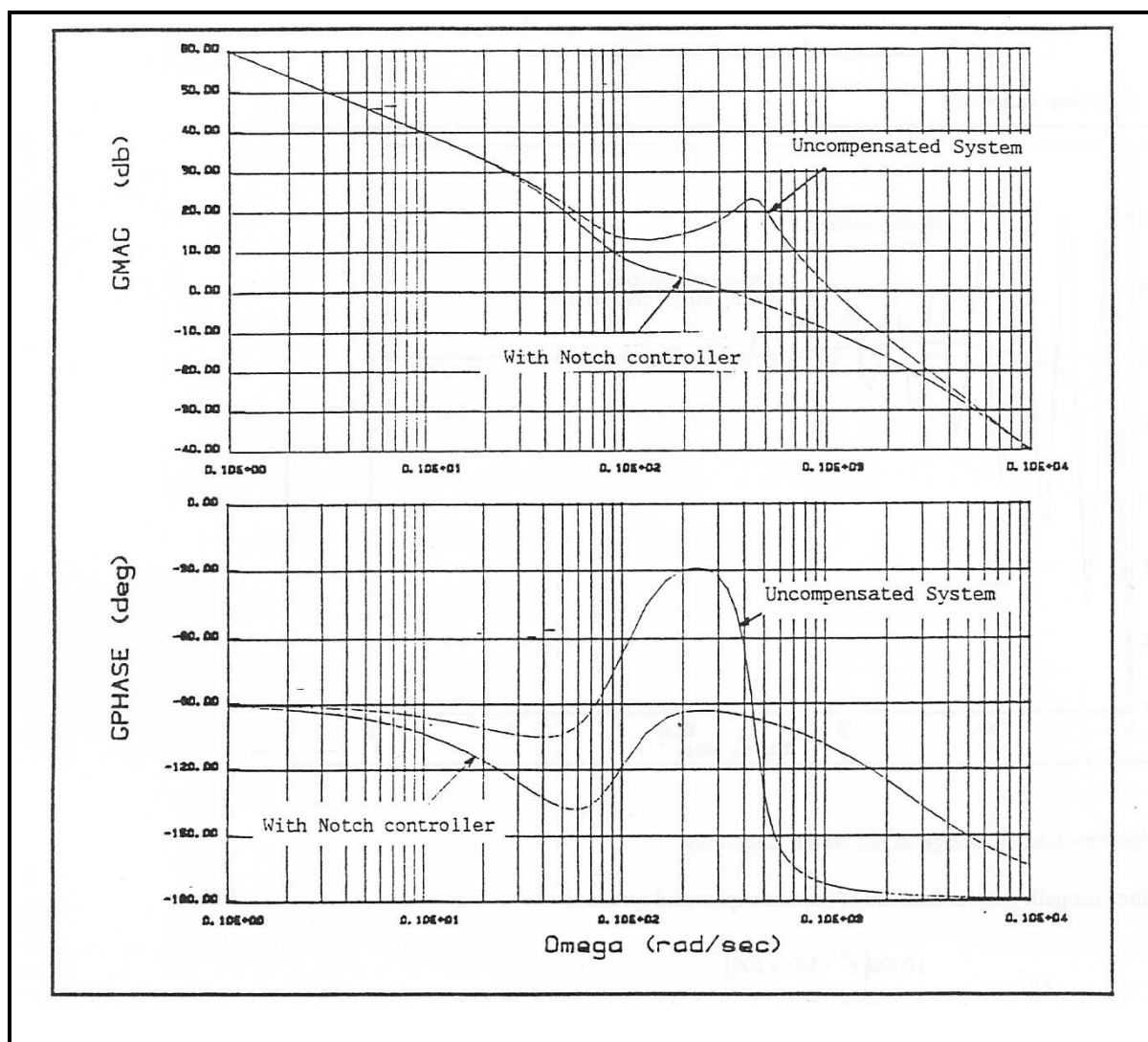
Notch Controller Transfer Function

$$G_c(s) = \frac{s^2 + 15.06s + 2025.6}{s^2 + 189.216s + 2025.6}$$

Forward-path Transfer Function

$$G(s) = \frac{10,000(s^2 + 10s + 100)}{s(s + 4.937)(s^2 + 189.22s + 2025.6)}$$

Bode Plots



Attributes of the frequency response: PM = 80.37 deg GM = infinite $M_r = 1.097$ BW = 66.4 rad/sec

Attributes of the frequency response of the system designed in part (b):

PM = 59.64 deg GM = infinite $M_r = 1.048$ BW = 126.5 rad/sec

11-55 (a) Process Transfer Function

$$G_p(s) = \frac{500(s+10)}{s(s^2 + 10s + 1000)}$$

The Bode plot is constructed below. The frequency-domain attributes of the uncompensated system are:

$$\text{PM} = 30 \text{ deg} \quad \text{GM} = \text{infinite} \quad M_r = 1.86 \quad \text{and} \quad \text{BW} = 3.95 \text{ rad/sec}$$

The unit-step response is oscillatory.

(b) Design of the Notch Controller

For the uncompensated process, the complex poles have the following constants:

$$\omega_n = \sqrt{1000} = 31.6 \text{ rad/sec} \quad 2\zeta\omega_n = 10 \quad \text{Thus} \quad \zeta = 0.158$$

The transfer function of the notch controller is

$$G_c(s) = \frac{s^2 + 2\zeta_z\omega_n s + \omega_n^2}{s^2 + 2\zeta_p\omega_n s + \omega_n^2}$$

For the zeros of $G_c(s)$ to cancel the complex poles of $G_p(s)$, $\zeta_z = \zeta = 0.158$.

From the Bode plot, we see that to bring down the peak resonance of $|G(j\omega_n)|$ in order

to smooth out the magnitude curve, the notch controller should provide approximately

−26 dB of attenuation. Thus, using Eq. (10-155),

$$\frac{\zeta_z}{\zeta_p} = 10^{\frac{-26}{20}} = 0.05 \quad \text{Thus} \quad \zeta_p = \frac{0.158}{0.05} = 3.1525$$

The transfer function of the notch controller is

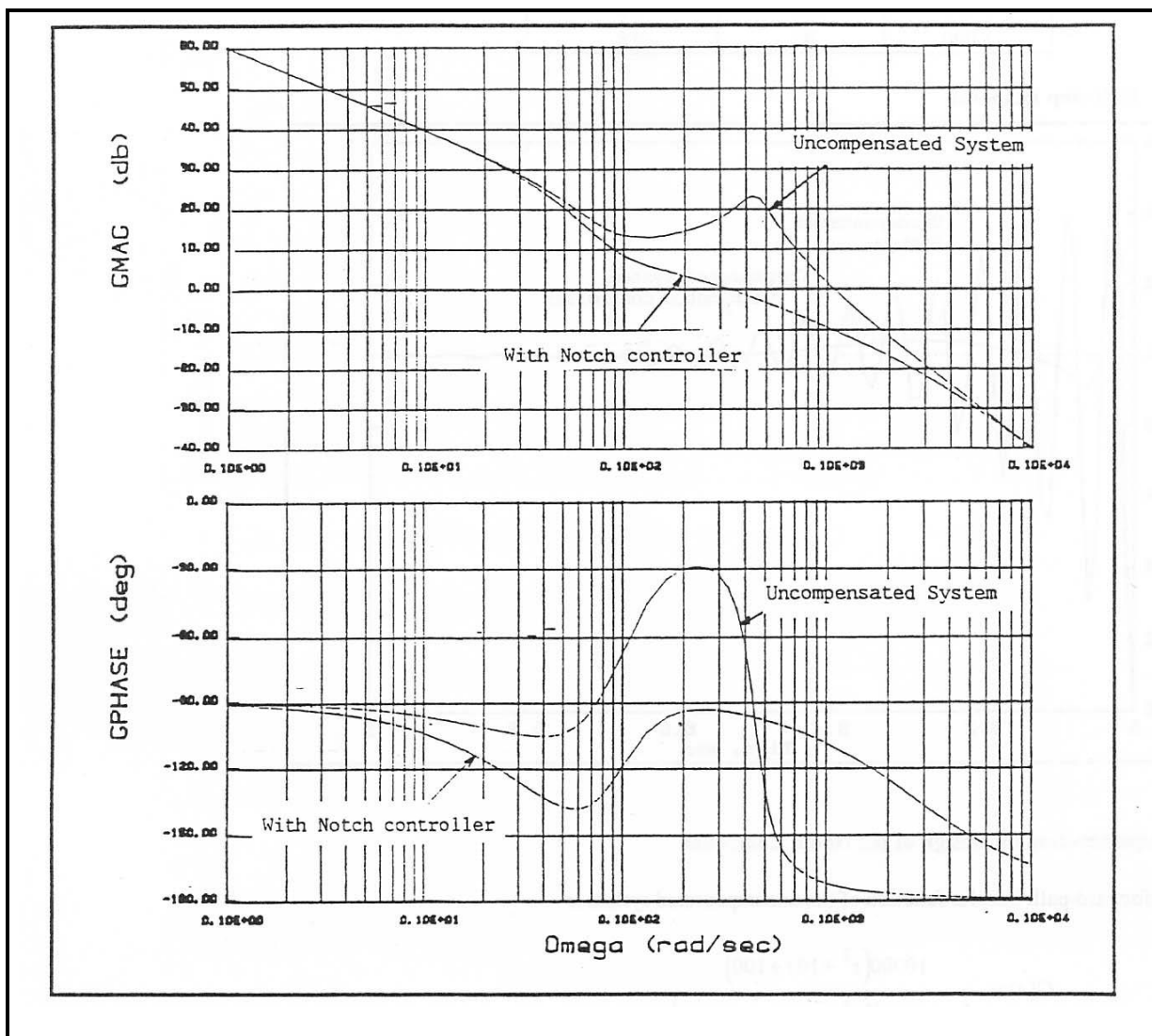
$$G_c(s) = \frac{s^2 + 10s + 1000}{s^2 + 199.08s + 1000} \quad G(s) = G_c(s)G_p(s) = \frac{500(s + 10)}{s(s^2 + 199.08s + 1000)}$$

The attributes of the compensated system are:

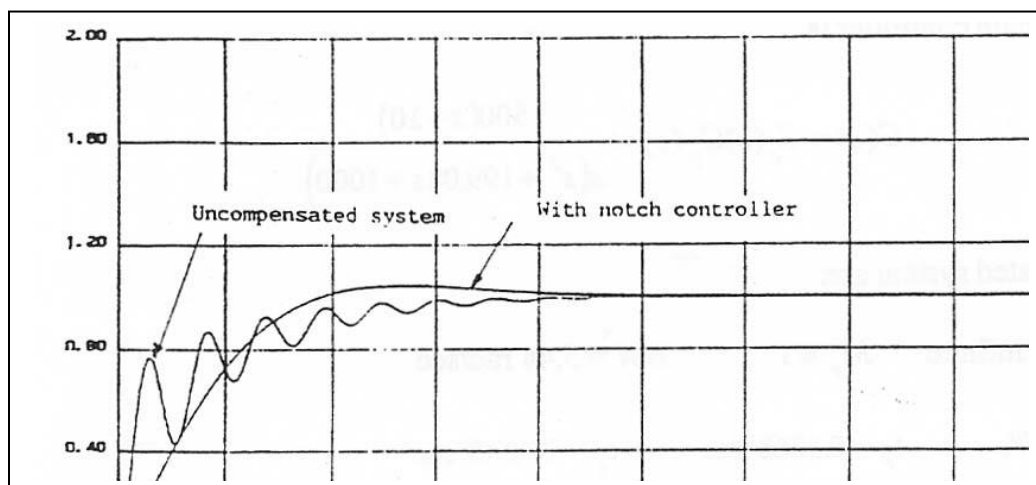
$$\text{PM} = 72.38^\circ \quad \text{GM} = \text{infinite} \quad M_r = 1 \quad \text{BW} = 5.44 \text{ rad/sec}$$

$$\text{Maximum overshoot} = 3.4\% \quad t_r = 0.3868 \text{ sec} \quad t_s = 0.4848 \text{ sec}$$

Bode Plots



Step Responses



(c) Time-domain design of the Notch Controller

With $\zeta_z = 0.158$ and $\omega_n = 31.6$, the forward-path transfer function of the compensated system is

$$G(s) = G_c(s)G_p(s) = \frac{500(s+10)}{s(s^2 + 63.2\zeta_p s + 1000)}$$

The following attributes of the unit-step response are obtained by varying the value of ζ_p .

ζ_p	$2\zeta\omega_n$	Max Overshoot (%)	t_r (sec)	t_s (sec)
1.582	100	0	0.4292	0.5859
1.741	110	0	0.4172	0.5657
1.899	120	0	0.4074	0.5455
2.057	130	0	0.3998	0.5253
2.215	140	0.2	0.3941	0.5152
2.500	158.25	0.9	0.3879	0.4840
3.318	209.7	4.1	0.3884	0.4848

When $\zeta_p = 2.5$ the maximum overshoot is 0.9%, the rise time is 0.3879 sec and the setting

time is 0.4840 sec. These performance attributes are within the required specifications.

11-56 Let the transfer function of the controller be

$$G_c(s) = \frac{20,000(s^2 + 10s + 50)}{(s + 1000)^2}$$

Then, the forward-path transfer function becomes

$$G(s) = G_c(s)G_p(s) = \frac{20,000K(s^2 + 10s + 50)}{s(s^2 + 10s + 100)(s + 1000)^2}$$

For $G_{cf}(s) = 1$, $K_v = \lim_{s \rightarrow 0} sG(s) = \frac{10^6 K}{10^8} = 50$ Thus the nominal $K = 5000$

For $\pm 20\%$ variation in K , $K_{\min} = 4000$ and $K_{\max} = 6000$. To cancel the complex closed-loop poles, we let

$$G_{cf}(s) = \frac{50(s + 1)}{s^2 + 10s + 50} \quad \text{where the } (s + 1) \text{ term is added to reduce the rise time.}$$

Closed-loop Transfer Function:

$$\frac{Y(s)}{R(s)} = \frac{10^6 K(s + 1)}{s(s^2 + 10s + 100)(s + 1000)^2 + 20,000K(s^2 + 10s + 50)}$$

Characteristic Equation:

$K = 4000$: $s^5 + 2010s^4 + 1,020,100s^3 + 9.02 \times 10^7 s^2 + 9 \times 10^8 + 4 \times 10^9 = 0$

Roots: $-97.7, -648.9, -1252.7, -5.35 + j4.6635, -5.35 - j4.6635$

Max overshoot $\cong 6.7\%$

Rise time < 0.04 sec

$$K = 5000: \quad s^5 + 2010s^4 + 1,020,100s^3 + 1.1 \times 10^8 s^2 + 1.1 \times 10^9 s + 5 \times 10^9 = 0$$

$$\text{Roots:} \quad -132.46, \quad 587.44, \quad -1279.6, \quad -5.272 + j4.7353, \quad -5.272 - j4.7353$$

$$\text{Max overshoot} \cong 4\%$$

$$\text{Rise time} < 0.04 \text{ sec}$$

$$K = 6000 \quad s^5 + 2010s^4 + 1,020,100s^3 + 1.3 \times 10^8 s^2 + 1.3 \times 10^9 s + 6 \times 10^9 = 0$$

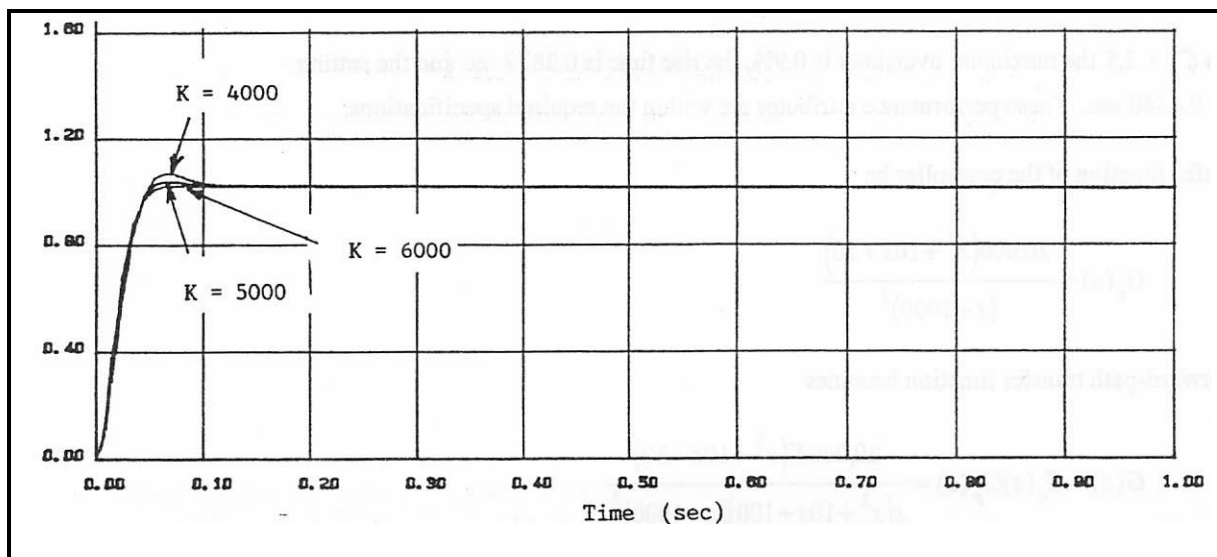
$$\text{Roots:} \quad -176.77, \quad -519.37, \quad -1303.4, \quad -5.223 + j4.7818, \quad -5.223 - j4.7818$$

$$\text{Max overshoot} \cong 2.5\%$$

$$\text{Rise time} < 0.04 \text{ sec}$$

Thus all the required specifications stay within the required tolerances when the value of K varies by plus and minus 20%.

Unit-step Responses



11-57 Let the transfer function of the controller be

$$G_c(s) = \frac{200(s^2 + 10s + 50)}{(s + 100)^2}$$

The forward-path transfer function becomes

$$G(s) = G_c(s)G_p(s) = \frac{200,000K(s^2 + 10s + 50)}{s(s + a)(s + 100)^2}$$

For $a = 10$,

$$K_v = \lim_{s \rightarrow 0} sG(s) = \frac{10^7 K}{10^5} = 100K = 100 \quad \text{Thus} \quad K = 1$$

Characteristic Equations: ($K = 1$)

$a = 10$: $s^4 + 210s^3 + 2.12 \times 10^5 s^2 + 2.1 \times 10^6 s + 10^7 = 0$

Roots: $-4.978 + j4.78, \quad -4.978 - j4.78, \quad -100 + j447.16, \quad -100 - j447.16$

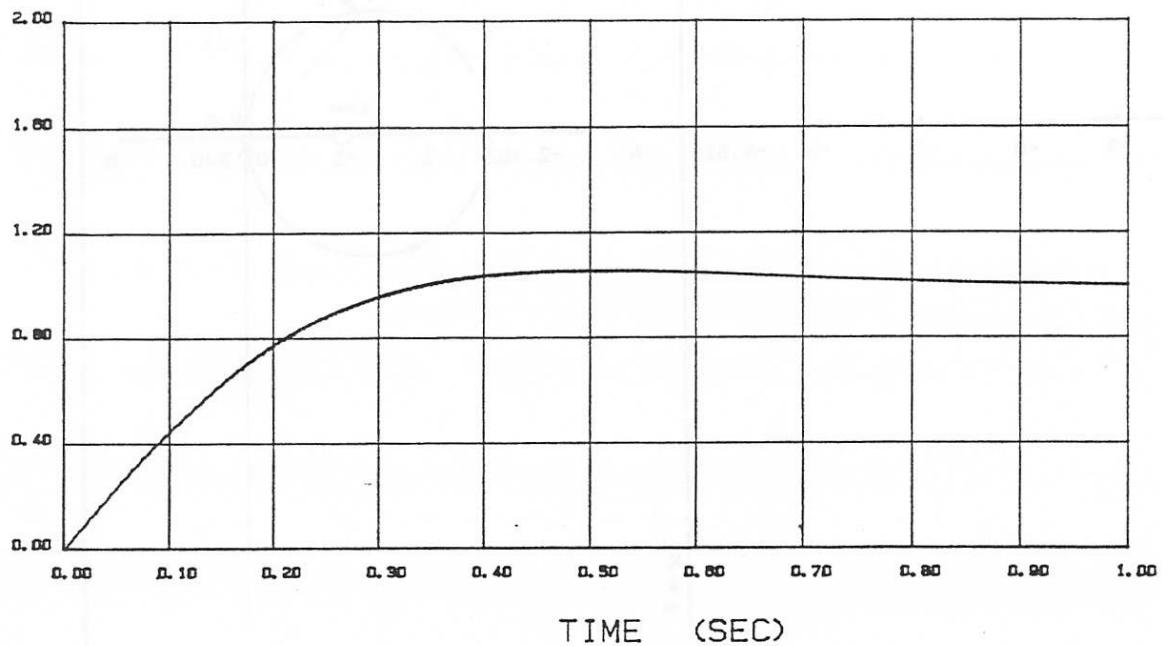
$a = 8$: $s^4 + 208s^3 + 2.116 \times 10^5 s^2 + 2.08 \times 10^6 s + 10^7 = 0$

Roots: $-4.939 + j4.828, \quad -4.939 - j4.828, \quad -99.06 + j446.97, \quad -99.06 - j446.97$

$a = 12$: $s^4 + 212s^3 + 2.124 \times 10^5 s^2 + 2.12 \times 10^6 s + 10^7 = 0$

Roots: $-5.017 + j4.73, \quad -5.017 - j4.73, \quad -100.98 + j447.36, \quad -100.98 - j447.36$

Unit-step Responses: All three responses for $\alpha = 8$, $\alpha = 10$, and 12 are similar.

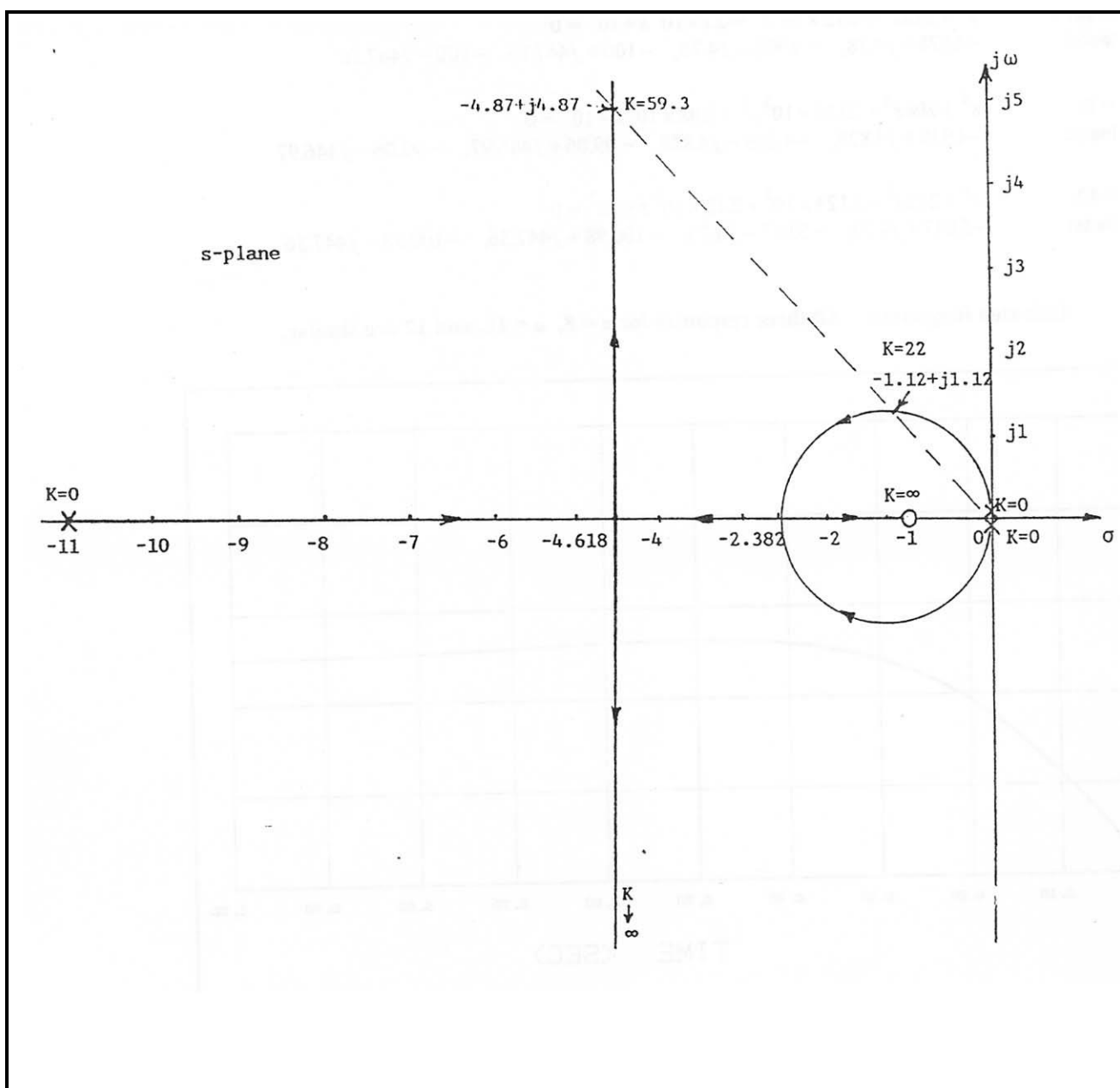


11-58 Forward-path Transfer Function:

$$G(s) = \frac{Y(s)}{E(s)} = \frac{K}{s(s+1)(s+10) + KK_t s} \quad K_v = \lim_{s \rightarrow 0} sG(s) = \frac{K}{10 + KK_t} = 1$$

Characteristic Equation: $s^3 + 11s^2 + (10 + KK_t)s + K = s^3 + 11s^2 + Ks + K = 0$

For root loci, $G_{eq}(s) = \frac{K(s+1)}{s^2(s+11)}$

Root Locus Plot (K varies)

The root loci show that a relative damping ratio of 0.707 can be realized by two values of K . $K = 22$ and 59.3. As stipulated by the problem, we select $K = 59.3$.

11-59 Forward-path Transfer Function:

$$G(s) = \frac{10K}{s(s+1)(s+10) + 10K_t s} \quad K_v = \lim_{s \rightarrow 0} sG(s) = \frac{10K}{10 + 10K_t} = \frac{K}{1 + K_t} = 1 \quad \text{Thus } K_t = K - 1$$

$$\text{Characteristic Equation: } s(s+1)(s+10) + 10K_t + 10K = s^3 + 11s^2 + 10Ks + 10K = 0$$

When $K = 5.93$ and $K_t = K - 1 = 4.93$, the characteristic equation becomes

$$s^3 + 11s^2 + 10.046s + 4.6 = 0$$

The roots are: -10.046 , $-0.47723 + j0.47976$, $-0.47723 - j0.47976$

11-60 Forward-path Transfer Function:

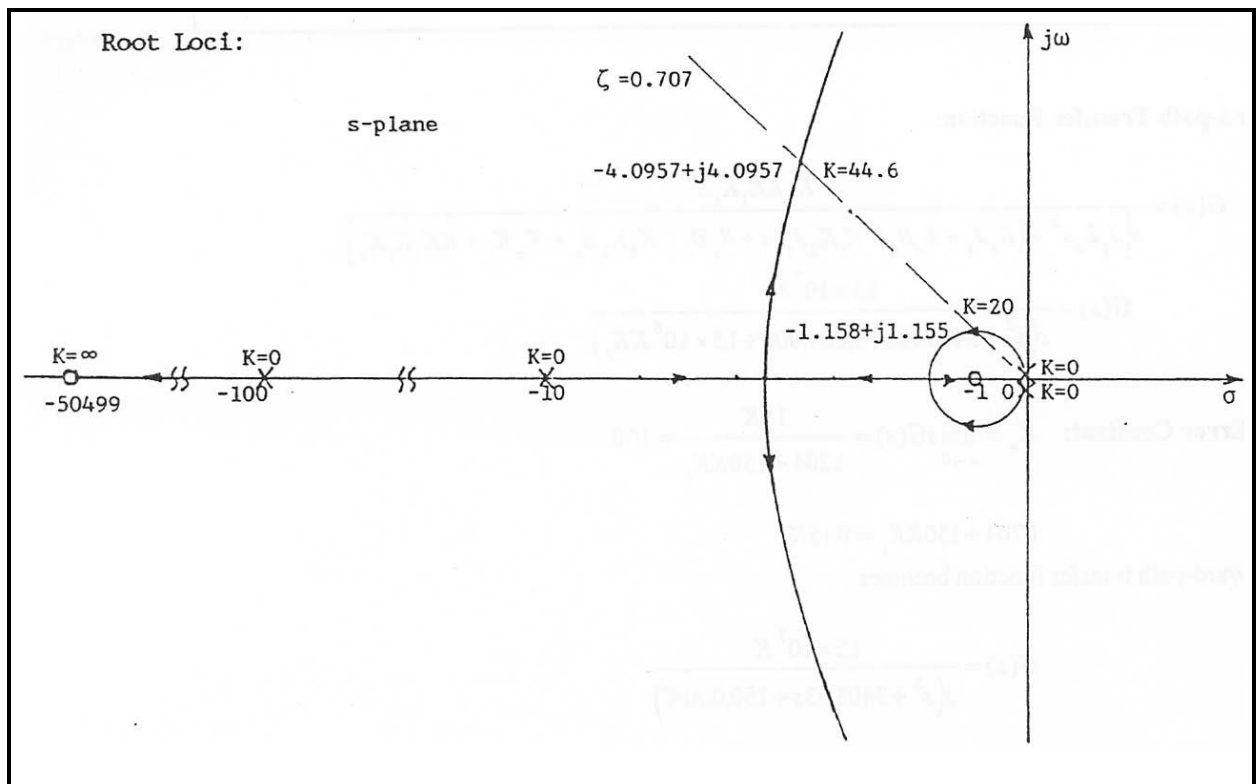
$$G(s) = \frac{K(1 + aTs)}{s((1 + Ts)(s^2 + 10s + KK_t))} \quad K_v = \lim_{s \rightarrow 0} sG(s) = \frac{1}{K_t} = 100 \quad \text{Thus } K_t = 0.01$$

Let $T = 0.01$ and $a = 100$. The characteristic equation of the system is written:

$$s^4 + 110s^3 + 1000s^2 + K(0.001s^2 + 101s + 100) = 0$$

To construct the root contours as K varies, we form the following equivalent forward-path transfer function:

$$G_{eq}(s) = \frac{0.001K(s^2 + 101,000s + 100,000)}{s^2(s+10)(s+100)} = \frac{0.001K(s+1)(s+50499)}{s^2(s+10)(s+100)}$$



From the root contour diagram we see that two sets of solutions exist for a damping ratio of 0.707.

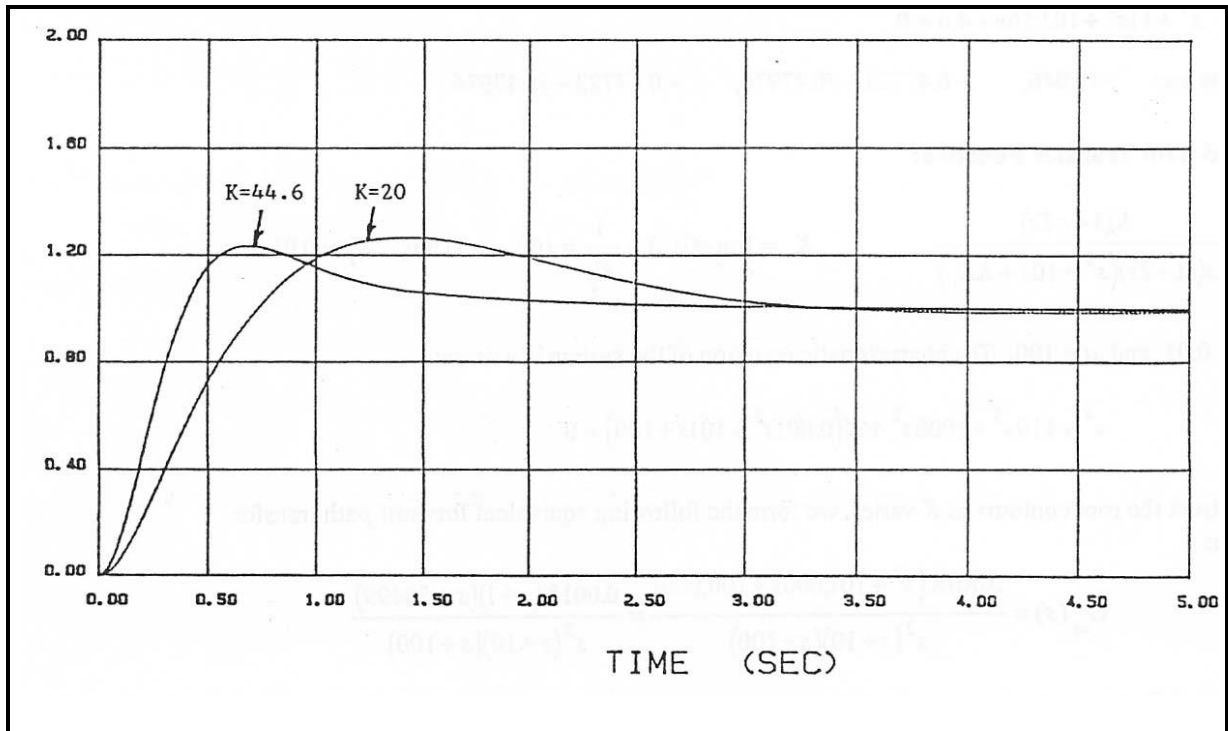
These are:

$K = 20$: **Complex roots:** $-1.158 + j1.155$, $-1.158 - j1.155$

$K = 44.6$: **Complex roots:** $-4.0957 + j4.0957$, $-4.0957 - j4.0957$

The unit-step responses of the system for $K = 20$ and 44.6 are shown below.

Unit-step Responses:

**11-61 Forward-path Transfer Function:**

$$G(s) = \frac{K_s K K_i K_i N}{s \left[J_t L_a s^2 + (R_a J_t + L_a B_t + K_1 K_2 J_t) s + R_a B_t + K_1 K_2 B_t + K_b K_i + K K_1 K_i K_t \right]}$$

$$G(s) = \frac{1.5 \times 10^7 K}{s \left(s^2 + 3408.33s + 1,204,000 + 1.5 \times 10^8 K K_t \right)}$$

Ramp Error Constant: $K_v = \lim_{s \rightarrow 0} sG(s) = \frac{15K}{1.204 + 150KK_t} = 100$

Thus $1.204 + 150KK_t = 0.15K$

The forward-path transfer function becomes

$$G(s) = \frac{1.5 \times 10^7 K}{s(s^2 + 3408.33s + 150,000K)}$$

Characteristic Equation: $s^3 + 3408.33s + 150,000Ks + 1.5 \times 10^7 K = 0$

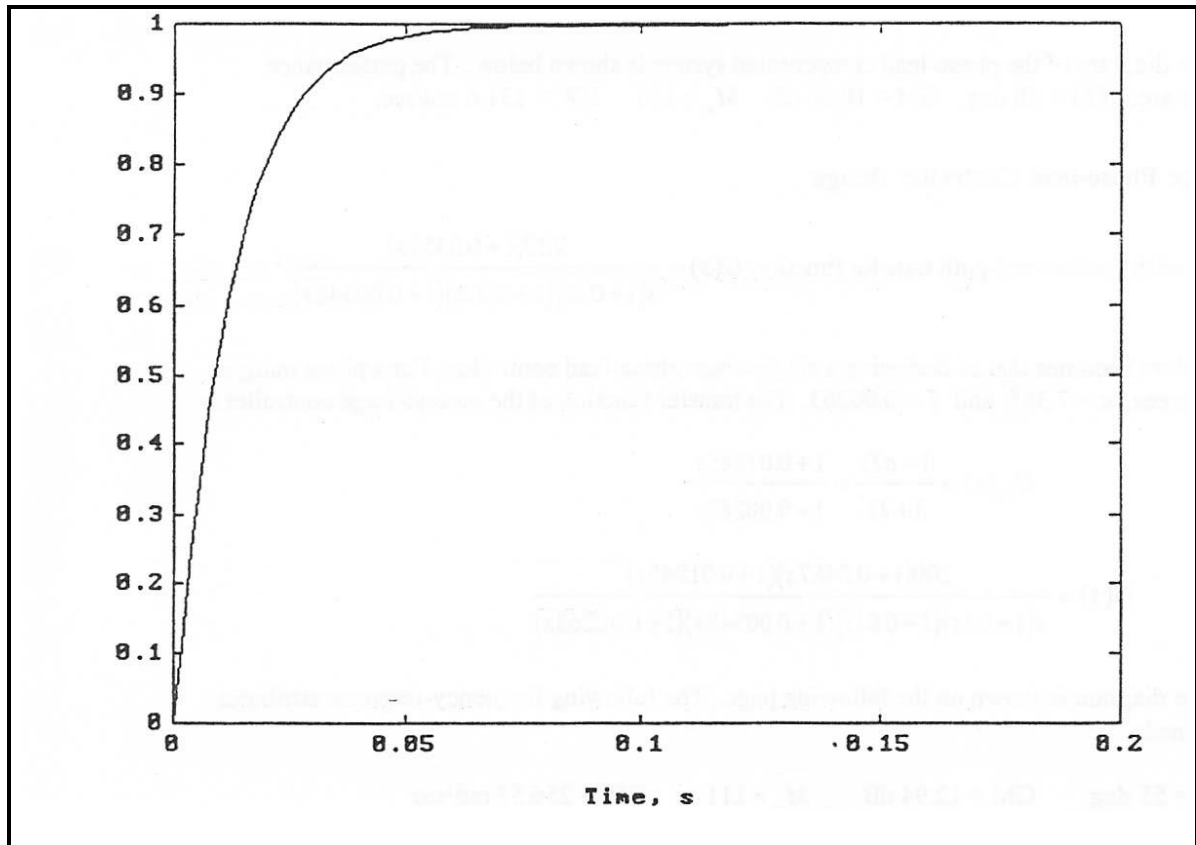
When $K = 38.667$ the roots of the characteristic equation are at

$$-0.1065, \quad -1.651 + j1.65, \quad -1.651 - j1.65 \quad (\zeta \cong 0.707 \text{ for the complex roots})$$

The forward-path transfer function becomes

$$G(s) = \frac{5.8 \times 10^8}{s(s^2 + 3408.33s + 5.8 \times 10^6)}$$

Unit-step Response



Unit-step response attributes: Maximum overshoot = 0 Rise time = 0.0208 sec Settling time = 0.0283 sec

11-62 (a) Disturbance-to-Output Transfer Function

$$\left. \frac{Y(s)}{T_L(s)} \right|_{r=0} = \frac{2(1+0.1s)}{s(1+0.01s)(1+0.1s) + 20K} \quad G_c(s) = 1$$

For $T_L(s) = 1/s$

$$\lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} sY(s) = \frac{1}{10K} \leq 0.01 \quad \text{Thus} \quad K \geq 10$$

(b) Performance of Uncompensated System. $K = 10$, $G_c(s) = 1$

$$G(s) = \frac{200}{s(1+0.01s)(1+0.1s)}$$

The Bode diagram of $G(j\omega)$ is shown below. The system is unstable. The attributes of the frequency response are: PM = -9.65 deg GM = -5.19 dB.

(c) Single-stage Phase-lead Controller Design

To realize a phase margin of 30 degrees, $\alpha = 14$ and $T = 0.00348$.

$$G_c(s) = \frac{1+aTs}{1+Ts} = \frac{1+0.0487s}{1+0.00348s}$$

The Bode diagram of the phase-lead compensated system is shown below. The performance attributes are: PM = 30 deg GM = 10.66 dB $M_r = 1.95$ BW = 131.6 rad/sec.

(d) Two-stage Phase-lead Controller Design

Starting with the forward-path transfer function $G(s) = \frac{200(1+0.0487s)}{s(1+0.1s)(1+0.01s)(1+0.00348s)}$

The problem becomes that of designing a single-stage phase-lead controller. For a phase margin or 55 degrees, $\alpha = 7.385$ and $T = 0.00263$. The transfer function of the second-stage controller is

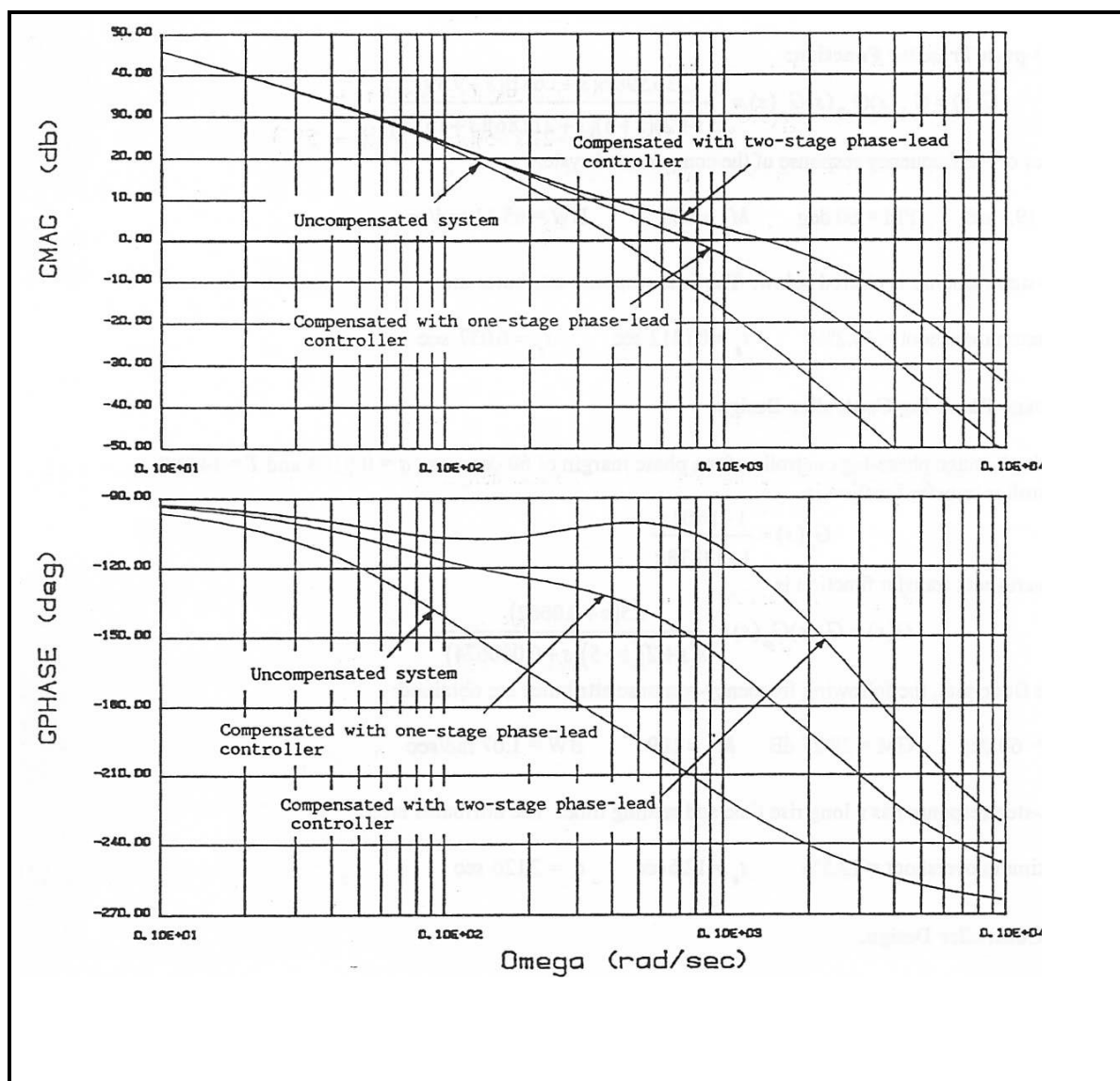
$$G_{c1}(s) = \frac{1+aTs}{1+Ts} = \frac{1+0.01845s}{1+0.00263s}$$

Thus
$$G(s) = \frac{200(1 + 0.0487s)(1 + 0.01845s)}{s(1 + 0.1s)(1 + 0.01s)(1 + 0.00348s)(1 + 0.00263s)}$$

The Bode diagram is shown on the following page. The following frequency-response attributes are obtained:

$$\text{PM} = 55 \text{ deg} \quad \text{GM} = 12.94 \text{ dB} \quad M_r = 1.11 \quad \text{BW} = 256.57 \text{ rad/sec}$$

Bode Plot [parts (b), (c), and (d)]



11-63 (a) Two-stage Phase-lead Controller Design.

The uncompensated system is unstable. PM = -43.25 deg and GM = -18.66 dB.

With a single-stage phase-lead controller, the maximum phase margin that can be realized affectively is 12 degrees. Setting the desired PM at 11 deg, we have the parameters of the single-stage phase-lead controller as $a = 128.2$ and $T_1 = 0.00472$. The transfer function of the single-stage controller

$$G_{c1}(s) = \frac{1 + aT_1s}{1 + T_1s} = \frac{1 + 0.6057s}{1 + 0.00472s}$$

Starting with the single-stage-controller compensated system, the second stage of the phase-lead controller is designed to realize a phase margin of 60 degrees. The parameters of the second-stage controller are: $b = 16.1$ and $T_2 = 0.0066$. Thus,

$$G_{c2}(s) = \frac{1 + bT_2s}{1 + T_2s} = \frac{1 + 0.106s}{1 + 0.0066s}$$

$$G_c(s) = G_{c1}(s)G_{c2}(s) = \frac{1 + 0.6057s}{1 + 0.00472s} \frac{1 + 0.106s}{1 + 0.0066s}$$

Forward-path Transfer Function:

$$G(s) = G_{c1}(s)G_{c2}(s)G_p(s) = \frac{1,236,598.6(s + 1.651)(s + 9.39)}{s(s + 2)(s + 5)(s + 211.86)(s + 151.5)}$$

Attributes of the frequency response of the compensated system are:

$$\text{GM} = 19.1 \text{ dB} \quad \text{PM} = 60 \text{ deg} \quad M_r = 1.08 \quad \text{BW} = 65.11 \text{ rad/sec}$$

The unit-step response is plotted below. The time-response attributes are:

$$\text{Maximum overshoot} = 10.2\% \quad t_s = 0.1212 \text{ sec} \quad t_r = 0.037 \text{ sec}$$

(b) Single-stage Phase-lag Controller Design.

With a single-stage phase-lag controller, for a phase margin of 60 degrees, $\alpha = 0.0108$ and $T = 1483.8$.

The controller transfer function is

$$G_c(s) = \frac{1 + 16.08s}{1 + 1483.8s}$$

The forward-path transfer function is

$$G(s) = G_c(s)G_p(s) = \frac{6.5(1 + 0.0662s)}{s(1 + 2s)(1 + 5s)(1 + 0.000674s)}$$

From the Bode plot, the following frequency-response attributes are obtained:

$$\text{PM} = 60 \text{ deg} \quad \text{GM} = 20.27 \text{ dB} \quad M_r = 1.09 \quad \text{BW} = 1.07 \text{ rad/sec}$$

The unit-step response has a long rise time and settling time. The attributes are:

$$\text{Maximum overshoot} = 12.5\% \quad t_s = 12.6 \text{ sec} \quad t_r = 2.126 \text{ sec}$$

(c) Lead-lag Controller Design.

For the lead-lag controller, we first design the phase-lag portion for a 40-degree phase margin.

The result is $\alpha = 0.0238$ and $T_1 = 350$. The transfer function of the controller is

$$G_{c1}(s) = \frac{1 + 8.333s}{1 + 350s}$$

The phase-lead portion is designed to yield a total phase margin of 60 degrees. The result is

$b = 4.8$ and $T_2 = 0.2245$. The transfer function of the phase-lead controller is

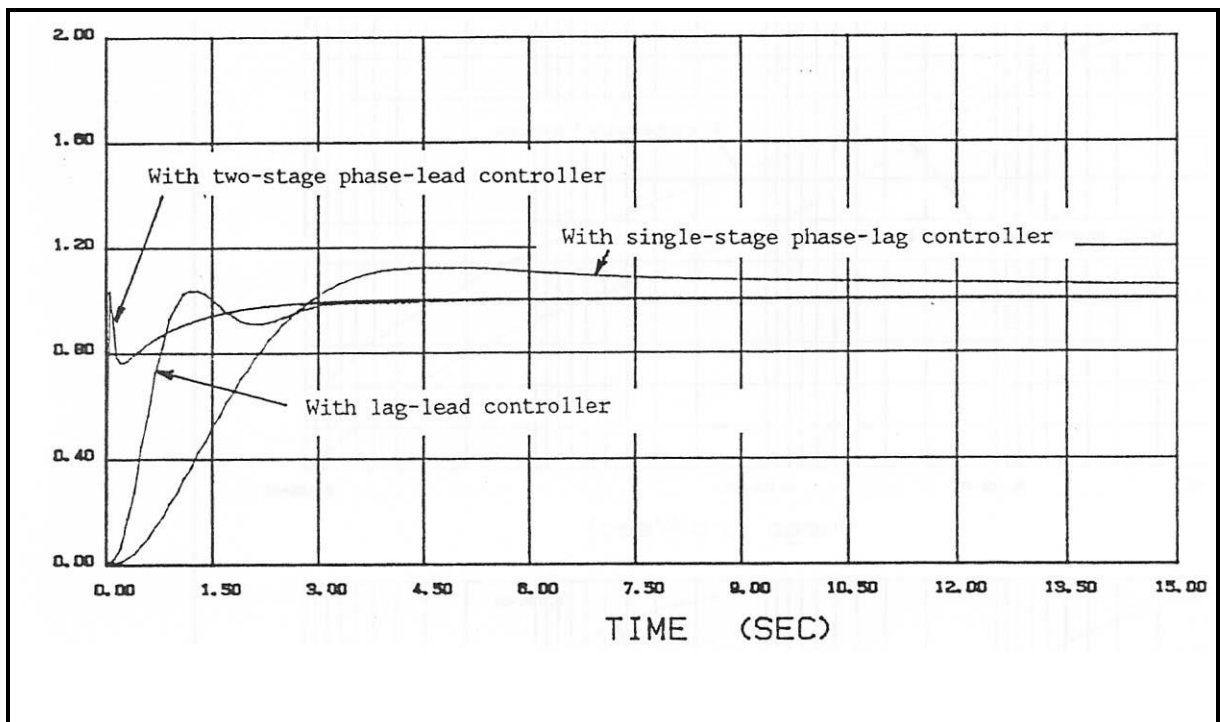
$$G_{c2}(s) = \frac{1 + 1.076s}{1 + 0.2245s}$$

The forward-path transfer function of the lead-lag compensated system is

$$G(s) = \frac{68.63(s + 0.12)(s + 0.929)}{s(s + 2)(s + 5)(s + 0.00286)(s + 4.454)}$$

Frequency-response attributes: PM = 60 deg GM = 13.07 dB $M_r = 1.05$ BW = 3.83 rad/sec

Unit-step response attributes: Maximum overshoot = 5.9% $t_s = 1.512$ sec $t_r = 0.7882$ sec



Unit-step Responses.

11-64 (a) The uncompensated system has the following frequency-domain attributes:

PM = 3.87 deg GM = 1 dB $M_r = 7.73$ BW = 4.35 rad/sec

The Bode plot of $G_p(j\omega)$ shows that the phase curve drops off sharply, so that the phase-lead controller would not be very effective. Consider a single-stage phase-lag controller. The phase

margin of 60 degrees is realized if the gain crossover is moved from 2.8 rad/sec to 0.8 rad/sec.

The attenuation of the phase-lag controller at high frequencies is approximately –15 dB.

Choosing an attenuation of –17.5 dB, we calculate the value of a from

$$20\log_{10} a = -17.5 \text{ dB} \quad \text{Thus} \quad a = 0.1334$$

The upper corner frequency of the phase-lag controller is chosen to be at $1/aT = 0.064$ rad/sec.

Thus, $1/T = 0.00854$ or $T = 117.13$. The transfer function of the phase-lag controller is

$$G_c(s) = \frac{1 + 15.63s}{1 + 117.13s}$$

The forward-path transfer function is

$$G(s) = G_c(s)G_p(s) = \frac{5(1 + 15.63s)(1 - 0.05s)}{s(1 + 0.1s)(1 + 0.5s)(1 + 117.13s)(1 + 0.05s)}$$

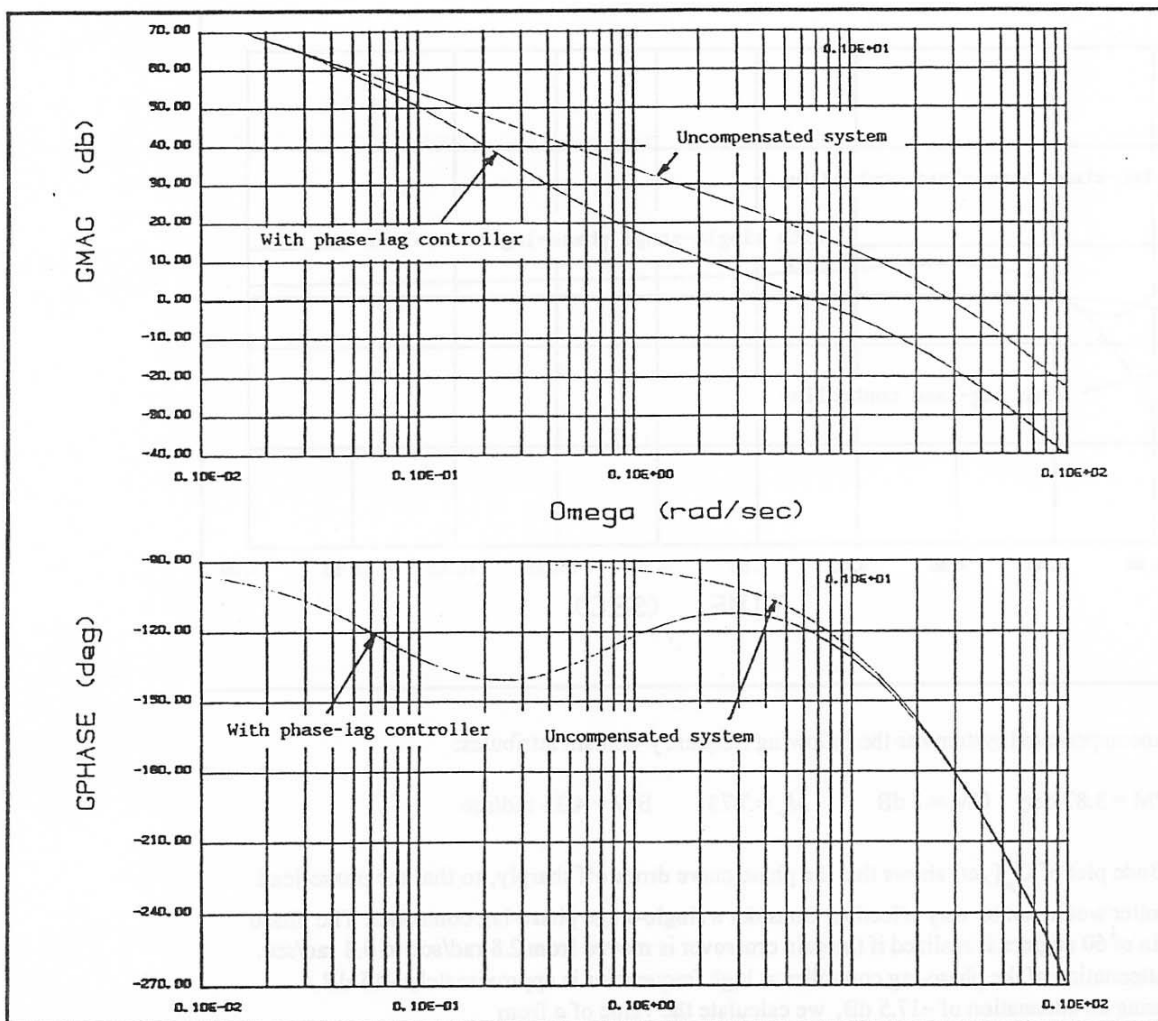
From the Bode plot of $G(j\omega)$, the following frequency-domain attributes are obtained:

$$\text{PM} = 60 \text{ deg} \quad \text{GM} = 18.2 \text{ dB} \quad M_r = 1.08 \quad \text{BW} = 1.13 \text{ rad/sec}$$

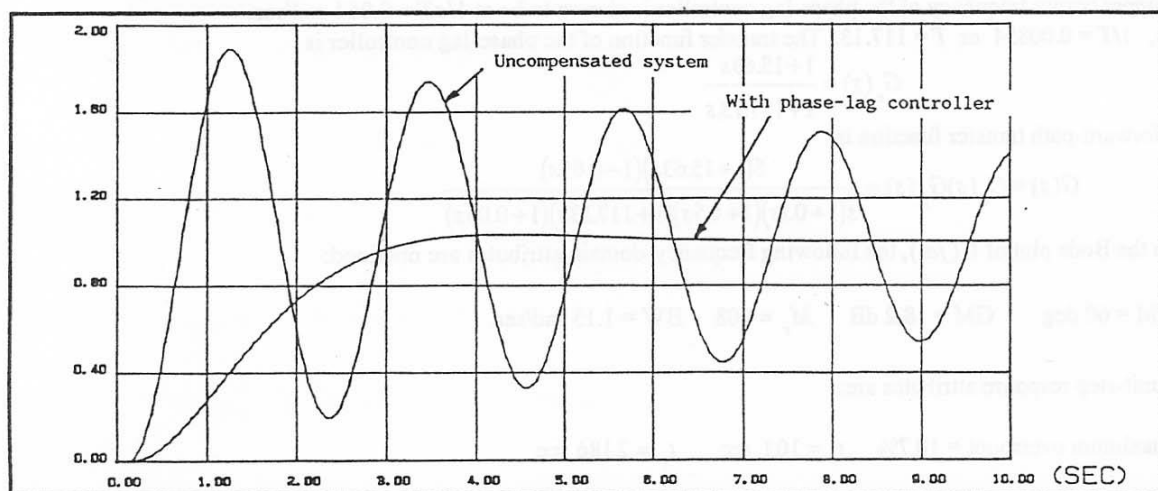
The unit-step response attributes are:

$$\text{maximum overshoot} = 10.7\% \quad t_s = 10.1 \text{ sec} \quad t_r = 2.186 \text{ sec}$$

Bode Plots



Step Responses



11-64 (b) Using the exact expression of the time delay, the same design holds. The time and frequency domain attributes are not much affected.

11-65 (a) Uncompensated System.

Forward-path Transfer Function:
$$G(s) = \frac{10}{(1+s)(1+10s)(1+2s)(1+5s)}$$

The Bode plot of $G(j\omega)$ is shown below.

The performance attributes are: PM = -10.64 deg GM = -2.26 dB

The uncompensated system is unstable.

(b) PI Controller Design.

Forward-path Transfer Function:
$$G(s) = \frac{10(K_p s + K_I)}{s(1+s)(1+10s)(1+2s)(1+5s)}$$

Ramp-error Constant: $K_v = \lim_{s \rightarrow 0} sG(s) = 10K_I = 0.1$ Thus $K_I = 0.01$

$$G(s) = \frac{0.1(1+100K_p s)}{s(1+s)(1+10s)(1+2s)(1+5s)}$$

The following frequency-domain attributes are obtained for various values of K_p .

K_p	PM (deg)	GM (dB)	M_r	BW (rad/sec)
0.01	24.5	5.92	2.54	0.13
0.02	28.24	7.43	2.15	0.13
0.05	38.84	11.76	1.52	0.14

0.10	50.63	12.80	1.17	0.17
0.12	52.87	12.23	1.13	0.18
0.15	53.28	11.22	1.14	0.21
0.16	52.83	10.88	1.16	0.22
0.17	51.75	10.38	1.18	0.23
0.20	49.08	9.58	1.29	0.25

The phase margin is maximum at 53.28 degrees when $K_p = 0.15$.

The forward-path transfer function of the compensated system is

$$G(s) = \frac{0.1(1+15s)}{s(1+s)(1+10s)(1+5s)(1+2s)}$$

The attributes of the frequency response are:

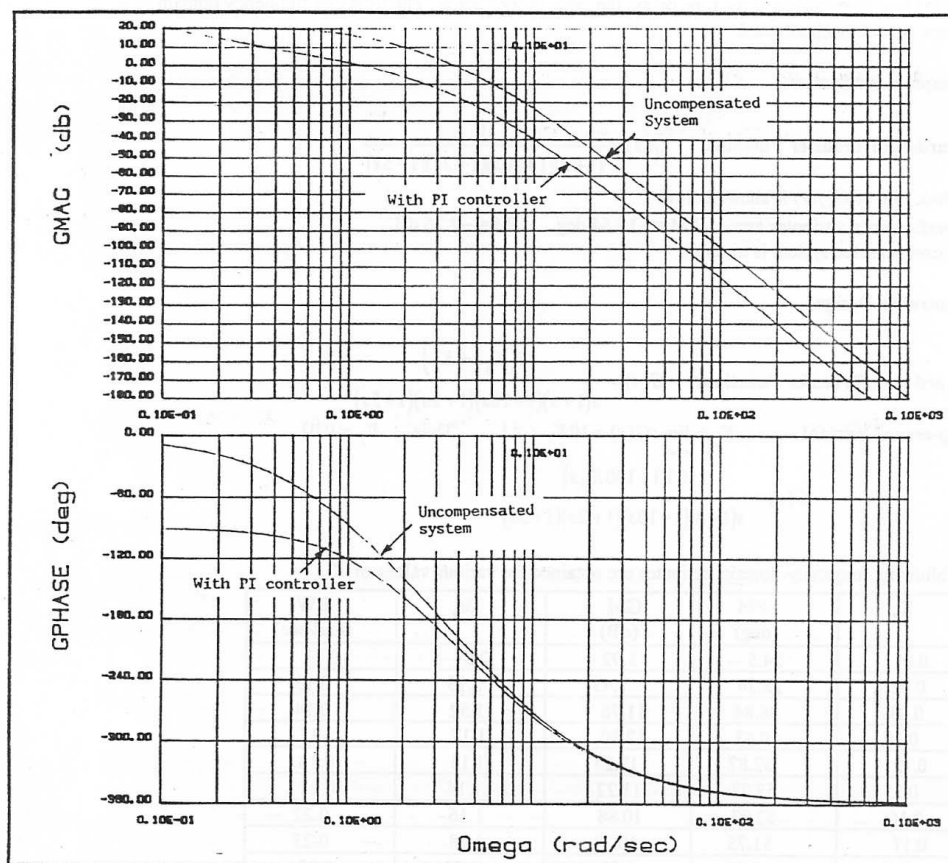
$$\text{PM} = 53.28 \text{ deg} \quad \text{GM} = 11.22 \text{ dB} \quad M_r = 1.14 \quad \text{BW} = 0.21 \text{ rad/sec}$$

The attributes of the unit-step response are:

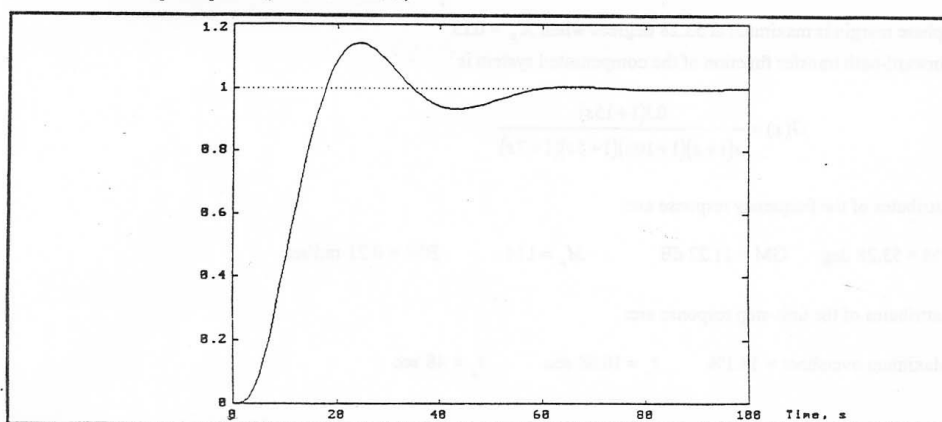
$$\text{Maximum overshoot} = 14.1\% \quad t_r = 10.68 \text{ sec} \quad t_s = 48 \text{ sec}$$

Bode Plots

Bode Plots



Step Response (with PI control)



11-65 (c) Time-domain Design of PI Controller.

By setting $K_I = 0.01$ and varying K_P we found that the value of K_P that minimizes the maximum overshoot of the unit-step response is also 0.15. Thus, the unit-step response obtained in part (b) is still applicable for this case.

11-66 Closed-loop System Transfer Function.

$$\frac{Y(s)}{R(s)} = \frac{1}{s^3 + (4 + k_3)s^2 + (3 + k_2 + k_3)s + k_1}$$

For zero steady-state error to a step input, $k_1 = 1$. For the complex roots to be located at $-1 + j$ and $-1 - j$,

we divide the characteristic polynomial by $s^2 + 2s + 2$ and solve for zero remainder.

$$\begin{array}{r} s + (2 + k_2) \\ s^2 + 2s + 2 \overline{) s^3 + (4 + k_3)s^2 + (3 + k_2 + k_3)s + 1} \\ \underline{s^3 + 2s^2 + 2s} \\ (2 + k_3)s^2 + (1 + k_2 + k_3)s + 1 \\ \underline{(2 + k_3)s^2 + (4 + 2k_3)s + 4 + 2k_3} \\ (-3 + k_2 - k_3)s - 3 - 2k_3 \end{array}$$

For zero remainder, $-3 - 2k_3 = 0$ Thus $k_3 = -1.5$

$-3 + k_2 - k_3 = 0$ Thus $k_2 = 1.5$

The third root is at -0.5 . Not all the roots can be arbitrarily assigned, due to the requirement on the steady-state error.

11-67 (a) Open-loop Transfer Function.

$$G(s) = \frac{X_1(s)}{E(s)} = \frac{k_3}{s[s^2 + (4 + k_2)s + 3 + k_1 + k_2]}$$

Since the system is type 1, the steady-state error due to a step input is zero for all values of k_1 , k_2 , and k_3

that correspond to a stable system. The characteristic equation of the closed-loop system is

$$s^3 + (4 + k_2)s^2 + (3 + k_1 + k_2)s + k_3 = 0$$

For the roots to be at $-1 + j$, $-1 - j$, and -10 , the equation should be:

$$s^3 + 12s^2 + 22s + 20 = 0$$

Equating like coefficients of the last two equations, we have

$$4 + k_2 = 12 \quad \text{Thus} \quad k_2 = 8$$

$$3 + k_1 + k_2 = 22 \quad \text{Thus} \quad k_1 = 11$$

$$k_3 = 20 \quad \text{Thus} \quad k_3 = 20$$

(b) Open-loop Transfer Function.

$$\frac{Y(s)}{E(s)} = \frac{G_c(s)}{(s+1)(s+3)} = \frac{20}{s(s^2 + 12s + 22)} \quad \text{Thus} \quad G_c(s) = \frac{20(s+1)(s+3)}{s(s^2 + 12s + 22)}$$

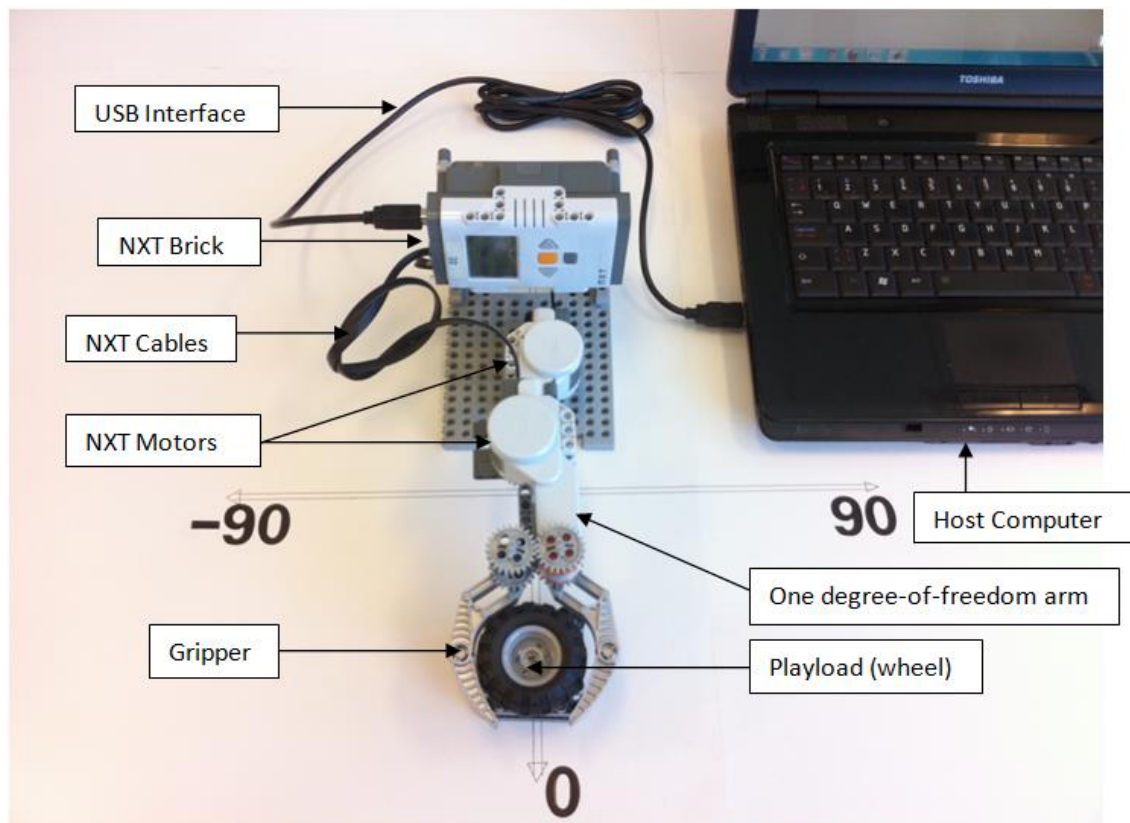
11-68 Term Project problems:**1 Introduction**

Figure 1: Robotic arm test bed with payload

This system, as shown in **Figure 1**, is composed of the dc motor used throughout chapter 5 and chapter 6. We connect a rigid beam to the motor shaft to create simple robotic system conduction a pick-and-place operation. A gripper is attached to the end of the arm and is able to grip onto a payload and drop it at as specified angular position. To limit the overshoot, settling time and rising time, let us design a controller for this system.

2 Proportional Control**Time domain analysis**

Consider the model of the robot arm position control system, where we added a proportional controller in the forward path. The new system block diagram is shown in **Figure 2**.

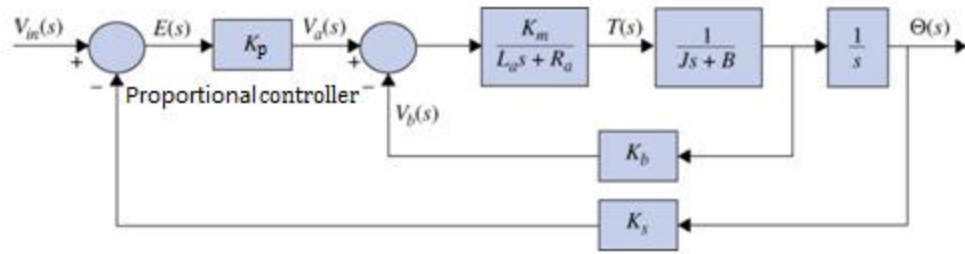


Figure 2: Block diagram of a position-control dc motor

The system transfer function is

$$\frac{\theta(s)}{\theta_{in}(s)} = \frac{\frac{K_p K_m K_s}{R_a}}{\left(\frac{L_a}{R_a} s + 1\right) \{J s^2 + \left(B + \frac{K_b K_m}{R_a}\right) s + \frac{K_p K_m K_s}{R_a}\}} \quad (1)$$

Where K_s is the sensor gain, and, as before, $\frac{L_a}{R_a}$ may be neglected for small L_a .

$$\frac{\theta(s)}{\theta_{in}(s)} = \frac{\frac{K_p K_m K_s}{R_a J}}{s^2 + \left(\frac{R_a B + K_b K_m}{R_a J}\right) s + \frac{K_p K_m K_s}{R_a J}} \quad (2)$$

The motor used in this experiment is a NXT dc motor with the following parameters:

K_m = Motor (torque) constant 0.25 Nm/A

K_b = Speed Constant 0.25V/rad/sec

B = viscous-friction coefficient 1.7×10^{-3} kg m²/sec

R_a = Armature resistance 2.27Ω

L_a = Armature inductance 0.23 mH

J = Armature moment of inertia 2.7×10^{-3} kg-m²

τ_m = Motor mechanical time constant 0.094 sec

K_s = Sensor gain 1

K_p = Gain of the proportional controller

Substituting the preceding values into the system transfer function, we get the closed-loop function is

$$T(s) = \frac{40.789(K_p)}{s^2 + 10.827s + 40.789(K_p)} \quad (3)$$

The characteristic equation is written

$$s^2 + 10.827s + 40.789(K_p) \quad (4)$$

The root loci of Eq. (4) are shown in **Figure 3**.

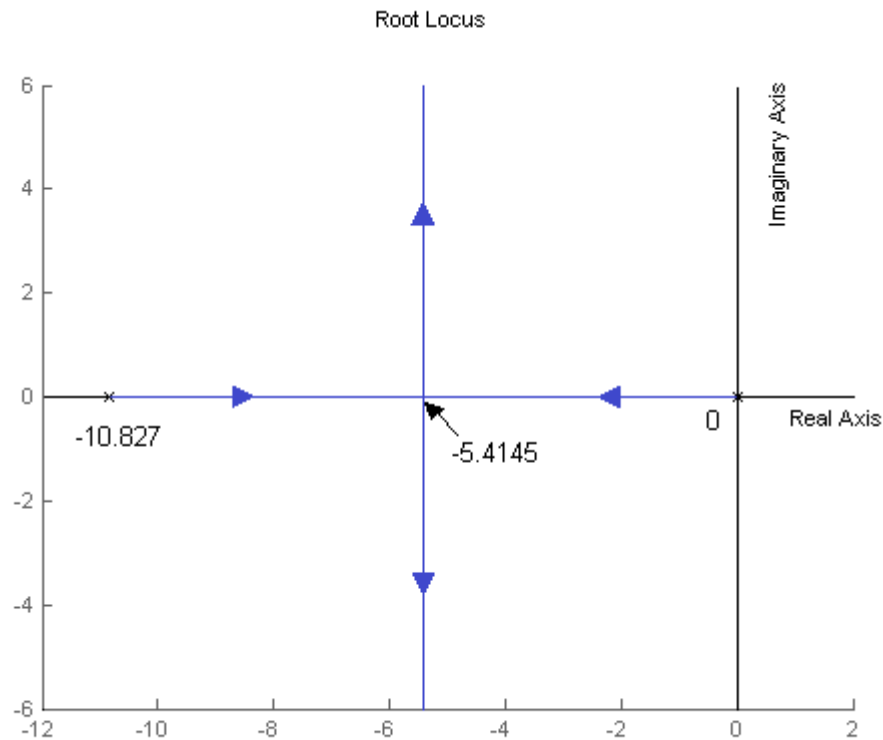


Figure 3: Root locus of Eq. (4)

By looking at the root loci in **Figure 3**, we see that, depending on the value of K_p , we can get two real or two complex-conjugate poles. When $K_p = 0.719$, we get a damping ratio of 1, and the system is critically damped. For values of K_p greater than 0.719, the system is underdamped, and for values of K_p less than 0.719, the system is overdamped. We also note that the system is stable for all the values of $K_p > 0$. The step response of the system for three different values of K_p are shown in **Figure 4**.

Toolbox 1

Figure 3 is obtained by the following sequence of MATLAB function

```
kp = 0;
```

```
num = [40.789];
```

```
den = [1 10.827 40.789*kp];
```

```
rlocus (num, den)
```

Figure 4 is obtained by the following sequence of MATLAB function

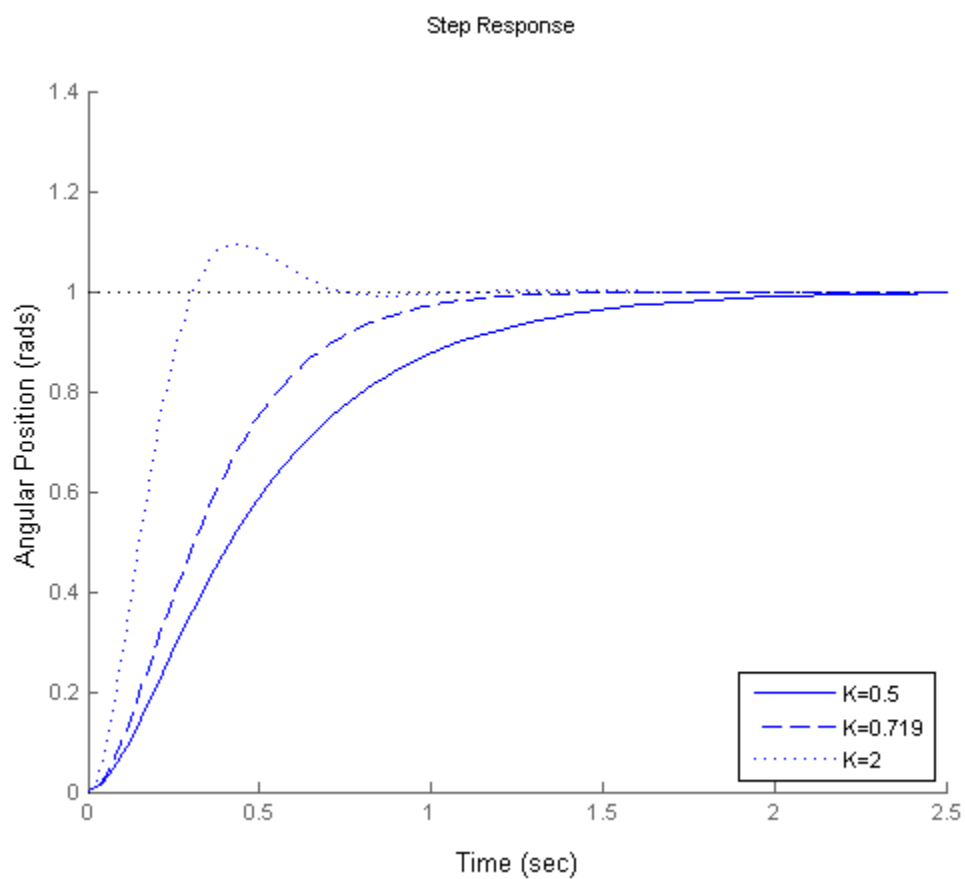


Figure 4: Unit-step responses with proportional control

Table 1 Attributes of the Unit-Step Response with Proportional control

K_P	Percent Overshoot	Settling Time (5%) t_s (sec)	Rise Time t_r (sec)	Steady-State Error Due to Unit Step
Gain = 2	9.5%	0.58 sec	0.21 sec	0
Gain = 0.719	0%	0.86 sec	0.59 sec	0
Gain = 0.5	0%	1.37 sec	0.96 sec	0

Table 1 summarizes the attributes of the system's unit-step response for three different values of K_P .

When poles are complex conjugates, as we increase K_P , the overshoot of the system increase, but the settling time and rise time of the system decrease. Also note that steady-state error due to unit-step input decreases as K_P increase.

Using the model show in **Figure 5-69**, the closed-loop position response of the motor with play load is simulated for a step input 1 rad or 57.296 degrees. The results are shown below in **Figure 5** for multiple proportional gains. The steady-state error due to unit step input decreased as K_P increase. The big difference between the simulation results and the real time NXT motor result may be caused by the viscous-friction coefficient. In reality, the friction may not be viscous. We make an approximation linear model for the viscous-friction coefficient.

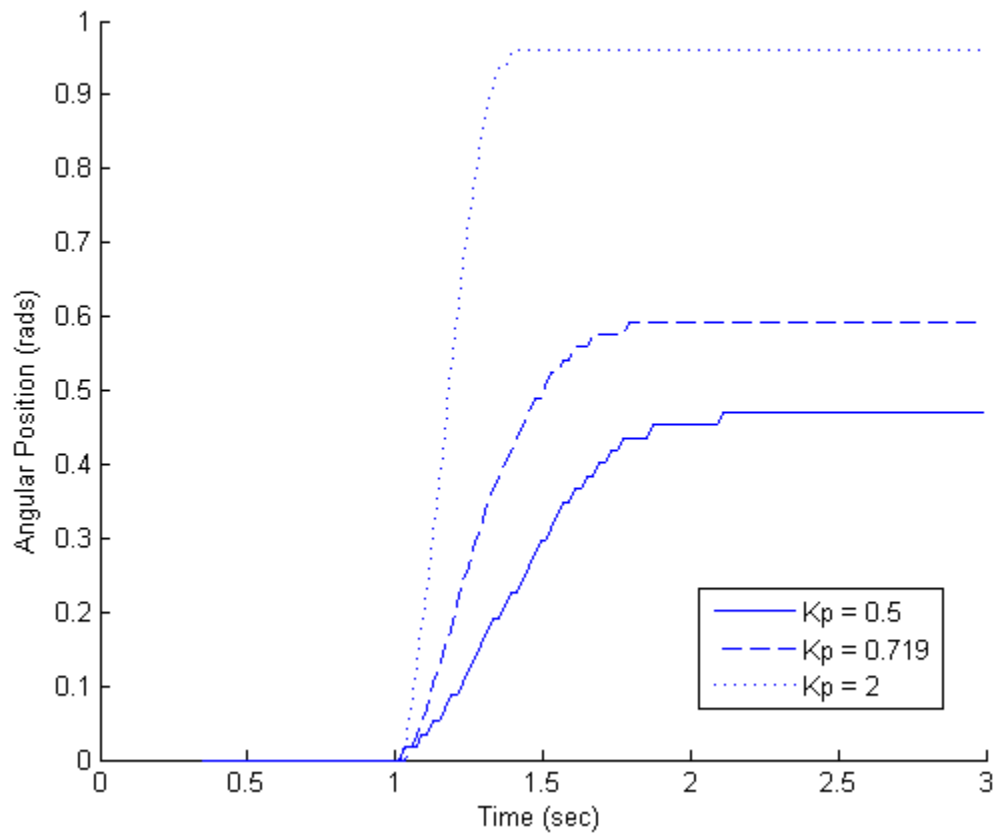


Figure 5: NXT motor closed loop position response results for multiple proportional gains

Frequency Domain Analysis

Now let us carry out the design of the proportional controller in the frequency domain. The bode plots of $T(s)$ for $K_p = 0.5$, 0.719 , and 2 are shown in **Figure 6**. The performance measures in the frequency domain for the compensated system with these controller parameters are tabulated in **Table 2**, along with the time domain attributes for comparison. The bode plots as well as the performance data were easily generated by using MATLAB tools. Use ACSYS component controls to reproduce the results in **Table 2**.

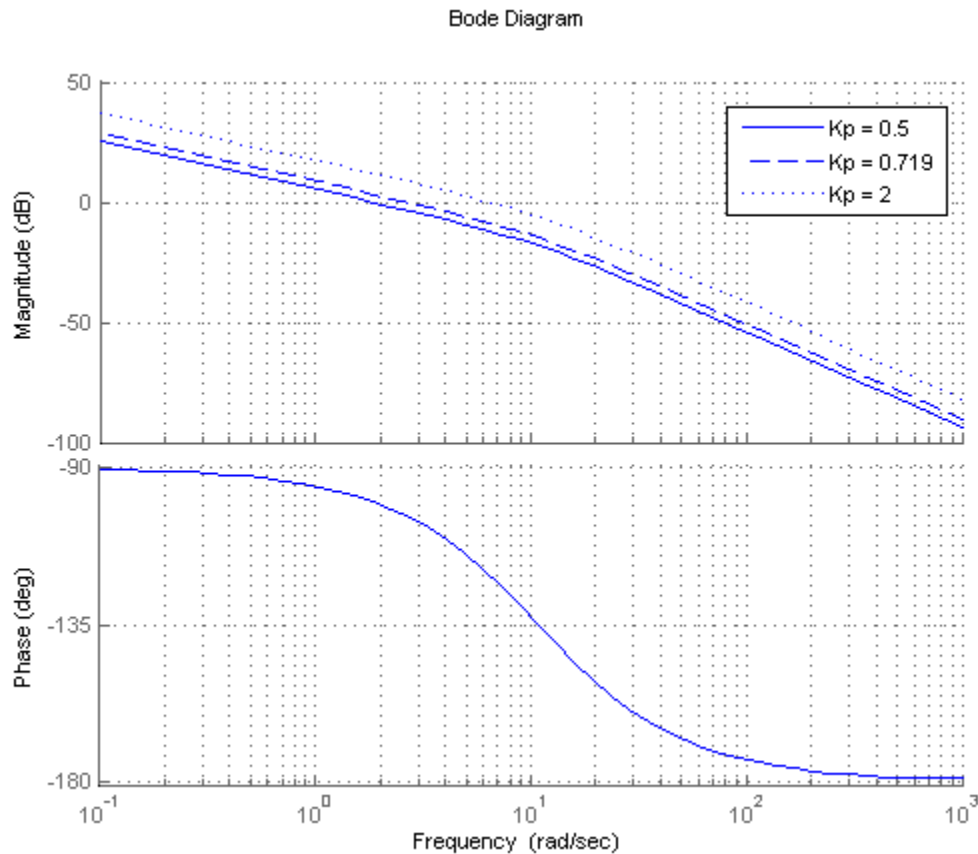


Figure 6: Bode plot of $G(s) = \frac{40.789K_p}{s^2 + 10.827s}$

The results in **Table 2** show that the gain margin is always infinite, and thus the relative stability is measured by the phase margin. This is one example where the gain margin is not an effective measure of the relative stability of the system. When K_p increase the magnitude plots shift up, the phase plots remain same.

Table 2 Frequency Domain Characteristics of the System with Proportional Controller

K_p	GM (dB)	PM (deg)	Gain CO (rad/sec)	BW (rad/sec)	M_r	t_r (sec)	t_s (sec)	Maximum Overshoot (%)
0.5	∞	80.3	1.85	1.86	1	0.96	0.58	0%
0.719	∞	76.1	2.61	2.68	1	0.59	0.86	0%
2	∞	59.1	6.39	6.47	1.04	0.21	0.58	9.5%

3 PD Control

Time Domain Analysis

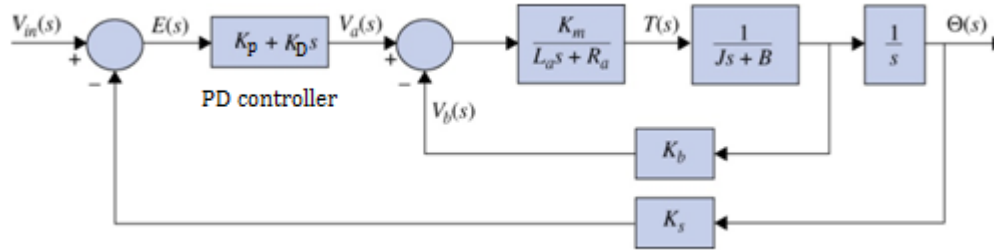


Figure 7: Block diagram of a PD control dc motor

Consider the second-order closed-loop transfer function (3), applying a PD controller, the closed loop transfer function becomes

$$T(s) = \frac{40.789(K_p + K_D s)}{s^2 + (10.827 + 40.789K_D)s + 40.789(K_p)} \quad (5)$$

The characteristic equation is written

$$s^2 + (10.827 + 40.789K_D)s + 40.789(K_p) \quad (6)$$

The forward-path transfer function is

$$G(s) = \frac{40.789(K_p + K_D s)}{s^2 + 10.827s} \quad (7)$$

The system block diagram is shown in **Figure 7**.

Now, let us set the performance specifications as follows:

Settling time $t_s \leq 0.3$ sec

Maximum overshoot $\leq 5\%$

Steady-state error due to unit-ramp input ≤ 0.05

We start by finding the steady-state error for a unit-ramp input:

$$e_{ss|ramp} = \lim_{s \rightarrow 0} \frac{1}{1 + sG(s)} = \frac{1}{1 + \frac{40.789K_p}{10.827}} \quad (8)$$

Therefore, for the system to have steady-state error due to unit ramp ≤ 0.05 , we need $K_p \geq 5$. The damping ratio of the system for $K_p = 5$ can be expressed as

$$\zeta = \frac{10.827 + 40.789K_D}{28.562} = 0.379 + 1.428 K_D \quad (9)$$

If we wish to have critical damping, $\zeta = 1$, the above equation gives $K_D = 0.435$, one thing we need to check is that this value satisfies the settling-time requirement. Settling time can be expressed as

$$t_s = \frac{8}{10.827 + 40.789K_D} \quad (10)$$

We see that, for $K_D \geq 0.388$, we have $t_s \leq 0.3$ sec. Therefore, with $K_D = 0.435$, we can satisfy the settling-time requirement. We note that Eq. (10) is approximation and damping ration comes closer to one, the actual settling time will be higher, therefore, K_D must be higher than 0.388 to satisfy the settling time for damping ratio of one. Nevertheless, we can still use the approximation and verify our answer by simulation once a value for K_D is found. Also, we need to make sure that, for the values for K_p and K_D , the system is stable. Applying the stability requirement, we find that for system stability

$$K_p > 0 \text{ and } K_D > -0.265$$

Alternatively, we can use the system's root contours to find K_p and K_D . We can apply the root-contour method to the characteristic equation in Eq. (6) to examine the effect of varying K_p and K_D . First, by setting K_D to zero, Eq. (6) becomes

$$S^2 + 10.827S + 40.789K_p = 0 \quad (11)$$

The root loci of Eq. (11) are shown in **Figure 8**.

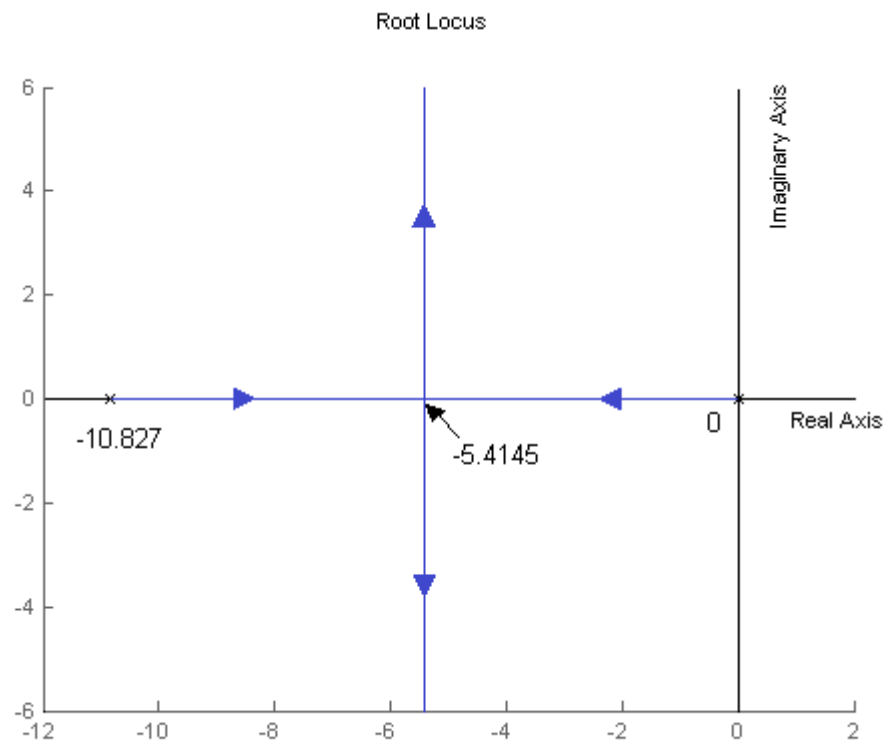


Figure 8: Root locus of Eq. (11)

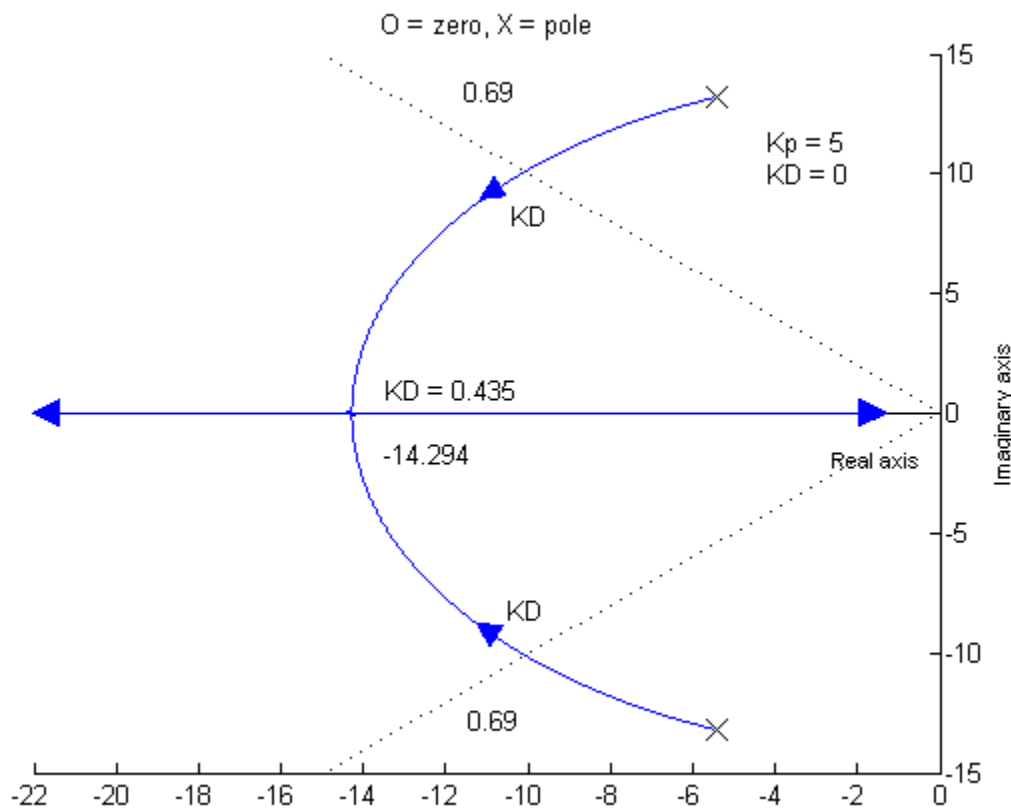


Figure 9: The root contours of Eq. (11) when $K_D = 0$

The code used to plot the root locus is given in Toolbox for $K_D = 0$.

The root contours for $K_p = 1$ and $K_p = 5$ based on the pole-zero configuration are shown in **Figure 9**. Note that we chose $K_p = 5$ to satisfy the steady-state error requirement. For $K_p = 5$ and $K_D = 0$, the characteristic equation roots are at $-5.41 \pm j13.22$ and the damping ratio of the closed-loop system is 0.379. When the value of K_D is increased, the two characteristic equation roots move toward the real axis along a circular arc. The dashed line shows the points on the s-plane with the constant damping ratio of 0.69, which corresponds to 5% overshoot. We see that this line intersects the root contour for $K_p = 5$ at $K_D = 0.245$. Because this value of K_D is not big enough to satisfy the settling-time requirement of less than 0.3 sec, we need a larger K_D . When K_D is increased to 0.435, the roots are real and equal at -14.294, and the damping is critical. At this

point $K_D \geq 0.388$; therefore, our settling-time requirement is met.

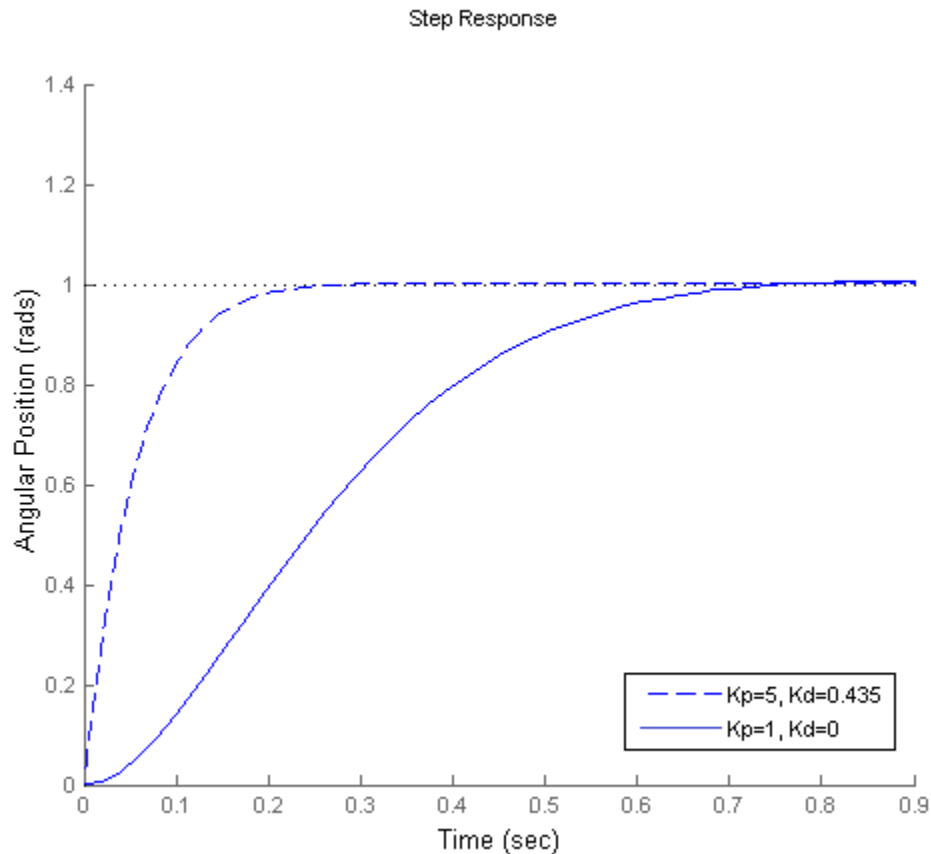


Figure 10: Unit-step responses with and without PD control using Toolbox2

Toolbox 2

Figure 10 is obtained by the following sequence of MATLAB function

```
num=[0.435*40.789 5*40.789];
den=[1 10.827+17.7432 203.9450];
step(num, den);
hold on
kp=1;
num = [0 40.789*kp];
den = [1 10.827 40.789*kp];
step (num , den)
```

Figure 10 shows the unit-step response of the closed-loop system without PD control and with $K_p = 5$ and $K_D = 0.435$. With the PD control, although K_D is chosen for critical damping, the maximum overshoot is 0%. The zero is at $s = -\frac{K_p}{K_D}$ for the closed-loop transfer function. It makes the response faster by lowering the rise time and the settling time of the system.

Table 3 Attributes of the Unit-Step Response with PD control

K_D	Percent Overshoot	Settling Time (5%) t_s (sec)	Rise Time t_r (sec)
Gain = 0.1	15%	0.35 sec	0.116 sec
Gain = 0.435	0%	0.16 sec	0.113 sec
Gain = 1	0%	0.27 sec	0.111 sec

Table 3 summarizes the attributes of the system's unit-step response for $K_p = 5$ and different values of K_D .

Using the model show in **Figure 5-69**, the closed-loop position response of the motor with play load is simulated for a step input 1 rad or 57.296 degrees. The results are shown below in **Figure 11** for $K_p = 5$ and $K_D = 0.435$. The percent overshoot is 0%. The 5% settling time is 0.48 sec. The rising time is 0.35 sec.

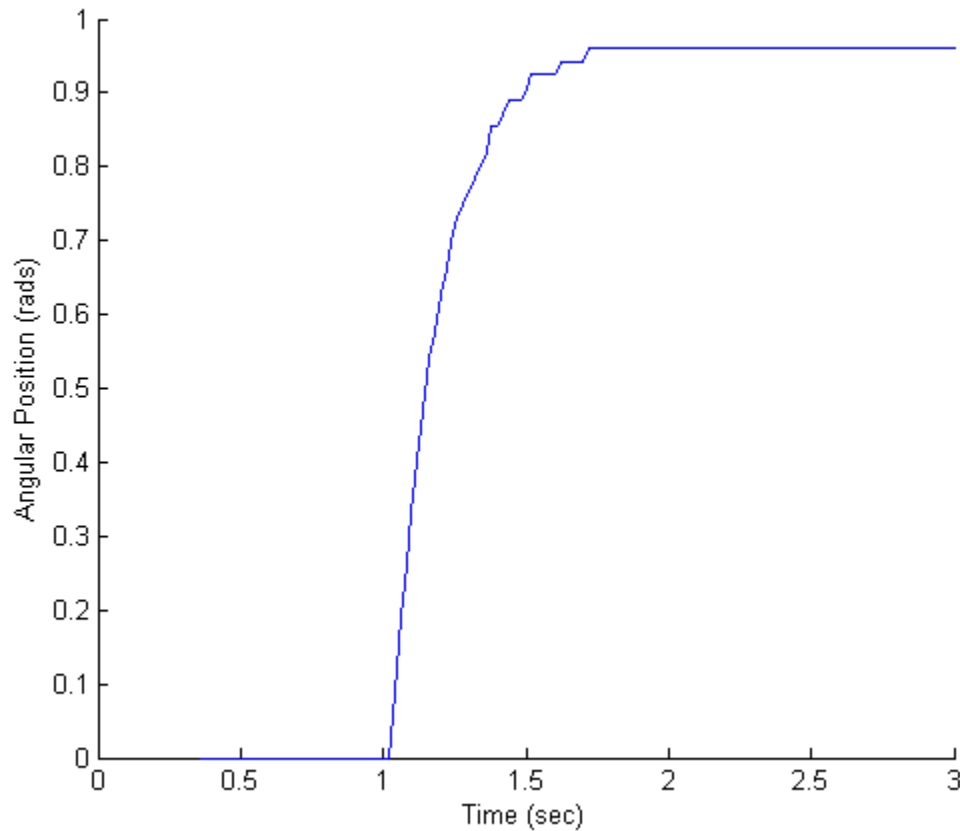


Figure 11: NXT motor closed loop position response results for $K_P = 5$, $K_D = 0.435$

Improve Transient Response via PD Control

We can improve transient response via PD control. In Proportional control, the maximum overshoot is 9.5% when the proportional gain equals to two. The root locus for the proportional control system is shown in **Figure 6**. 9.5% overshoot is equivalent to $\zeta = \sqrt{\frac{\ln(0.095)^2}{\pi^2 + \ln(0.095)^2}} = 0.599$

, we search along that damping ratio line for an odd multiple of 180° and find that the dominant, second order pair of poles is at $-5.41 \pm j7.23 = \omega_o(-\zeta \pm \sqrt{\zeta^2 - 1})$. So, the angle equal to

$$\tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{-\zeta}\right) = 126.8^\circ.$$

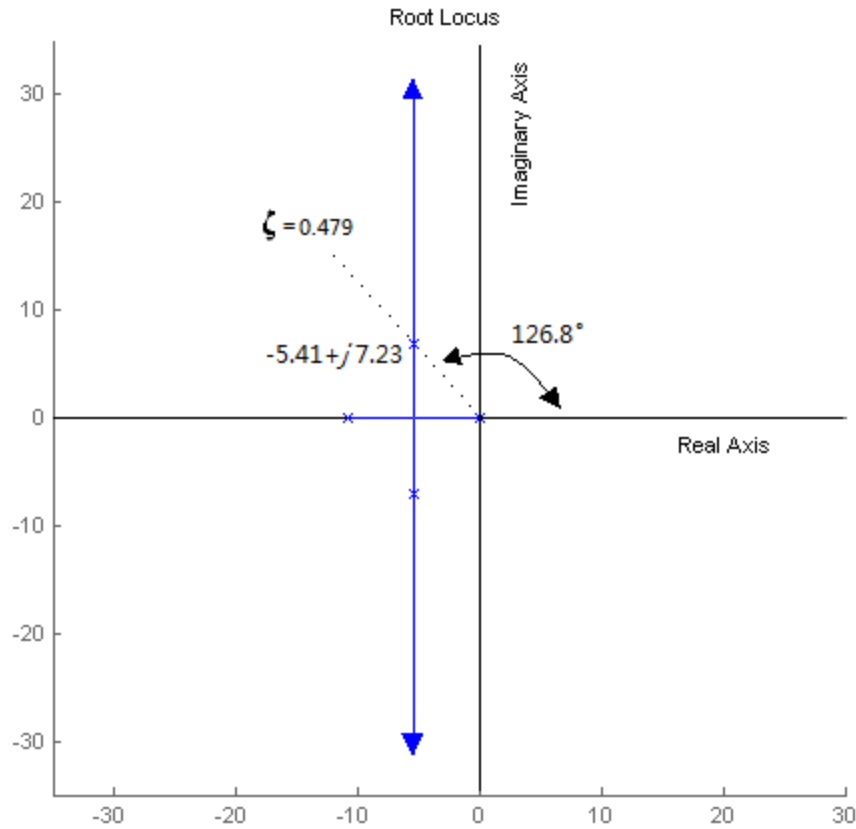


Figure 12: Root locus for proportional control system shown in Figure 2

Thus, the settling time of the uncompensated system is

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{5.41} = 0.739$$

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{7.23} = 0.435$$

The transient and steady state error characteristics of the uncompensated system are summarize in **Table 4**.

Table 4 Uncompensated and compensated system characteristic

	Uncompensated	Compensated
Plant and compensator	$\frac{40.789K_p}{s(s+10.827)}$	$\frac{40.789(K_p + K_D s)}{s(s+10.827)}$
Dominant poles	$-5.41 \pm j7.23$	$-10.840 \pm j14.49$
K_p	2	0.265
ζ	0.599	0.599
ω_n	9.03	18.06
%OS	9.5	9.5
T_s	0.739	0.369
T_p	0.435	0.217
Zero	none	-30.174

Now we proceed to compensate the system. First we find the location of the compensated system's dominant poles. In order to have a twofold reduction in the settling time, the compensated system's settling time will be half of 0.739. The new settling time will be 0.369. Therefore, the real part of the compensated system's dominant, second order pole is

$$\sigma = \frac{4}{T_s} = \frac{4}{0.369} = 10.840$$

Figure 12 shows the designed dominant, second order pole, with a real part equal to and an imaginary part of

$$\omega_d = 10.840 \tan(180^\circ - 126.8^\circ) = 14.490$$

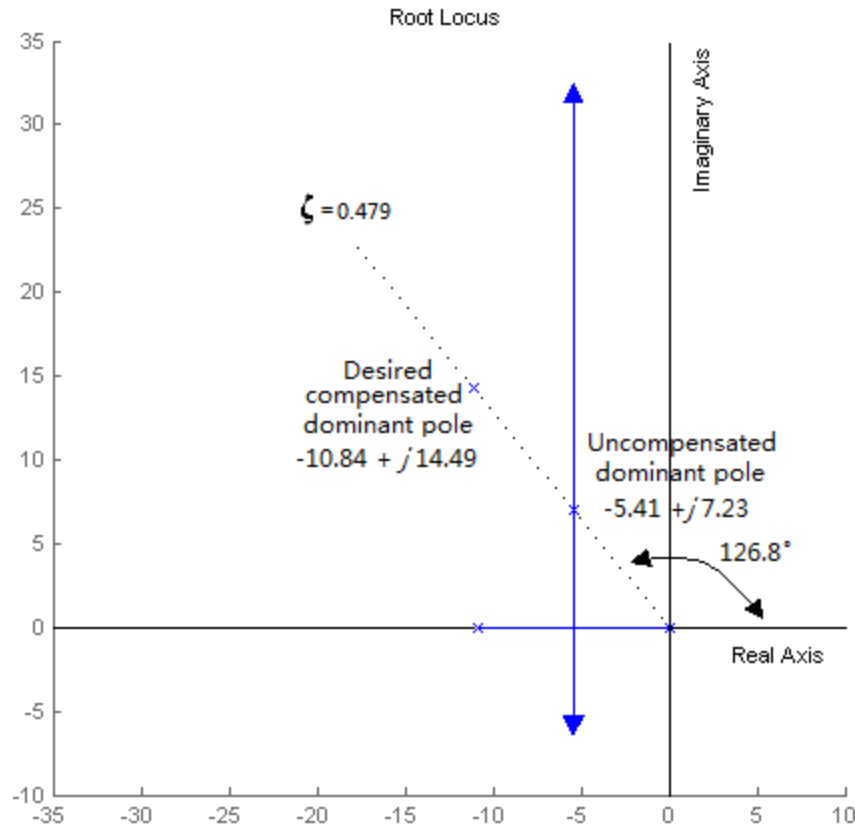


Figure 13: Compensated dominant pole superimposed over the uncompensated root locus

Next we design the location of the compensator zero. Input the uncompensated system's poles and zeros in the root locus program as well as the design point $-10.840 \pm j14.490$ as a test point. The result is the sum of the angles to the design point of all the poles and zeros of the uncompensated system except for those of the compensator zero itself. The difference between the result obtained and 180° is the angular contribution required of the compensator zero. Using the open loop poles shown in **Figure 13** and the test point $-10.840 + j14.490$, which is the desired dominant second order pole, the angular contribution of the pole at 0 is 126.8° ; the angular contribution of the pole at -10.827 is $\tan^{-1}\left(\frac{14.490}{-10.840 - (-10.827)}\right) = 90.05^\circ$. The sum of the angle equals to $90.05^\circ + 126.8^\circ = 216.85^\circ$. Hence, the angular contribution required from the compensator zero for the test point to be on the root locus is $+216.85^\circ - 180^\circ = 36.85^\circ$. The geometry is shown in **Figure 14**, where we now must solve for $-\sigma$, which is the location of the compensator zero.

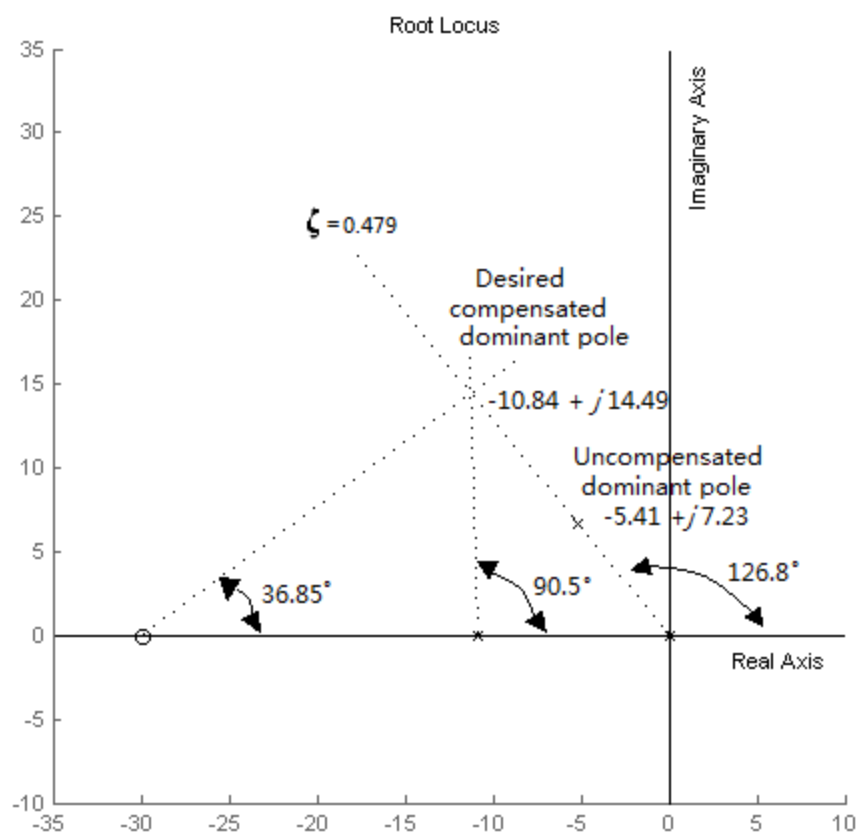


Figure 14: Evaluating the location of the compensating zero

Form the **Figure 14**,

$$\frac{14.49}{\sigma - 10.840} = \tan 36.85^\circ$$

Thus, we get $\sigma = 30.174$. The complete root locus for the compensated system is shown in **Figure 15**.

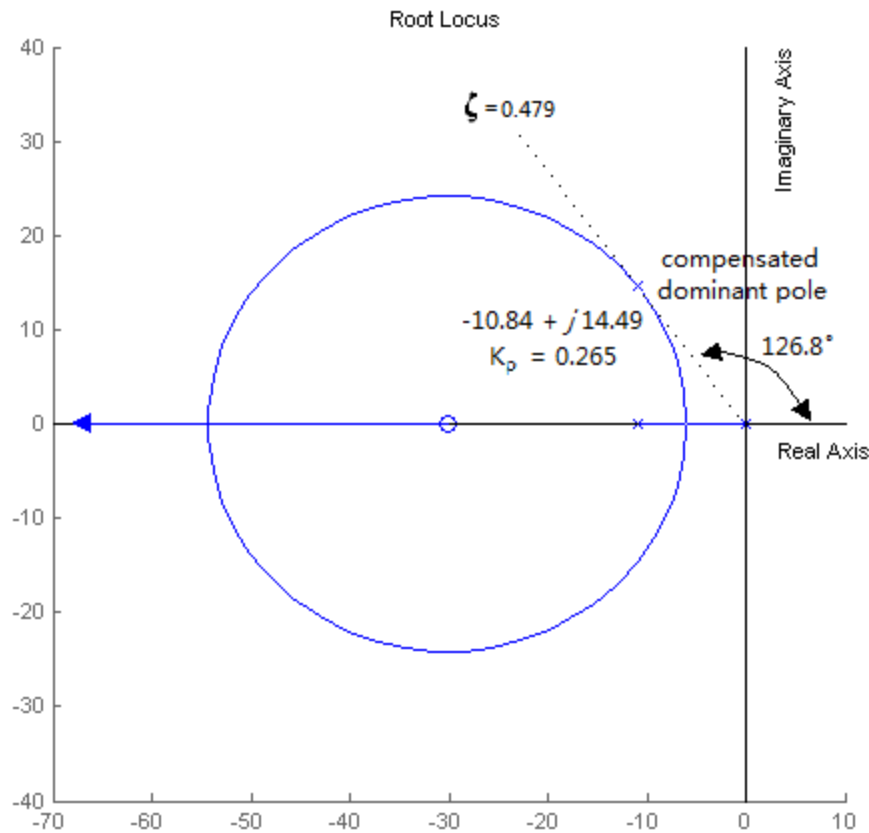


Figure 15: Root locus for the compensated system

Table 4 summarizes the results for both the uncompensated system and the compensated system. The simulation results can be obtained using Matlab or ACSYS. The percent overshoot differs by 3% between the uncompensated and compensated systems, while there is approximately a twofold improvement in speed as evaluated from the settling time.

The final results are displayed in **Figure 16**, which compares the uncompensated system and the faster compensated system.

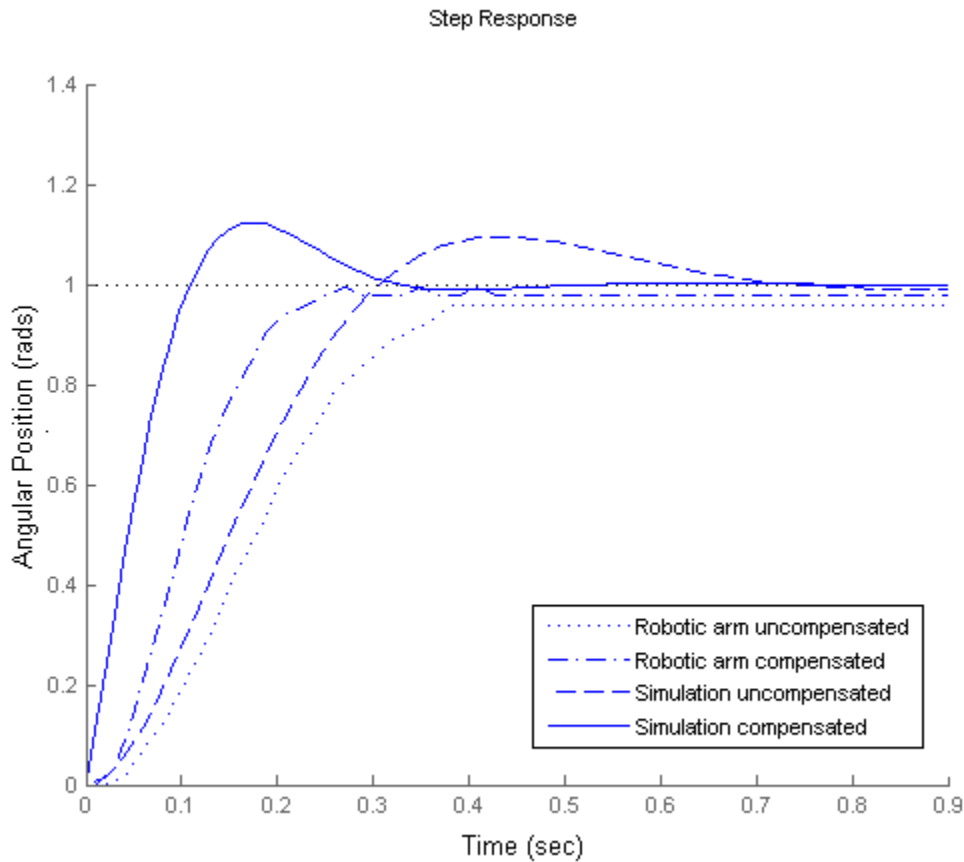


Figure 16: Uncompensated and compensated system step response

Using the model show in **Figure 5-69**, the closed-loop position response of the motor with play load is simulated for a step input 1 rad or 57.296 degrees with $K_p = 2$, $K_D = 0$ and $K_p = 8$, $K_D = 0.265$. The final results are displayed in **Figure 16**, which compares the robotic arm uncompensated system and the faster compensated system. **Table 5** summaries the results for both uncompensated and compensated system step response.

Table5 Robotic arm system step response comparison

		Percent Overshoot	Settling Time (5%)	Peak Time	Steady State Error
Simulated Position Response	$K_p = 2$ $K_D = 0$	10	0.587	0.431	0
	$K_p = 8$ $K_D = 0.265$	12	0.25	0.186	0
Robotic Arm Position Response	$K_p = 2$ $K_D = 0$	0	0.35	0.27	4
	$K_p = 8$ $K_D = 0.265$	1.8	0.20	0.4	2.3

Frequency Domain Analysis

Now let us carry out the design of the PD controller in the frequency domain. **Figure 17** shows the bode plot of $G(s)$ in Eq.7 with $K_P = 5$ and $K_D = 0$. The phase margin of the uncompensated system is 41.2° , and the resonant peak M_r is 1.43. These values correspond to lightly damped system. Let us give the following performance criteria:

Steady state error due to a unit ramp input ≤ 0.05

Phase margin $\geq 80^\circ$

Resonant peak $M_r \leq 1.05$

BW ≤ 20 rad/sec

The bode plots of $G(s)$ for $K_p = 5$ and $K_D = 0, 0.1, 0.435$, and 1 are shown in **Figure17**. The performance measures in the frequency domain for the compensated system with these controller parameters are tabulated in **Table 6**, along with the time domain attributes for comparison. The bode plots as well as the performance data were easily generated by using MATLAB tools. Use ACSYS component controls to reproduce the results in Table.

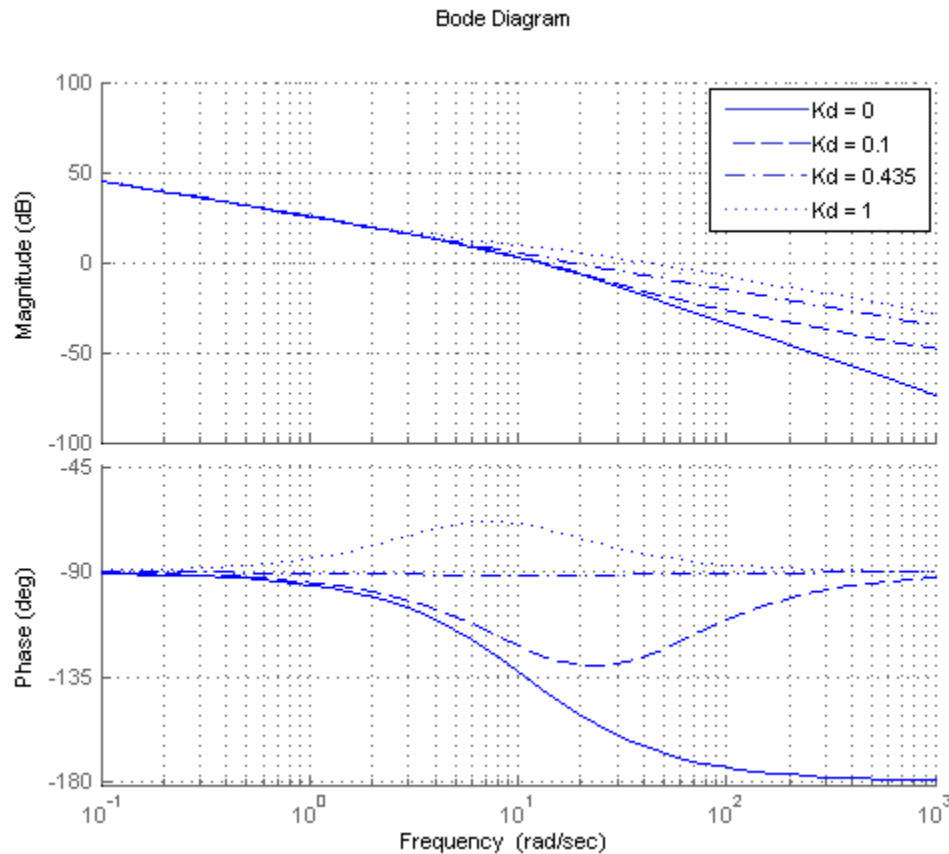


Figure 15: Bode plot of $G(s) = \frac{40.789(K_p + K_D s)}{s^2 + 10.827s}$

The results in **Table 6** show that the gain margin is always infinite, and thus the relative stability is measured by the phase margin. This is one example where the gain margin is not an effective measure of the relative stability of the system. When $K_D = 0.435$, which corresponds to critical damping, the phase margin is 88.5° , the resonant peak M_r is 1, and BW is 18. The performance requirements in the frequency domain are all satisfied. Other effects of the PD control are that the BW and the gain-crossover frequency are increased. The phase-crossover frequency is always infinite in this case.

Table 6 Frequency Domain Characteristics of the System with PD Controller

K_D	GM (dB)	PM (deg)	Gain CO (rad/sec)	BW (rad/sec)	M_r	t_r (sec)	t_s (sec)	Maximum Overshoot (%)
0	∞	41.2	11.9	12.4	1.43	0.117	0.54	28%
0.1	∞	54.8	12.5	12.6	1.14	0.116	0.35	15%
0.435	∞	88.5	18.4	18	1	0.113	0.16	0%
1	∞	98.1	38.9	39.7	1	0.111	0.27	0%

4 PI Control

Time Domain Analysis

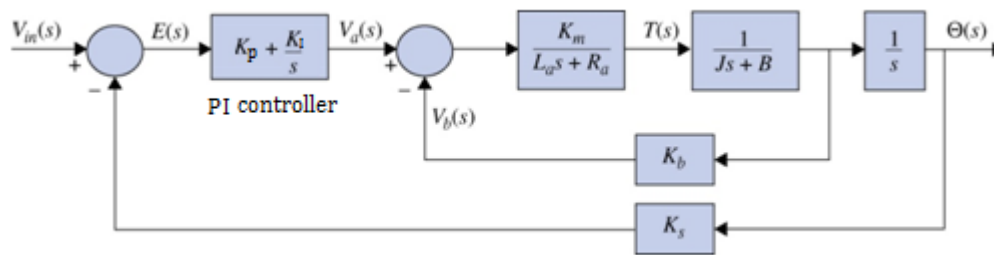


Figure 16: Block diagram of a PI control dc motor

Consider the second-order closed-loop transfer function (3), applying a PD controller, the forward-path transfer function becomes

$$G(s) = \frac{40.789K_p(s + \frac{K_I}{K_p})}{s^2(s + 10.827)} \quad (12)$$

The closed loop transfer function becomes

$$T(s) = \frac{40.789(K_p s + K_I)}{s^3 + 10.827s^2 + 40.789(K_p s) + 40.789K_I} \quad (13)$$

The characteristic equation is written

$$s^3 + 10.827s^2 + 40.789(K_p s) + 40.789K_I \quad (14)$$

The system block diagram is shown in **Figure 18**.

Now, let us set the performance specifications as follows:

Settling time $t_s \leq 1.5$ sec

Rise time $t_r \leq 0.3$ sec

Maximum overshoot $\leq 10\%$

Steady-state error due to parabolic input ≤ 0.7

We start by finding the steady-state error for a unit-ramp input:

$$e_{ss|parabolic} = \lim_{s \rightarrow 0} \frac{1}{s^2 G(s)} = \frac{1}{\frac{40.789K_I}{10.827}} \quad (15)$$

Therefore, for the system to have steady-state error due to parabolic ramp ≤ 0.7 , we need $K_I \geq 0.379$. Later we need to make sure that the value of K_I used is above 0.379.

Applying Routh's test to Eq. 14 yields the result that the system is stable for $0 < \frac{K_I}{K_p} < 10.827$.

This means that, if the zero of $G(s)$ be placed too far to the left in the left-half s -plane, the system will be unstable.

Let us place the zero at $-K_I/K_p$ relatively close to the origin. For this case, the most significant poles of the forward-path transfer function without the PI controller are at -10.827 and 0. Thus K_I/K_p should be chosen so that the following condition is satisfied

$$\frac{K_I}{K_p} \ll 10.827 \quad (16)$$

The root loci of Eq. (12) with $K_I/K_p = 0.3$ are shown in **Figure 19**. Notice that one of the poles always has a value close to zero while the other two poles behave the same as those shown in **Figure 8**, which is for Eq. (11).

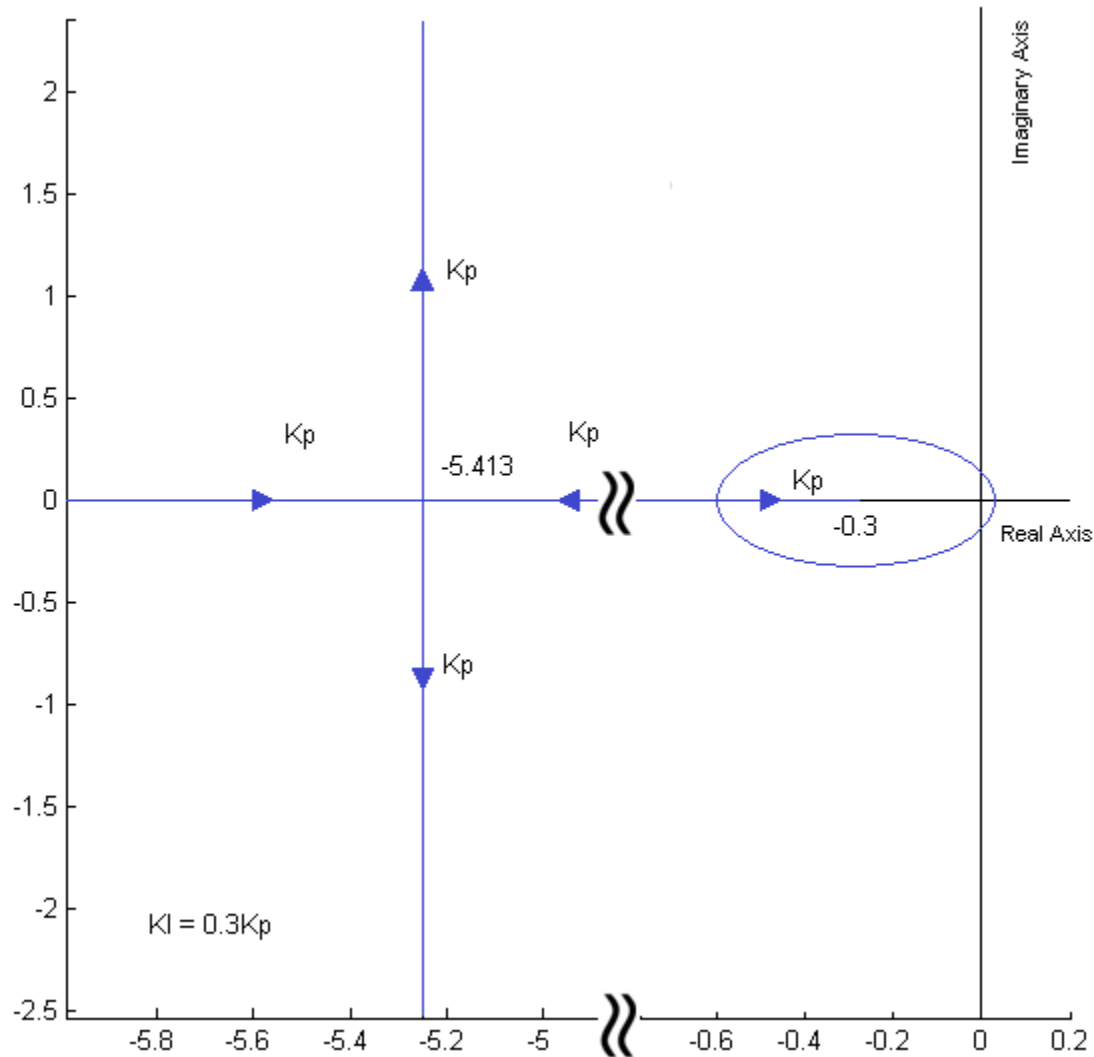


Figure 17: Root loci of Eq. (13)

With the condition in Eq. (16) satisfied, the pole near zero is effectively cancelled by the zero at $-K_I/K_p$, and we are left with the poles of Eq. (11). Therefore, Eq. (12) can be approximated by

$$G(s) \cong \frac{40.789K_p}{(s^2 + 10.827s + 40.789K_p)} \quad (17)$$

Let us assume we wish to have a relative damping ratio of 0.7. From Eq. (17), the required value of K_p for the damping ratio is 1.47. Thus with $K_p = 1.47$ and $K_I = 0.44$, we find the roots of the characteristic equation of Eq. (14):

$$s = -0.31 - 5.26 + j5.39 \text{ and } -5.26 - j5.39$$

In this case we see that the real pole of the closed-loop system is very close to the zero at $-K_I/K_p$ so that the transient due to the real pole is negligible, and the system dynamics are essentially dominated by the two complex poles; therefore, $K_p = 1.47$ will give us a damping ratio that is close to 0.7. In general, when s takes on values along the operating points on the complex portion of the root loci, we can neglect the effect of the real pole of the closed-loop system and use Eq. (17) to find the system characteristics.

Table 7 summarizes the attributes of the unit step response for various values of $-K_I/K_p$ with $K_p = 1.47$, which corresponds to a relative damping ratio of 0.7. The results verify that PI control reduces overshoot at the expense of longer rise time.

Figure 20 shows the unit-step response of the robot arm position control system with $K_p = 1.47$ and $K_I = 0.44$.

Table7 Attributes of the Unit-Step Response with PI control

K_I/K_p	K_I	K_p	Percent Overshoot	Settling Time (5%) $t_s(\text{sec})$	Rise Time $t_r(\text{sec})$
0	0	5	28%	0.54	0.127
3	4.41	1.47	46%	1.38	0.192
1	1.47	1.47	22%	1.44	0.238
0.5	0.74	1.47	14%	1.33	0.266
0.3	0.44	1.47	10%	1.01	0.297
0.1	0.15	1.47	6%	0.7	0.312
0.05	0.074	1.47	5%	0.37	0.314

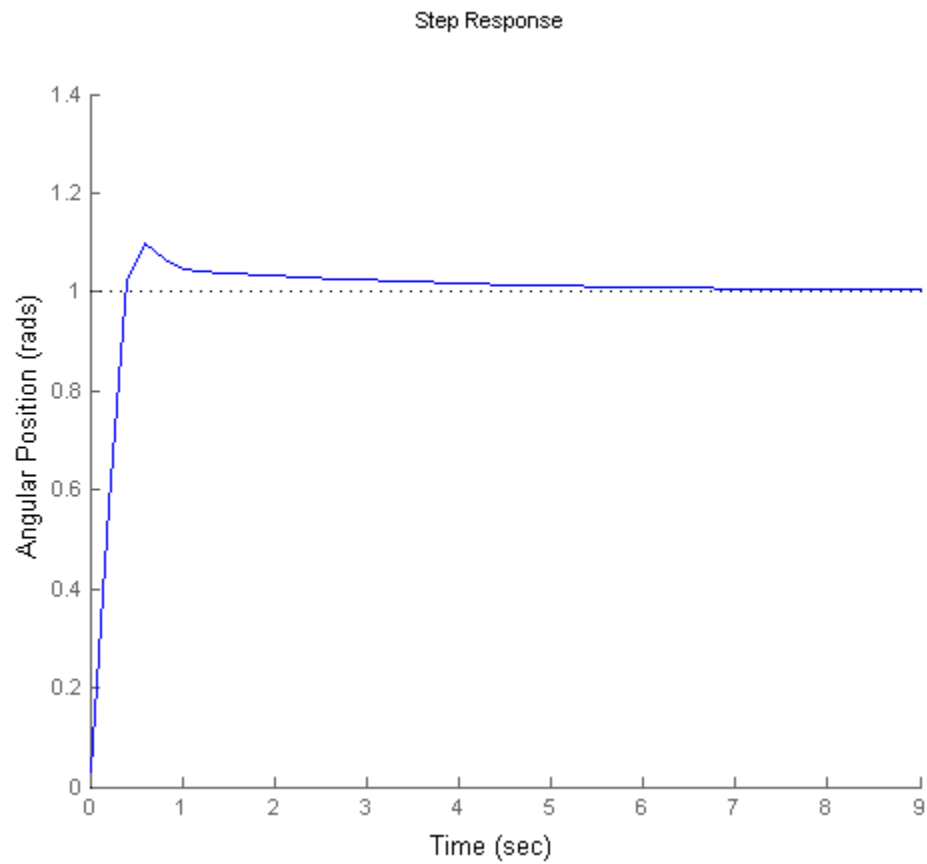


Figure 18: Unit responses of the system with PI control

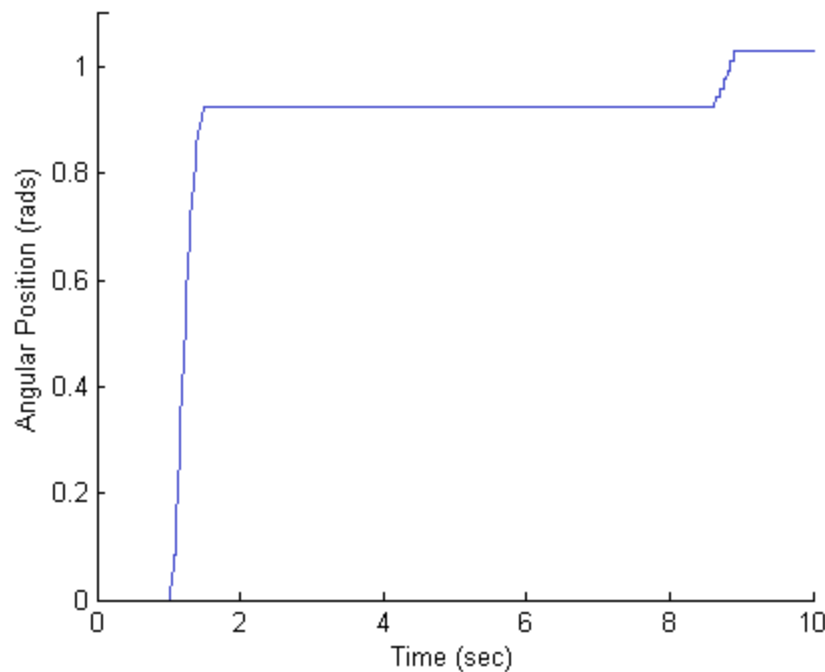


Figure 19: NXT motor closed loop position response results for $K_P = 1.47$, $K_I = 0.44$

Using the model shown in **Figure 5-69**, the closed-loop position response of the motor with play load is simulated for a step input 1 rad or 57.296 degrees. The results are shown below in **Figure 21** for $K_p = 1.47$ and $K_I = 0.44$. The percent overshoot is 0%; the 5% settling time is 7.76 sec; the rising time is 0.36 sec. The integral gain can eliminate the steady state error. We can add the integral gain to the proportional controller to eliminate the steady state error for the NXT motor closed loop response results as in **Figure 5**. The NXT motor closed loop response results for $K_D = 1$, and $K_p = 0.5$, 0.733 and 2 are shown in **Figure 22**. Comparing **Figure 22** and **Figure 5**, we can tell the steady error decreased with the PI controller.

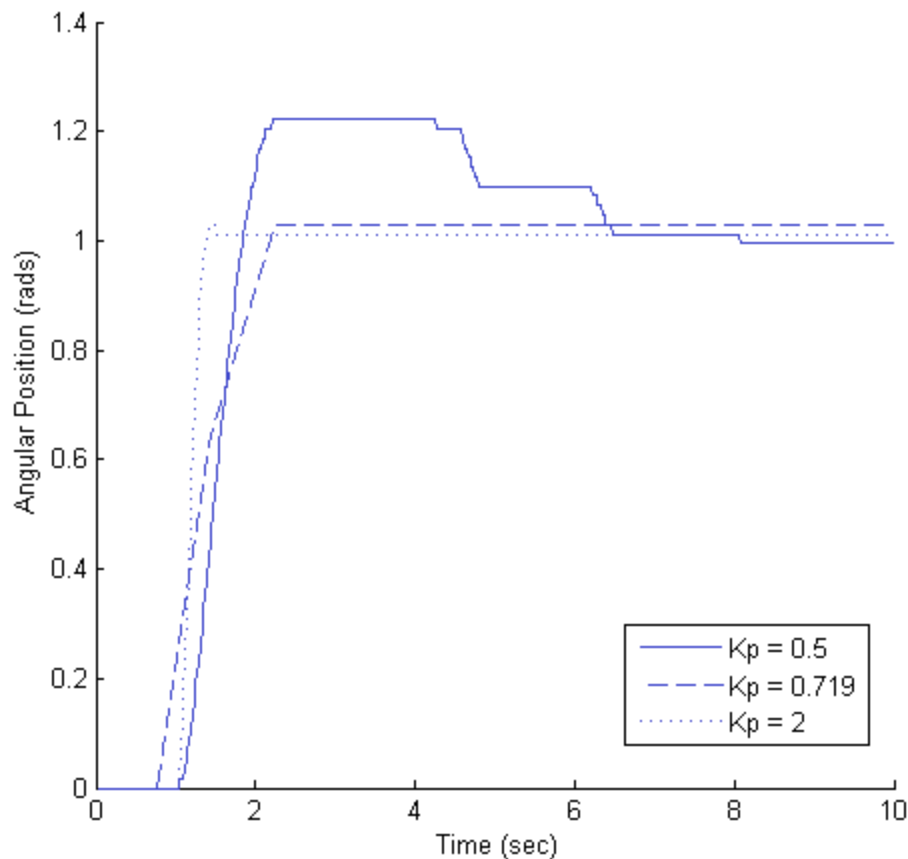


Figure 20: NXT motor closed loop position response results for $K_I = 1$, and $K_P = 0.5, 0.719$ and 2

Frequency Domain Design

Now let us carry out the design of the PI controller in the frequency domain. The forward path transfer function of the uncompensated system is obtained by setting $K_P = 5$ and $K_I = 0$ in the $G(s)$ in Eq. 12, and the bode plot is shown in **Figure 23**. The phase margin is 41.2° , and the gain crossover frequency is 12.34 rad/sec. Let us specify that the required phase margin should be at least 65° , and this is to be achieved with the PI controller. We conduct the following steps:

Look for the new gain crossover frequency ω'_g at which the phase margin of 65° is realized. From **Figure 23**, ω'_g is found to be 4.99 rad/sec. The magnitude of $G(j\omega)$ at this frequency is 10.63 dB. Thus, the PI controller should provide an attenuation of -10.63 dB at $\omega'_g = 170$ rad/sec. Solving for K_P , we get

$$K_P = 5 * 10^{-|G(j\omega'_g)|_{dB}/20} = 5 * 10^{10.63/20} = 5 * 0.294 = 1.47$$

Notice that, in the time domain design conducted earlier, K_p was selected to be 1.47 so that the relative damping ratio of the complex characteristic equation roots will be approximately 0.7.

Let us choose $K_p = 1.47$, so that we can compare the design results of the frequency domain with those of the time domain design obtained earlier. Once K_p is determined,

$$K_I = \frac{w'_g K_p}{10} = \frac{4.99 \cdot 1.47}{10} = 0.73$$

As pointed out earlier, the value of K_I is not rigid, as long as the ratio K_I/K_p is sufficiently smaller than the magnitude of the pole of $G(s)$ at 10.827. As it turns out, the value of K_I given above is not sufficiently small for this system.

Table8 Attributes of the Unit-Step Response with PI control

K_I/K_p	K_I	K_p	GM (dB)	PM (deg)	M_r	BW (rad/sec)	Gain CO (rad/sec)	Phase CO (rad/sec)
0	0	5	∞	41.2	1.43	12.4	11.9	∞
3	4.41	1.47	$-\infty$	34.5	1.78	5.59	5.54	0
1	1.47	1.47	$-\infty$	53.7	1.20	5.1	4.9	0
0.5	0.74	1.47	$-\infty$	59.3	1.10	5.04	4.87	0
0.3	0.44	1.47	$-\infty$	61.7	1.06	5.03	4.86	0
0.1	0.15	1.47	$-\infty$	63.9	1.02	5.02	4.85	0

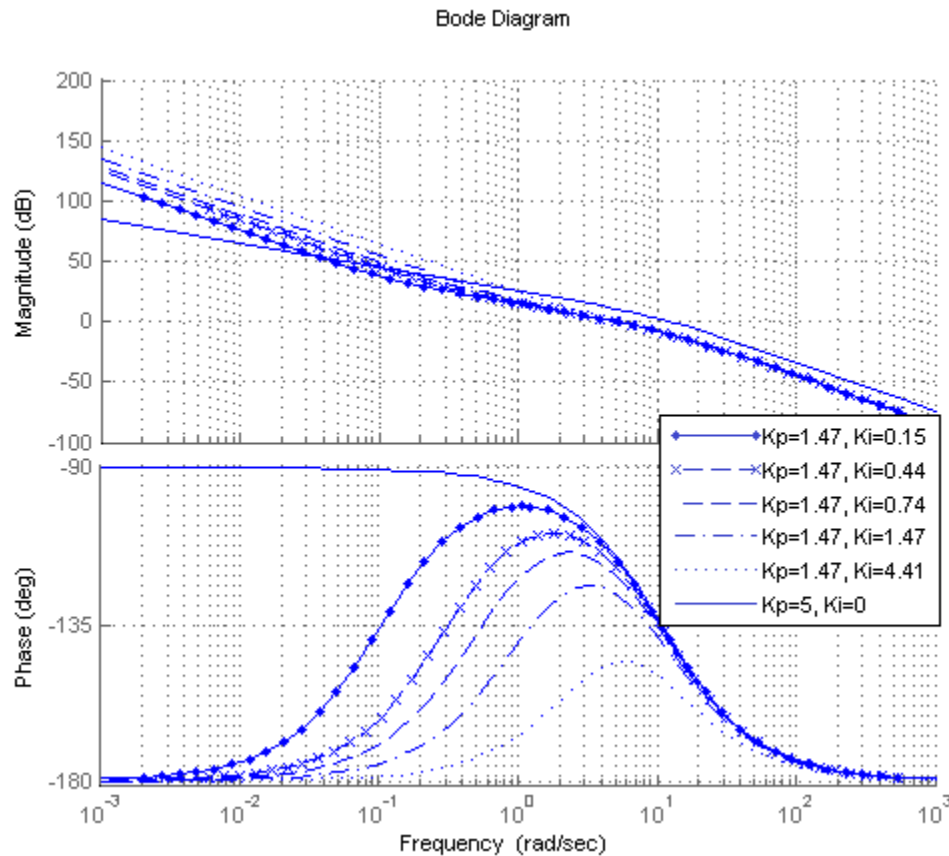


Figure 21: Bode Plot of $G(s) = \frac{40.789K_p(s + \frac{K_I}{K_p})}{s^2(s + 10.827)}$

Toolbox 3

Figure 23 is obtained by the following sequence of MATLAB function

```
ki = [ 0.15 0.44 0.74 1.47 4.41];
```

```
kp=1.47;
```

```
for i=1:length(ki)
```

```
num=[40.789*kp 40.789*ki(i)];
```

```
den=[1 10.827 0 0];
```

```
bode(num, den)
```

```
hold on
```

```
end
```

```
grid
```

```
hold on
```

```
num=[40.789*5];
```

```
den=[1 10.827 0];
```

```
bode (num, den)
```

The bode plot of the forward path transfer function with $K_p = 1.47$ and $K_I = 0.15, 0.44, 0.74, 1.47$, and 4.41 are shown in **Figure 23**. **Table 8** shows the frequency domain properties of the uncompensated system and the compensated system with various values of K_I . Notice that, for values of K_I/K_p that are sufficiently small, the phase margin, M_r , BW, and gain-crossover frequency all vary little.

It should be noted that the phase margin of the system can be improved further by reducing the value of K_p below 1.47 . However, the bandwidth of the system will be further reduced.

5 PID control

Time Domain Analysis

Consider the second-order closed-loop transfer function discussed previously. Applying a PID controller, the forward-path transfer function becomes

$$G(s) = \frac{40.789(K_{p1} + K_{D1}s)(K_{p2} + \frac{K_{I2}}{s})}{s(s + 10.827)} \quad (18)$$

The system block diagram is shown in **Figure 24**.

Let the time-domain performance requirements are as follows:

Rise time $t_r \leq 0.3$ sec

Settling time $t_s \leq 1.5$ sec

Maximum overshoot $\leq 5\%$

Steady-state error due to parabolic input ≤ 0.7

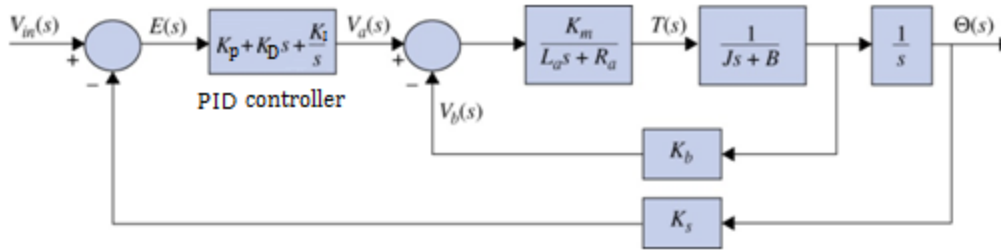


Figure 22: System block diagram with PID controller

We realize that we need a PID controller to fulfill the above requirements. First, we apply the PD control with the transfer function $K_{p1} + K_{D1}s$. The forward-path transfer function becomes

$$G(s) = \frac{40.789(K_{p1} + K_{D1}s)}{s(s + 10.827)} \quad (19)$$

Table 3 shows that the PD controller that can be obtained from the settling time stand point is with $K_{D1} = 0.435$ and $K_{P1} = 5$, and the maximum overshoot is 0 sec. The rise time and settling time are well within required values. Next we add the PI controller, and the forward-path transfer function becomes

$$G(s) = \frac{17.743K_{p2}(s + 11.494)(s + \frac{K_{I2}}{K_{p2}})}{s(s^2 + 10.827s)} \quad (20)$$

Following the guidelines of choosing a relatively small value for K_{I2}/K_{p2} , we let $K_{I2}/K_{p2} = 0.3$. Eq. (20) becomes

$$G(s) = \frac{17.743K_{p2}(s + 11.494)(s + 0.3)}{s(s^2 + 10.827s)} \quad (21)$$

Table 9 gives the tie domain performance characteristics along with the roots of the characteristic equation for various values of K_{p2} .

Setting $K_{p1} = 5$, $K_{D1} = 0.435$, $K_{p2} = 1$, and $K_{I2} = 0.3K_{p2} = 0.3$, the following results are obtained for the parameters of the PID controller.

$$K_P = 5$$

$$K_I = 0.3$$

$$K_D = 0.435$$

Figure 25 shows the unit-step responses of the system with the PID controller.

Table9 Attributes of the Unit-Step Response with PID control

K_p2	Percent Overshoot	Settling Time (5%) t_s (sec)	Rise Time t_r (sec)	Roots of Characteristic Equation
2	1%	0.075 sec	0.06 sec	-0.3 -11.84 -34.17
1.5	1%	0.09 sec	0.072 sec	-0.3 -12.07 -25.06
1	2%	0.15 sec	0.12 sec	-0.31 -14.12±j0.96
0.7	2%	0.21 sec	0.14 sec	-0.31 -11.47±j2.81
0.5	3%	0.27 sec	0.19 sec	-0.31 -9.69±j2.15
0.2	6%	1.92 sec	0.45 sec	-0.33 -3.55 -10.5

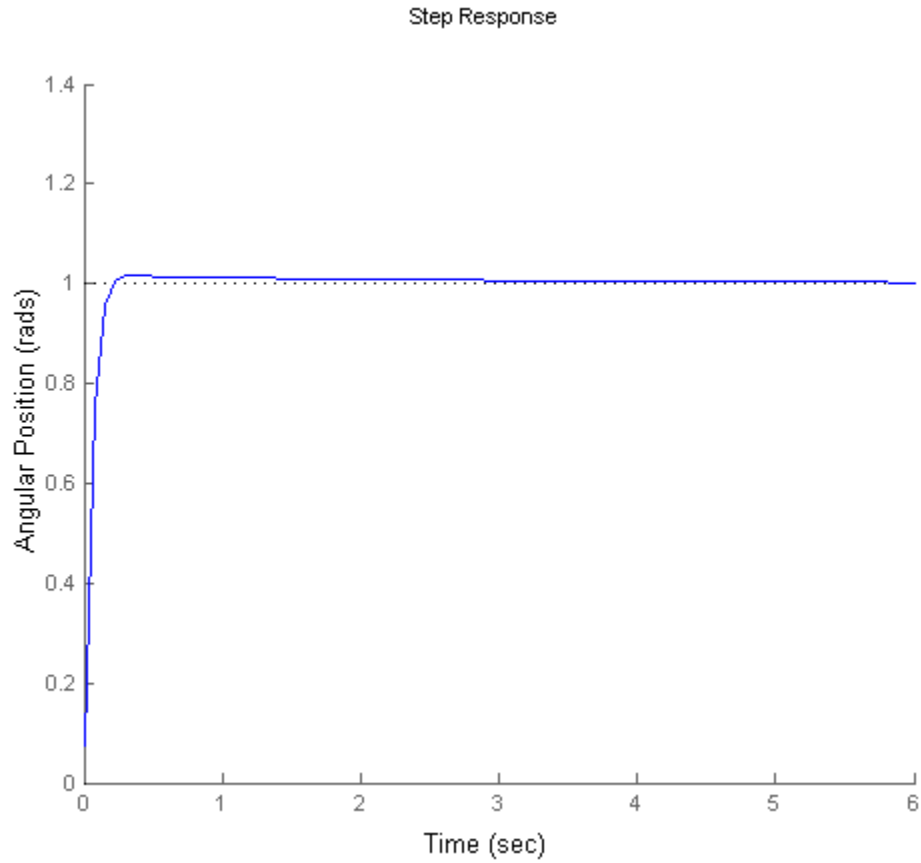


Figure 23: Unit-step response with PID control

Using the model show in **Figure 5-69**, the closed-loop position response of the motor with play load is simulated for a step input 1 rad or 57.296 degrees. The results are shown below in **Figure 26** with $K_p = 5$, $K_I = 0.3$, $K_D = 0.435$. The percent overshoot is 0%; the 5% settling time is 0.62 sec; the rising time is 0.42 sec.

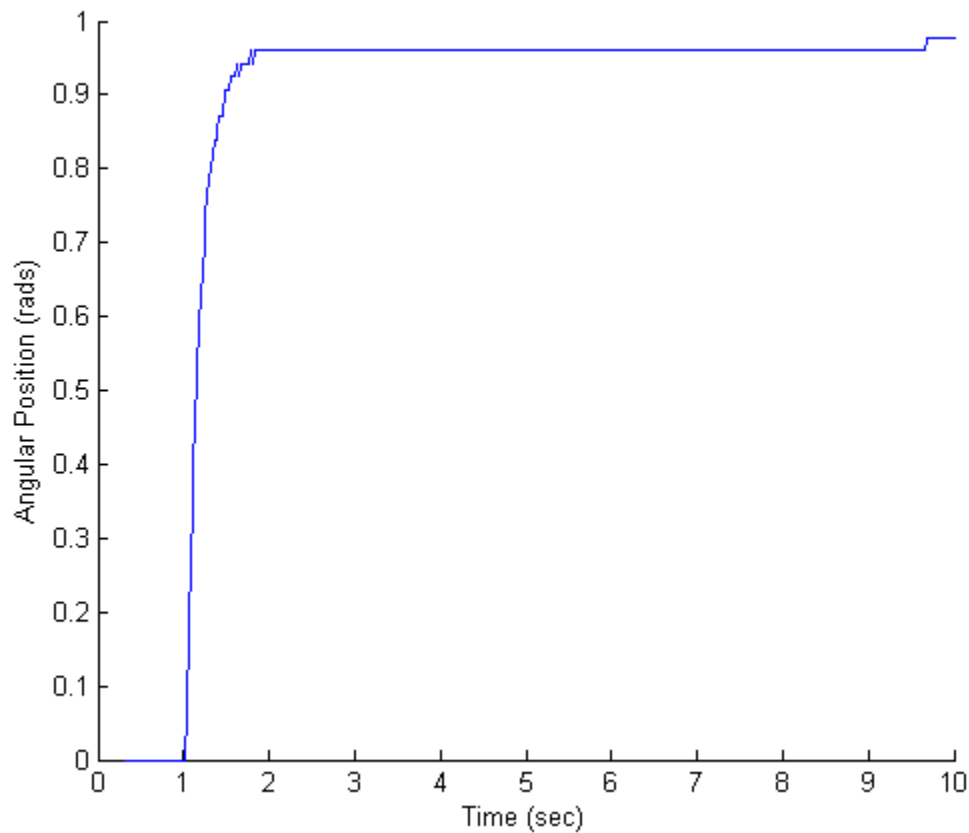


Figure 24 : NXT motor closed loop position response results for $K_P=5$, $K_I=0.3$, $K_D=0.435$

Frequency Domain Analysis

Now let us carry out the design of the PID controller in the frequency domain. **Figure 12** shows the bode plot of $G(s)$ in Eq.7 with $K_P=5$ and $K_D=0.435$. The phase margin of the uncompensated system is 88.5° , and the resonant peak M_r is 1.

Table10 Frequency domain performance of system with PID controller

K_{p2}	K_{I2}	GM (dB)	PM (deg)	M_r	BW (rad/sec)	t_r (sec)	t_s (sec)	Maximum Overshoot(%)
1	0	∞	88.5	1	18	0.113	0.16	0%
1.5	0.45	$-\infty$	88.1	1.01	26.9	0.072	0.09	1%
1	0.3	$-\infty$	87.5	1.01	18	0.12	0.15	2%
0.5	0.15	$-\infty$	86.5	1.03	9.2	0.19	0.27	3%

The bode plot of the forward path transfer function with PD and PID controller are shown in **Figure 27**. **Table 10** shows the frequency domain properties of the compensated system with $K_D = 0.435$, and various value of K_p and K_I .

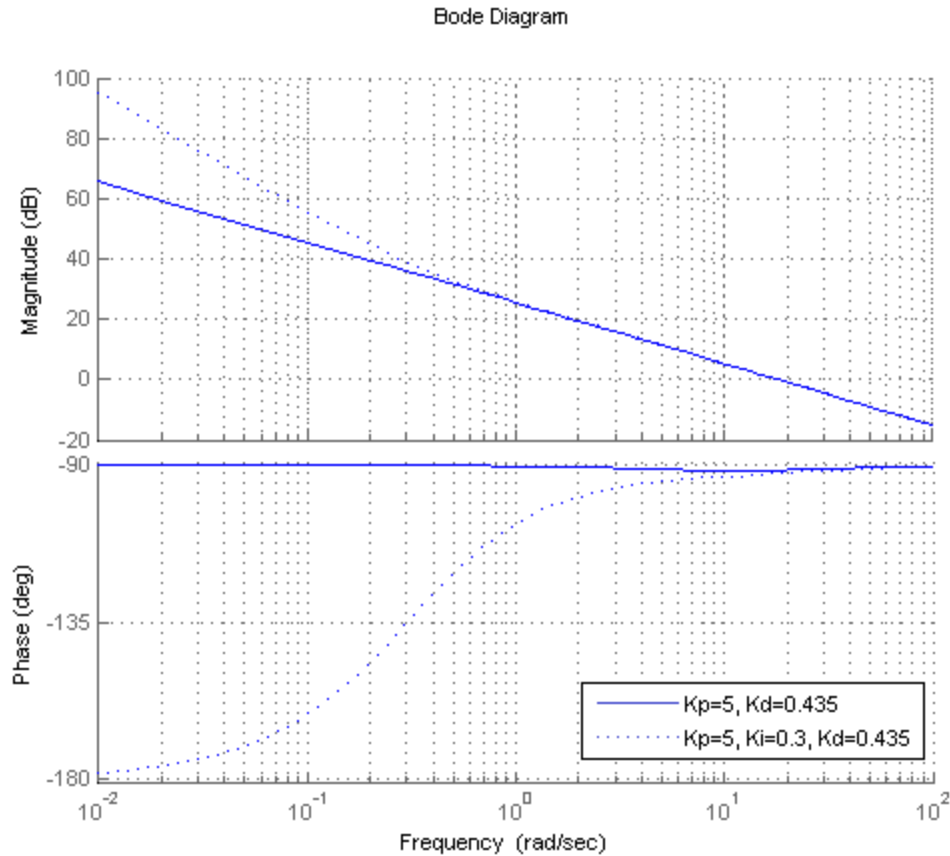


Figure 25: Bode plot of the system with PD and PID controllers

6 Phase Lead Controller

Time Domain Analysis

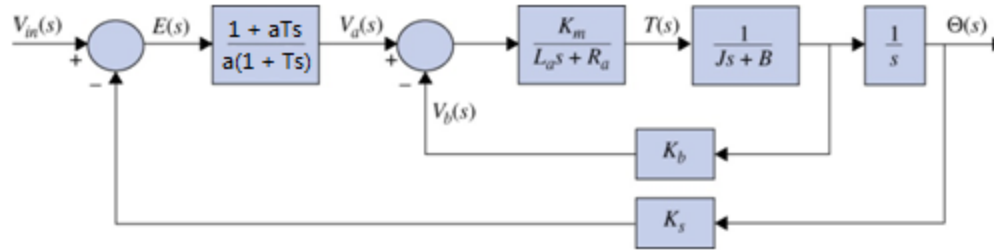


Figure 26: Block diagram of a phase lead control dc motor

Consider the second-order closed-loop transfer function (3), the transfer function of the motor is

$$G(s) = \frac{40.789}{s(s+10.827)} \quad (22)$$

Applying a phase-lead controller, the forward-path transfer function is

$$G(s) = \frac{40.789(1+aTs)}{as(s+10.827)(1+Ts)} \quad (a > 1) \quad (23)$$

Now, let us set the performance specifications as follows:

The maximum overshoot of the step response should be less than 5% or as small as possible.

Rise time $t_r \leq 1$ sec.

Settling time $t_s \leq 1.5$ sec.

The system block diagram is shown in **Figure 28**.

We can use the root-contour method to show the effects of varying T of the phase-lead controller. The ACSYS MATLAB tool was used for this root contour. Let us first set $a=2$. T varies from 0.01 to 100 in 1000 steps as shown in **Figure 29**. The forward-path transfer function of the compensated system with $a=2$ is written

$$G(s) = \frac{40.789(1+2Ts)}{2s(s+10.827)(1+Ts)} \quad (24)$$

Lead Lag Controller Design Tool

This is a tool for the design of a Phase-Lead or Phase-Lag controller. The transfer function for the controller takes the form:

$$C \frac{(aTs+1)}{(Ts+1)}$$

C is a constant which is entered in the C(s)= box in the transfer function input panel on the left of the Acsys Template window; a is entered in the edit box below; T will vary between the max and min values specified below. For each value of the varying parameter T the poles of the resulting closed loop transfer function will be plotted.

Lead Lag Controller Values

a:

T Min:

T Max:

Number of Steps:

Figure 27: Lead lag controller Design Tool

The root contour clearly shows that the system is stable when T varies in **Figure 30**. **Table 11** shows the attributes of the unit-step response when the value of T varies from 0 to 0.5. The ACSYS MATLAB tool was used for the calculations of the time response. The results show that overshoot is 0% for all these T values, although the rise and settling times increase continuously as T increase. With a=2, the design criteria meets when T = 0.01.

Table 11: Attributes of Unit-step response of system with Phase-Lead Controller

a=2	Percent Overshoot	Settling Time (5%) t_s (sec)	Rise Time t_r (sec)
T=0	0%	1.35 sec	0.96 sec
T=0.01	0%	1.4 sec	0.99 sec
T=0.1	0%	1.62 sec	1.24 sec
T=0.5	0%	2.21 sec	1.15 sec

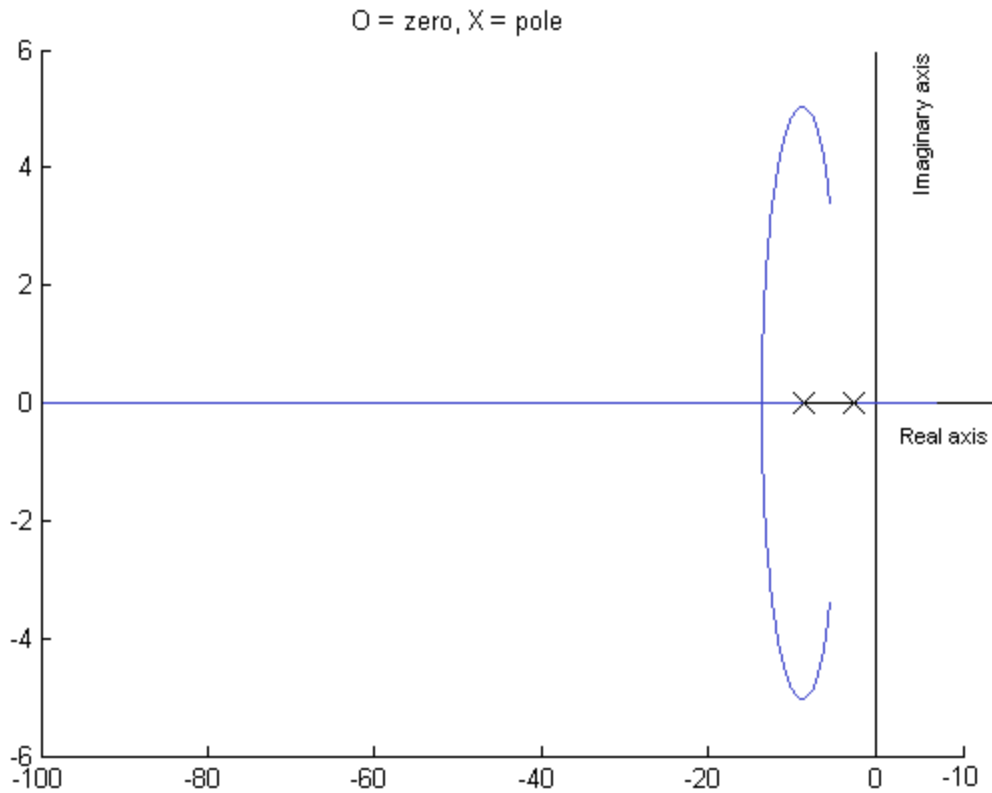


Figure 28: Root contours of the Robot Arm system with a phase-lead controller

Frequency Domain Analysis

Now, let us consider the frequency domain design. Using the ACSYS software, the bode plot with $a=2$, $T=0.01$ is shown in **Figure 31**.

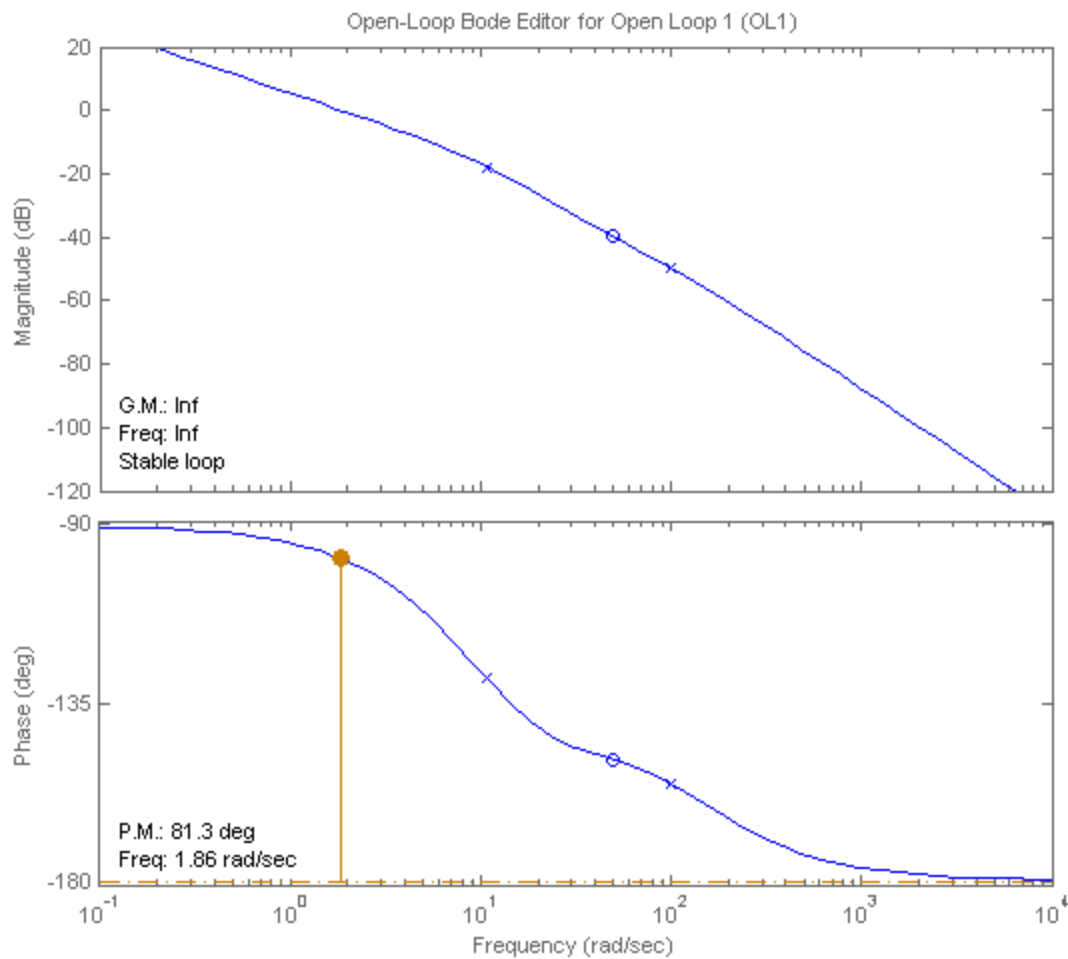


Figure 29: Bode plot of the phase-lead compensation system

7 Phase Lag Controller

Time Domain Analysis

Consider the second-order closed-loop transfer function (3), the transfer function of the motor is

$$G(s) = \frac{40.789}{s(s+10.827)} \quad (24)$$

Applying a phase-lag controller, the forward-path transfer function is

$$G(s) = \frac{40.789(1+aTs)}{as(s+10.827)(1+Ts)} \quad (a < 1) \quad (25)$$

Now, let us set the performance specifications as follows:

The maximum overshoot of the step response should be less than 5% or as small as possible.

Rise time $t_r \leq 1$ sec.

Settling time $t_s \leq 1.5$ sec.

The system block diagram is shown in **Figure 28**.

We can use the root-contour method to show the effects of varying T of the phase-lag controller. The ACSYS MATLAB tool was used for this root contour. Let us first set $a=0.5$. T varies from 0.01 to 100 in 1000 steps as shown in **Figure 32**. The forward-path transfer function of the compensated system with $a=0.5$ is written

$$G(s) = \frac{40.789(1+0.5Ts)}{0.5s(s+10.827)(1+Ts)} \quad (26)$$

Lead Lag Controller Design Tool

This is a tool for the design of a Phase-Lead or Phase-Lag controller. The transfer function for the controller takes the form:

$$C^*(aTs+1)/(Ts+1)$$

C is a constant which is entered in the $C(s)=$ box in the transfer function input panel on the left of the Acsys Template window; a is entered in the edit box below; T will vary between the max and min values specified below. For each value of the varying parameter T the poles of the resulting closed loop transfer function will be plotted.

Lead Lag Controller Values

a :

T Min:

T Max:

Number of Steps:

Figure 30: Lead lag controller Design Tool

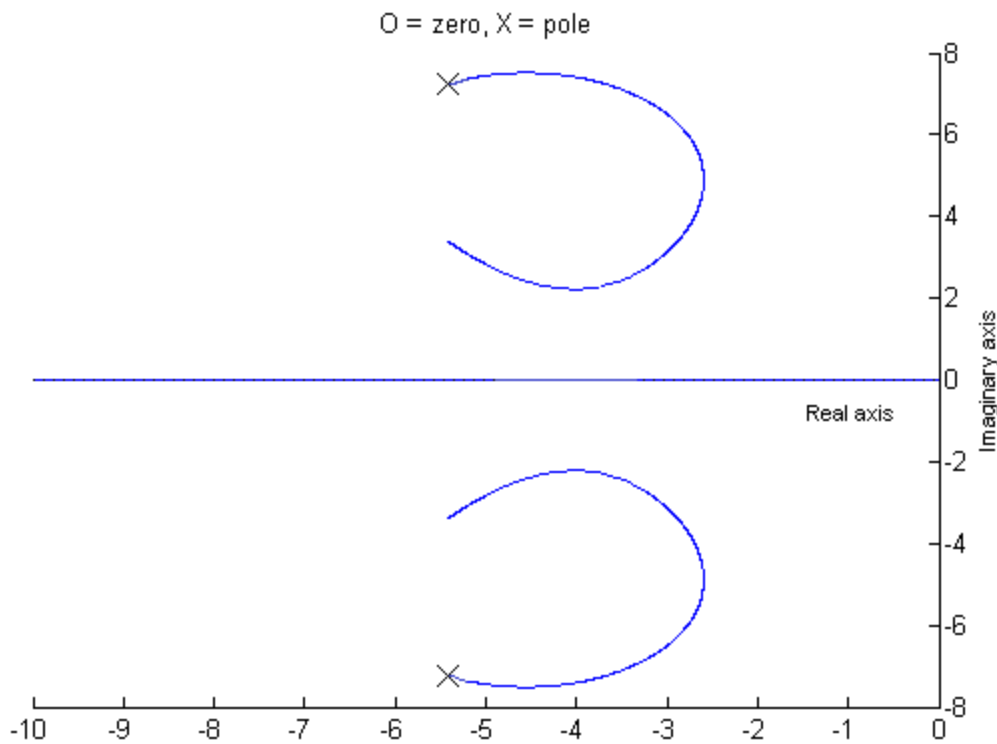


Figure 31: Root contours of the system with a phase-lag controller for $a=0.5$

The root contour clearly shows that the system is stable when T varies in **Figure 33**. **Table 12** shows the attributes of the unit-step response when the value of T varies from 0 to 0.5. The ACSYS MATLAB tool was used for the calculations of the time response. The results show that the smallest maximum overshoot is obtained when $T = 0$, although the rise and settling times increase continuously as T increase. However, the smallest value of the maximum overshoot is 9%, which exceeds the design specification.

Table 12: Attributes of Unit-step response of system with Phase-Lag Controller

$a=0.5$	Percent Overshoot	Settling Time (5%) t_s (sec)	Rise Time t_r (sec)
$T=0$	9%	0.57 sec	0.21 sec

T=0.01	11%	0.59 sec	0.20 sec
T=0.1	21%	0.72 sec	0.23 sec
T=0.5	20%	1.08 sec	0.29 sec

Next, we set $a = 0.8$. T varies from 0.01 to 100 in 1000 steps. The forward-path transfer function of the compensated system with $a=0.8$ is written

$$G(s) = \frac{40.789(1+0.8Ts)}{0.8s(s+10.827)(1+Ts)} \quad (26)$$

The root contour clearly shows that the system is stable when T varies in **Figure 34**. **Table 13** shows the attributes of the unit-step response when the value of T varies from 0 to 0.5. The ACSYS MATLAB tool was used for the calculations of the time response. The results show that the smallest maximum overshoot is obtained when $T = 0$, although the rise and settling times increase continuously as T increase. The percent overshoot is less than 5% when $T \leq 0.1$.

Table 13: Attributes of Unit-step response of system with Phase-Lag Controller

a=0.8	Percent Overshoot	Settling Time (5%) t_s (sec)	Rise Time t_r (sec)
T=0	3%	0.44 sec	0.32 sec
T=0.01	3%	0.45 sec	0.33 sec
T=0.1	5%	0.64 sec	0.33 sec
T=0.5	6%	0.91 sec	0.35 sec

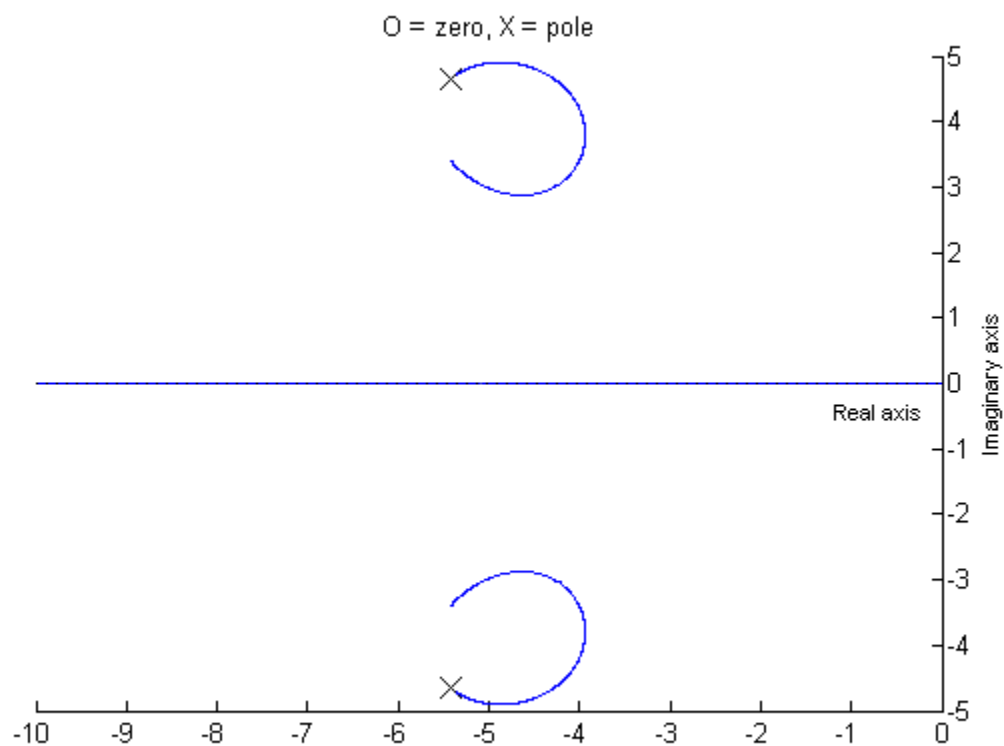


Figure 32: Root contours of the system with a phase-lag controller for $a=0.8$

Frequency Domain Analysis

Now, let us consider the frequency domain design. Using the ACSYS software, the bode plot with $a=0.8$, $T=0.01$ is shown in **Figure 35**.

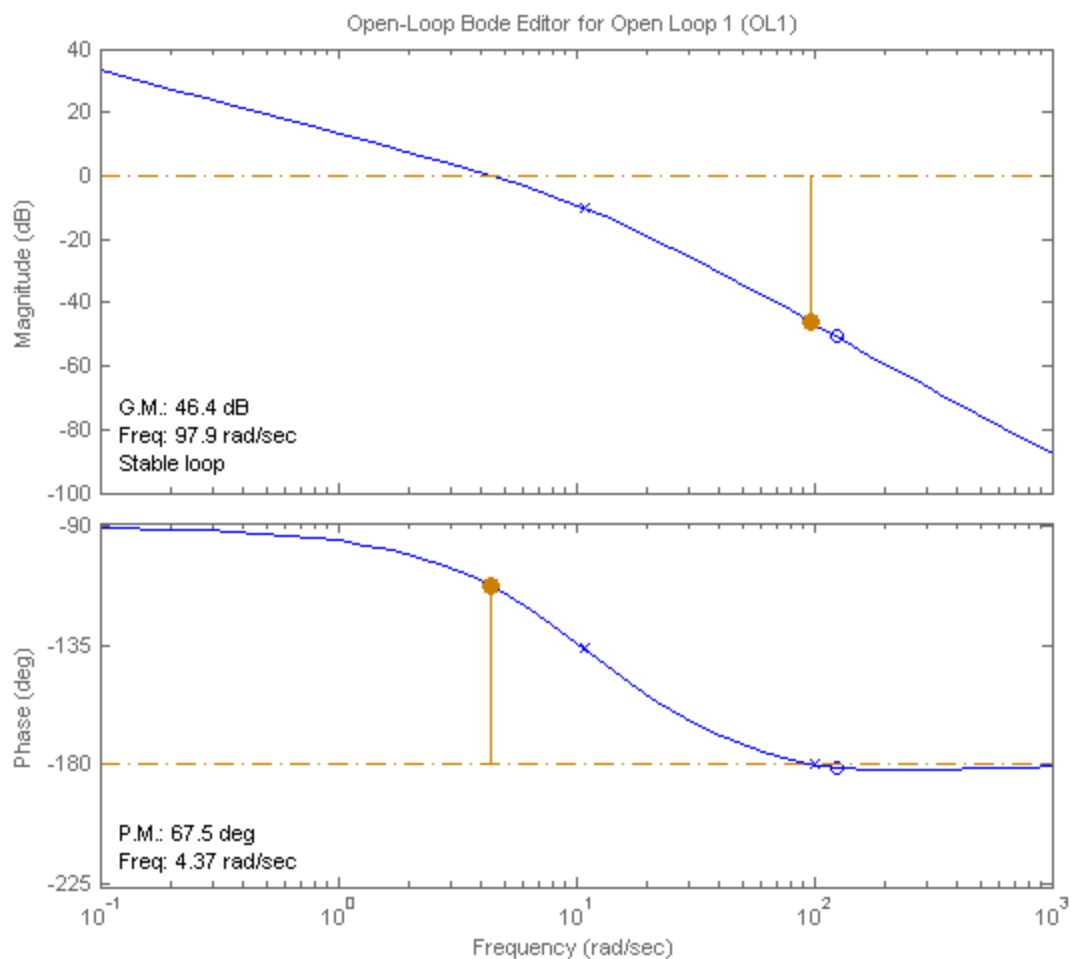


Figure 33: Bode plot of the phase-lag compensation system

8 Improving Transient Response via PD Control for the Third Order System

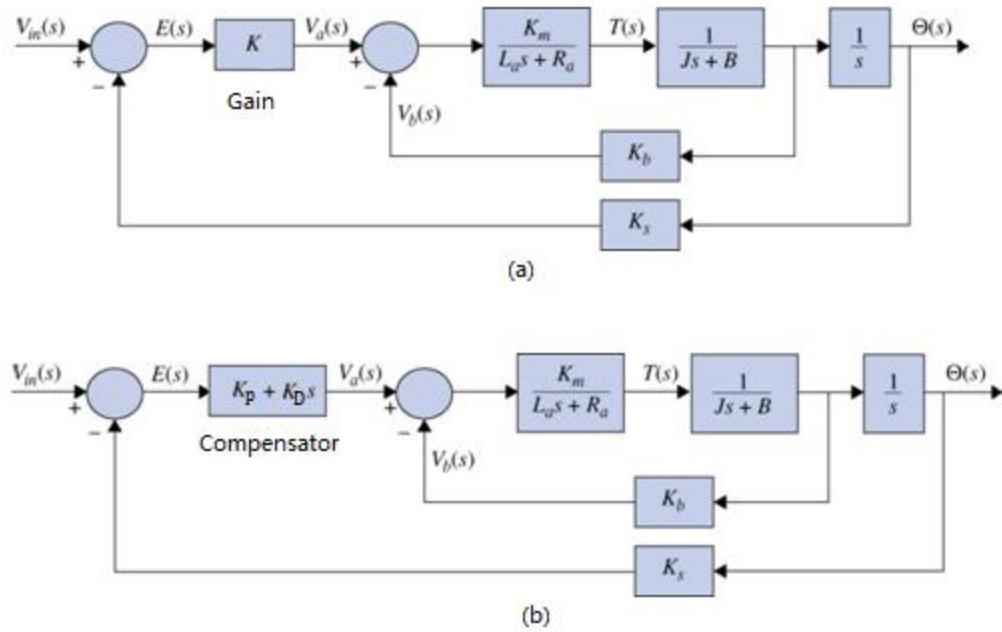


Figure 36: Closed loop system for dc motor: a. before compensation; b. after differential compensation

Now let us consider the third-order attitude control system described as in 5-7- 3.

$$G(s) = \frac{\Theta(s)}{\Theta_{in}(s)} = \frac{\frac{KK_m K_s}{R_a}}{\left(\frac{L_a}{R_a}s + 1\right) \left\{ Js^2 + \left(B + \frac{K_b K_m}{R_a}\right)s + \frac{KK_m K_s}{R_a} \right\}} \quad (27)$$

Substituting the preceding values into the system transfer function, the closed-loop transfer function of the uncompensated system of **Figure 36(a)** is

$$\frac{\Theta_y(s)}{\Theta_r(s)} = \frac{4.079 \times 10^5 K}{s^3 + 10010.827s^2 + 108270s + 4.079 \times 10^5 K} \quad (28)$$

The forward-path transfer function becomes

$$G(s) = \frac{4.079 \times 10^5 K}{s(s + 10.827)(s + 10000)} \quad (29)$$

Applying the PD controller of Eq. (29), the forward-path transfer function of the integral compensated system of **Figure 36 (b)** becomes

$$G(s) = \frac{4.079 \times 10^5 (K_p + K_D s)}{s^2 (s + 10.827)(s + 10000)} \quad (30)$$

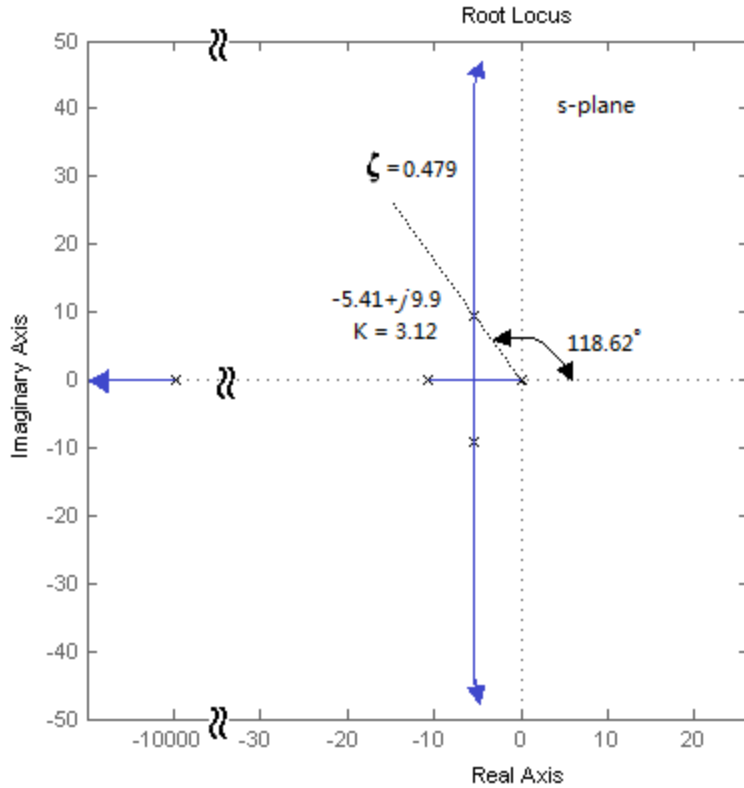


Figure 37: Root locus for uncompensated system shown in Figure 36. (a)

Let us first evaluate the performance of the uncompensated system operating with 18% overshoot. The root locus for the uncompensated system is shown in **Figure 41**. 18% overshoot

is equivalent to $\zeta = \sqrt{\frac{\ln(0.18)^2}{\pi^2 + \ln(0.18)^2}} = 0.479$, we search along that damping ratio line for an odd multiple of 180° and find that the dominant, second order pair of poles is at

$-5.41 \pm j9.9 = \omega_o(-\zeta \pm \sqrt{\zeta^2 - 1})$. So, the angel equal to $\tan^{-1}(\frac{\sqrt{1-\zeta^2}}{-\zeta}) = 118.62^\circ$. Thus, the settling

time of the uncompensated system is

$$T_s = \frac{4}{\zeta \omega_n} = \frac{4}{5.41} = 0.739$$

Since our evaluation of percent overshoot and settling time is based upon a second order approximation, we must check the assumption by finding the third pole and justifying the second order approximation. Our third pole is smaller than -10000, which is very far from the $j\omega$ axis as the dominant, second order pair. We conclude that our approximation is valid. The transient and steady state error characteristics of the uncompensated system are summarized in **Table 14**.

Table 14 Uncompensated and compensated system characteristic

	Uncompensated	Compensated
Plant and compensator	$\frac{4.079 \times 10^5 K}{s(s+10.827)(s+10000)}$	$\frac{4.079 \times 10^5 K(s+47.139)}{s(s+10.827)(s+10000)}$
Dominant poles	$-5.41 \pm j9.9$	$-10.840 \pm j19.865$
K	3.12	0.261
ζ	0.479	0.514
ω_n	11.3	22.4
%OS	18	15.2
T_s	0.739	0.369
T_p	0.317	0.163
Third pole	-10000	-10000
Zero	none	-47.139
comments	Second-order approximation OK	Pole-zero Not cancelling

Now we proceed to compensate the system. First we find the location of the compensated system's dominant poles. In order to have a twofold reduction in the settling time, the compensated system's settling time will be half of Eq. 6. The new settling time will be 0.369. Therefore, the real part of the compensated system's dominant, second order pole is

$$\sigma = \frac{4}{T_s} = \frac{4}{0.369} = 10.840$$

Figure 37 shows the designed dominant, second order pole, with a real part equal to and an imaginary part of

$$\omega_d = 10.840 \tan(180^\circ - 118.62^\circ) = 19.865$$

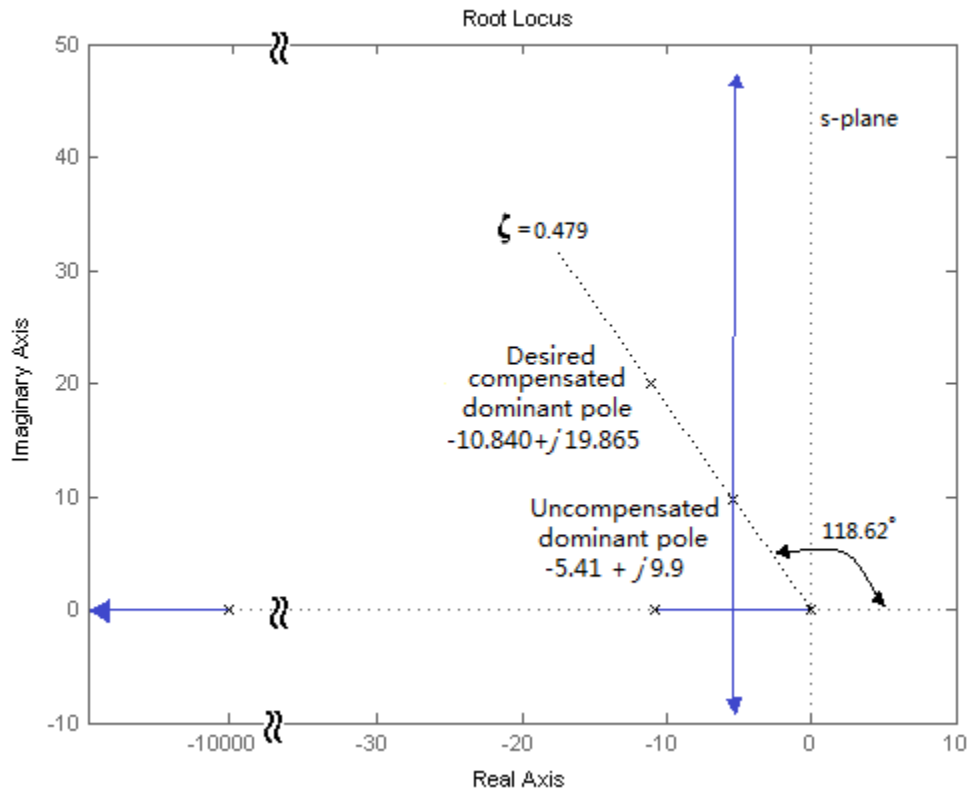


Figure 348: Compensated dominant pole superimposed over the uncompensated root locus

Next we design the location of the compensator zero. Input the uncompensated system's poles and zeros in the root locus program as well as the design point $-10.840 \pm j19.865$ as a test point. The result is the sum of the angles to the design point of all the poles and zeros of the uncompensated system except for those of the compensator zero itself. The difference between the result obtained and 180° is the angular contribution required of the compensator zero. We can ignore the third pole at -10000 because the angular contribution is almost zero. Using the open loop poles shown in **Figure 38** and the test point $-10.840 + j19.865$, which is the desired dominant second order pole, the angular contribution of the pole at -10000 is approximately 0° ; the angular contribution of the pole at 0 is 118.61° ; the angular contribution of the pole at -10.827 is $\tan^{-1}\left(\frac{19.865}{-10.840 - (-10.827)}\right) = 90.07^\circ$. The sum of the angle equals to

$0^\circ + 90.07^\circ + 118.62^\circ = 208.69^\circ$. Hence, the angular contribution required from the compensator zero for the test point to be on the root locus is $+208.69^\circ - 180^\circ = 28.69^\circ$. The geometry is shown in **Figure 39**, where we now must solve for $-\sigma$, which is the location of the compensator zero.

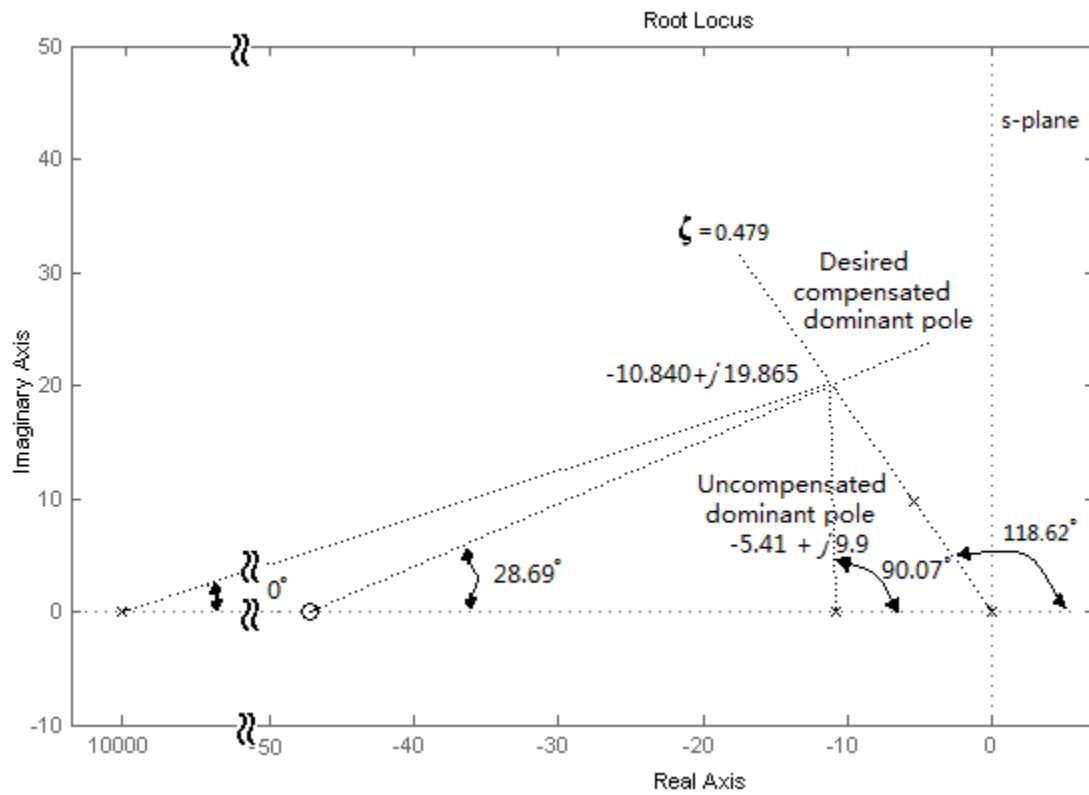


Figure 39: Evaluating the location of the compensating zero

Form the **Figure 39**,

$$\frac{19.865}{\sigma - 10.840} = \tan 28.69^\circ$$

Thus, we get $\sigma = 47.139$. The complete root locus for the compensated system is shown in **Figure 40**.

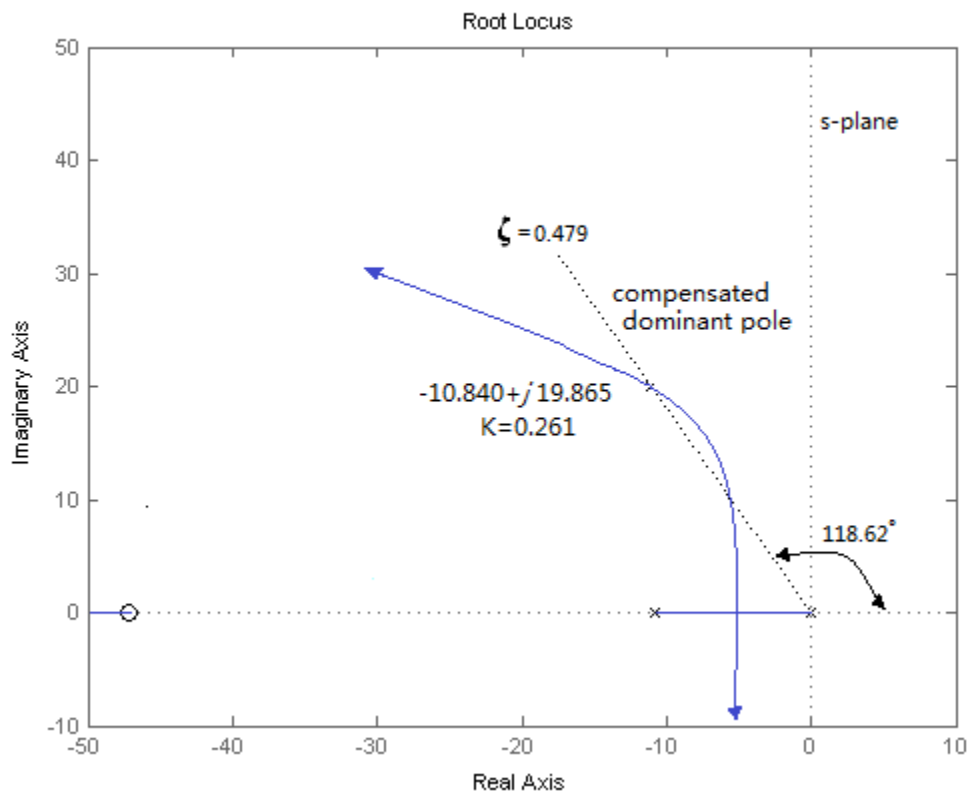


Figure40: Root locus for the compensated system

Table 14 summarizes the results for both the uncompensated system and the compensated system. For the uncompensated system, the estimate of the transient response is accurate since the third pole is more than one hundred times larger than the real part of the dominant, second-order pair. The second order approximation for the compensated system, however, may be invalid because there is no approximate closed loop third pole and zero cancellation between the closed loop pole at -10.827 and the closed loop zero at -47.139 . The results of a simulation are shown in **Table 15**. The simulation results can be obtained using Matlab or ACSYS. The percent overshoot differs by 3% between the uncompensated and compensated systems, while there is approximately a twofold improvement in speed as evaluated from the settling time.

The final results are displayed in **Figure 41**, which compares the uncompensated system and the faster compensated system.

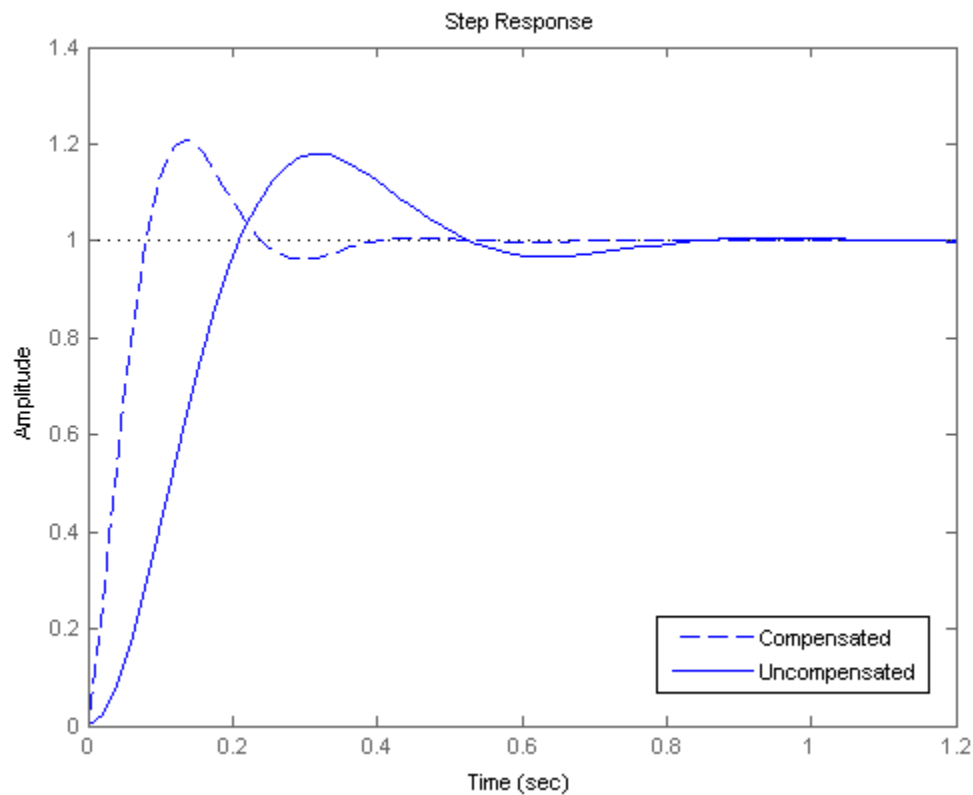


Figure 41: Uncompensated and compensated system step response

Using the model show in **Figure 5-69**, the closed-loop position response of the motor with play load is simulated for a step input 1 rad or 57.296 degrees. The results are shown below in **Figure 42** for $K_p = 3.12$, $K_D = 0$ and $K_p = 12.303$, $K_D = 0.261$. **Table 15** summaries the results for both uncompensated and compensated system step response.

Table15 Robotic arm system step response comparison

		Percent Overshoot	Settling Time (5%)	Peak Time	Steady State Error
Simulated Position Response	$K_p = 3.12$ $K_D = 0$	18	0.47	0.314	0
	$K_p = 12.303$ $K_D = 0.261$	21	0.213	0.136	0
Robotic Arm Position Response	$K_p = 3.12$ $K_D = 0$	1.7	0.36	0.32	4.7
	$K_p = 12.303$ $K_D = 0.261$	5.2	0.23	0.21	0.52

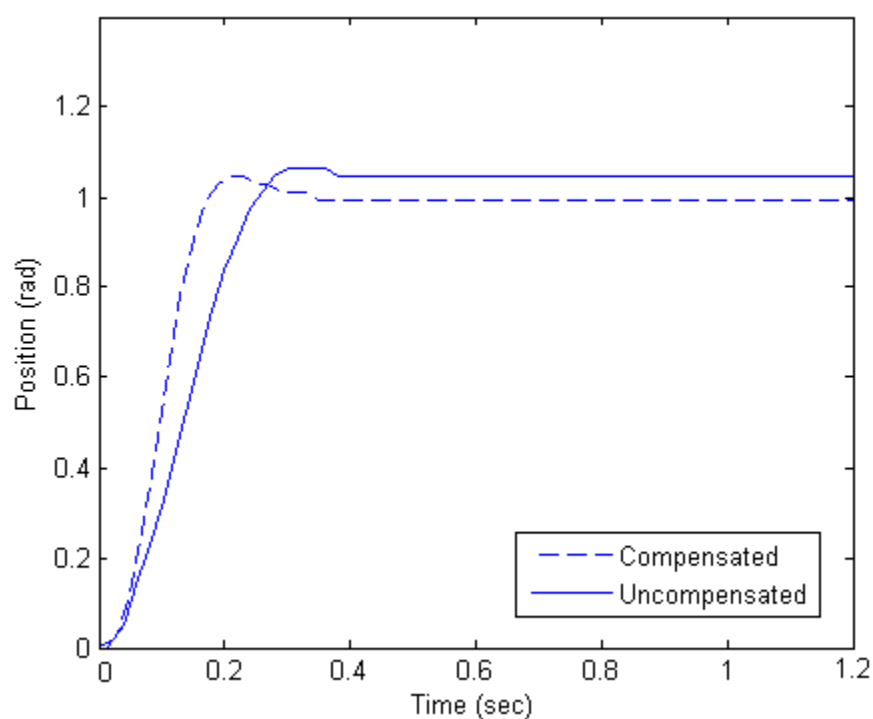


Figure42: Robotic arm uncompensated system and the faster compensated system